Probing Convex Polygons with Half-planes[1]

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Abstract: A *half-plane probe* through a polygon measures the area of intersection between a half-plane and the polygon. We develop techniques based on x-ray probing to determine convex n-gons in $7n + 7$ half-plane probes. We also show $n + 1$ half-plane probes are sufficient to verify a specified convex polygon and prove linear lower bounds for determination and verification.

Keywords: Theory of robotics, computational geometry, probing, half-planes, x-rays, convexity, complexity.

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1. Introduction

Identifying and understanding objects from sensory data is a fundamental problem in robotics and computer vision. Although imaging devices provide a tremendous amount of information, for reasons of economy and robustness, simple sensors are often used. Recently, there has been interest in studying the geometric properties of sensor models. The *finger probe* [123456] determines the first point of intersection between a directed line and a polygon. The *x-ray probe* [7] measures the length of intersection between a polygon P and the line l. For both of these sensing models there exist strategies for determining a convex n-gon in a number of probes linear in n. In this paper, we define a new probing model, the *half-plane probe* and give a linear probing strategy for using it.

A half-plane probe returns the area of intersection between a closed half-plane h and a polygon P. We define $h(l)$ to be the area of intersection between P and the closed half-plane to the left of the directed line l. There is a close relationship between x-ray and half-plane probes which we exploit to develop a linear half-plane probing strategy. The approach of this paper is similar to [7] but the resulting strategy requires different and more interesting geometric arguments to prove its correctness.

The original inspiration for studying half-plane probes was the famous story of Archimedes determining whether the king's crown was gold or silver by measuring the volume of water it displaced. Such dunks in the tub are really half-space probes. More importantly, half-plane probing problems have application to tomography [8] and remote sensing, such as the lunar occultation observations used to map astrostellar radio sources [9]. The instruments for measuring such radio sources have a lower resolution than desired, so each measurement represents the total amount of energy over an area. By waiting until the moon passes over a portion of the region and measuring how much the energy is reduced, detailed maps of the source can be produced. This is similar to our notion of a half-plane probe, although one cannot actively select the probing directions with such instruments. Finally, the study of half-planes proves important in unifying many problems in geometric probing [1011].

We assume that we are given a point O within P, which identifies the general location of P in the plane. Without such a point to provide a general idea of where P is, it is not clear how to find P in a finite number of probes. For convenience we assume O is within the interior of P.

A collection of half-plane probes through an object provides us with a great deal of information about it but not directly with the coordinates of a point on the surface. Half-plane probes have the advantage that they in some sense reflect the entire structure of the polygon in every probe. Thus they provide the possibility of extending probing results to simple polygons, since unlike with finger and x-ray probes concave edges are potentially verifiable.

This paper will give upper and lower bounds on the number of probes needed for the following two problems: to determine convex n-gons and to verify a conjectured convex polygon.

2. An Upper Bound for Half-Plane Probing

To obtain the vertices of P from half-plane probes, we have found it useful to think in terms of groups of probes which work together. This section considers different classes of probes, what powers and limitations they possess and how they interact to lead to probing strategies. These classes are designed to achieve the complementary goals of recognizing and determining edge pairs.

2.1. Origin Probes

The first class of probes are *origin probes*, a set of half-plane probes bounded by lines all aimed through a common point O within the object. Any half-plane probe which intersects a convex polygon and whose bounding line avoids vertices will pass through exactly two edges of the object. As proven in [7] the largest possible number of such edge pairs is n.

Each half-plane is defined by a directed line. We can therefore consider the complete set of origin probes through a point $O = (0, 0)$ to be defined by $y = tx$, where $t = \tan(\theta)$, $2\pi < \theta \le 0$. These define a function $f(t) = h(tx) - h(0)$, where $h(l)$ is the area returned by half-plane probe l. This function will contain enough information to determine the edges split by the probe (henceforth called the *probed edges*), except for special cases. Here, we consider $f(t)$ for a wedge defined by two lines and containing the origin, where each line contains an edge of the polygon.

Lemma 1: Let l_1: $y = m_1 x + b_1$ and l_2: $y = m_2 x + b_2$ be two distinct lines, $m_1, m_2 \ne 0$, and let $z = f(t)$ be defined as above. Then

$$Azt^2 + Bzt + Cz + Dt^2 + Et = 0 \ ,$$

where $A = 2m_1 m_2$, $B = -2({m_1}^2 m_2 + m_1 {m_2}^2)$, $C = 2{m_1}^2{m_2}^2$, $D = {b_2}^2 m_1 - {b_1}^2 m_2$, and $E = {m_2}^2{b_1}^2 - {m_1}^2{b_2}^2$.

Proof: Consider the situation in Figure 1, where both edges intersect the x-axis. This involves no loss of generality, since a rotation of the axes can always be performed. Hence, we need not consider the case where either slope is 0. For any t, the area swept out between $y = 0$ and $y = tx$ is the sum of the areas of the two triangles defined by $y = 0$, $y = tx$, and either l_1 or l_2. The value of $z = f(t)$ is defined to be the difference in area between the two triangles, $z = A_1 - A_2$. More formally,

$$z = \frac{t{b_1}^2}{2m_1(m_1 - t)} - \frac{t{b_2}^2}{2m_2(m_2 - t)} \ .$$

Multiplying through by the denominators and simplifying gives the result. □

We note that $f(t)$ is infinite when t is between m_1 and m_2. This complication does not occur when probing polygons since additional edges occur in this range. We use Lemma 1 to determine the equations of the lines that contain edge pairs. If we have a sufficient number of origin probes through a common edge pair, then we can determine

Figure 1: Defining f(t), the probes through an edge pair.

the function $f(t)$ for this edge pair. From $f(t)$, we then deduce the equations of the lines.

The function $f(t)$ is determined by five constants: A, B, C, D, and E. It follows that, in general, five probes through a pair of edges are enough to determine the function. Since all five constants are functions of the four line parameters they cannot all be independent and indeed, $C = A^2/2$. Given A, B, D, and E, we solve for the parameters of the equations obtaining:

$$m_1 = m_2 = \frac{-B \pm \sqrt{B^2 - 2A^3}}{2A}$$

$$b_1{}^2 = \frac{E + m_1 D}{m_2(m_2 - m_1)}, \quad \text{and} \quad b_2{}^2 = \frac{E + m_2 D}{m_1(m_2 - m_1)}.$$

From these equations several limitations on our ability to reconstruct the edges become apparent. Since b_1 and b_2 are squared, we obtain no information on the sign of the intercepts. Further, m_1 and m_2 are not distinguished from each other, meaning we cannot determine which intercept belongs to which line. More seriously, b_1 and b_2 are undefined when $m_1 = m_2$. Thus any probing strategy using origin probes must take special action to handle parallel edges.

However, to exploit Lemma 1 we must ensure our probes intersect the same edge pair. Unfortunately, we cannot verify $f(t)$ by making more probes.

Lemma 2: There is no constant k such that k half-plane probes consistent with $f(t)$ implies that the probes pass through the same pair of edges of P.

Proof: Consider a regular $2k$-gon with center at O. All probes through O give $f(t) = 0$, regardless of whether they intersect the same edge pair. □

It would be nice to generalize the proof of Lemma 2 to non-parallel edge pairs. Verifying edge pairs is the motivation for parallel probes, discussed below.

2.2. Parallel Probes

Parallel probes are a set of half-plane probes defined by lines of identical slope and direction. A complete collection of parallel probes of a given slope θ results in a cumulative area histogram $C(P, \theta)$ of the area of the object. The derivative of $C(P, \theta)$ at any point gives the value of the x-ray probe defined by the probing line. Further, the complete derivative of $C(P, \theta)$ gives $C'(P, \theta)$, the *Steiner symmetral* of Hammer's x-ray problem [712] as shown in Figure 2. Using a method similar to x-ray probes, they provide a mechanism for verifying edge pairs.

Theorem 3: Four parallel half-plane probes through an edge pair are sufficient to verify the edge pair.

Proof: Without loss of generality, let us consider four parallel half-plane probes perpendicular to the line $y = 0$. Label these X_1, X_2, X_3, X_4 in order of increasing x coordinate and let p_i be the x-intercept of X_i, $1 \le i \le 4$. Let $h(X_i)$ denote the area to the left of probe X_i. Now, we define three points M_1, M_2, M_3 which we assert are collinear if and only if X_1, X_2, X_3, X_4 are all incident on the same edge pair:

$$M_1 = (\frac{p_1 + p_2}{2}, \frac{h(X_2) - h(X_1)}{p_2 - p_1})$$

$$M_2 = (\frac{p_2 + p_3}{2}, \frac{h(X_3) - h(X_2)}{p_3 - p_2})$$

Figure 2: A polygon P with $C(P, 0)$ and its derivative $C'(P, 0)$.

$$M_3 = (\frac{p_3 + p_4}{2}, \frac{h(X_4) - h(X_3)}{p_4 - p_3})$$

First, we show that if X_1, X_2, X_3, X_4 all intersect the same edge pair, then M_1, M_2, and M_3 are collinear. Suppose the two lines containing the edge pair have equations $L_i(x) = m_i x + b_i$, $i = 1, 2$, and let $L(x) = L_2(x) - L_1(x)$. By the trapezoidal rule,

$$h(X_2) - h(X_1) = (\frac{(L_2(p_2) - L_1(p_2)) + (L_2(p_1) - L_1(p_1))}{2})(p_2 - p_1) .$$

Using the above and the definition of M_1, we get

$$M_1 = (\frac{p_1 + p_2}{2}, \frac{(m_2 - m_1)(p_1 + p_2)}{2} + b_2 - b_1) .$$

This implies $M_1 \in L$. Likewise, M_2 and M_3 are on L. This shows that the three points are collinear if the probes intersect the same edge pair.

Now we show that if the four probes do not intersect the same edge pair, then M_1, M_2, and M_3 cannot be collinear. Let q_i be the intersection of X_i and convex curve c representing a chain of edges of P, as in Figure 3. Note that M_i is on or above the line $q_i q_{i+1}$, but on or below c. The line $M_1 M_2$ must pass on or below q_2 by convexity, but must be on or above c at q_3 and beyond, again by convexity. Thus M_3 cannot be above the line $M_1 M_2$ if the chain of edges c is convex, and M_3 is on $M_1 M_2$ only if c is a straight line between q_1 and q_4, meaning the four probes intersect the same edge pair. □

Figure 3: Non-edge pairs violating convexity or collinearity.

2.3. Determining a Boundary Point

Since we know how to verify and determine edge pairs, we can now proceed to locate a point on the boundary of P.

First, we must identify a section of P through which we can parallel probe. We assume knowledge of a point O within P; to parallel probe we must find another such point to ensure all our probes intersect P. We start by sending two horizontal probes through O, one directed above $y = 0$ and one directed below. The total of their values is the area A of P. For each subsequent probe P, this information enables us to obtain the area on both sides of P.

Three additional probes will be sufficient to identify a section to probe. Let A_0 be the area above $y = 0$. We note that the shape of size A_0 minimizing the maximum of both height and horizontal spread above $y = 0$ will be a square centered on the y-axis and resting on the x-axis. Thus at least one of the probes $x = \pm\sqrt{A_0}/2$ or $y = \sqrt{A_0}$ must intersect P and with $x = 0$ or $y = 0$, respectively, determines a section to parallel probe.

Four parallel probes on an edge pair are sufficient to verify that edge pair. We now repeatedly send parallel probes within this section, in any order, until one edge pair has received 4 such probes. This must occur within $3(n-2)+4 = 3n-2$ parallel probes, since there are at most $n-1$ edge pairs. Since we will be probing in either a horizontal or vertical direction, at least one of the preliminary probes will have intersected P, so it can also be used as a parallel probe. Thus at most $3n-3$ additional probes are needed to verify one edge pair.

2.3.1. Bounding the Extent of the Polygon

To determine the edges, we must origin probe the edge pair. We have identified an open ended strip or *section* which contains an edge pair. Any point within the interior of this section is a candidate for the origin. To origin probe we must ensure that our probes intersect P within the section we have defined. This means aiming the probes at a sufficiently steep angle, which can be determined by simple trigonometry once an upper bound on the distance between the origin and the intersections between the edge pair and the section has been determined.

By sending probes defined by lines g and h perpendicular to the parallel probing direction, we can partition the area of P into three sections, the area between the two probes, above the probes, and below the probes, denoted α, β, γ, respectively. Up to two of the areas α, β and γ may be 0, depending on whether g and h intersect P. By choosing g and h to lie on opposite sides of O, we ensure that $\alpha > 0$. If β and γ are 0, g and h represent bounds on the edges of P. If not, we are interested in the convex object which intersects g and h and maximizes its height subject to the area constraints.

Lemma 4: There exists a triangle with base on h with area β above g and α between g and h.

Proof: We will construct such a triangle of height η, to be determined below. Let the distance between g and h be 1. Then the part of the triangle above g is a similar triangle with height $\eta - 1$. Since the two triangles are similar, the ratio of their areas is

$$\frac{(\eta - 1)^2}{\eta^2} = \frac{\beta}{\alpha + \beta}$$

If we choose $\eta = \dfrac{\alpha + \beta + \sqrt{(\alpha + \beta)\beta}}{\alpha}$ and the base of the triangle equal to $2(\alpha + \beta) - 2\sqrt{(\alpha + \beta)\beta}$ then the triangle fulfills the two conditions. □

Lemma 5: If there exists a convex figure different from a triangle that fulfills the requirements of Lemma 4, then there is a triangle that is higher and also fulfills the requirements.

Proof: Notice that this triangle has height η defined above, which implies the height of any other convex figure is less than η, if the assertion is true.

Let t be the topmost point of the figure. Construct a triangle Δ such that (a) its base lies in h and t is its topmost point, (b) it is similar to a triangle of height η and area $\alpha + \beta$, and (c) the part of Δ above line g is contained in the part of the convex figure above g.

Triangle Δ exists since we can choose the sides so they intersect interval (a, b) (see Figure 4) which is the intersection between g and the convex figure. For consider the triangle whose two sides go through a and b; its upper part is contained in the figure's upper part, and its lower part contains the figure's lower part. Thus this triangle Δ contains another triangle with the right proportions, that is, the intersection between Δ and g must be within the interval (a, b).

The upper part of Δ is smaller than β which shows that Δ has to be increased in order to realize total area $\alpha + \beta$. Thus η is greater than the height of the figure which is

Figure 4: Constructing a similar triangle Δ to the height bound.

the same as the height of Δ. □

By similarly considering the areas α and γ, the other side of P can be bounded and the origin probes aimed.

2.3.2. Parallel Edges

For x-ray probes, parallel edges proved to be a substantial obstacle, since the intercepts were undefined and could not be determined by additional probes through the edge pair, as the length of intersection is a function only of the separation between the edge pair and the angle of the probe [7]. We now give a procedure for determining the intercepts of parallel edges, using half-plane probes.

In the case of parallel edges, from the degenerate $f(t)$ we can determine the slopes and with the probes defining the section, the distance between the two parallel edges. To complete our knowledge, we must determine the intercepts. By performing a rotation on P so that the parallel edges of P are perpendicular to the x-axis, we can obtain the situation in Figure 5. Let p be a point known to be outside P, determined via the techniques of the previous section. Both edges can be determined from d, the distance from p to l_1. We define d' to be distance between l_1 and l_2, which has been determined by the parallel probes through l_1 and l_2. Aiming two probes with slopes 0 and m through p, l_1, and l_2 determines the area τ of the trapezoid between l_1 and l_2. This trapezoid represents the difference between two similar triangles so:

$$\tau = \frac{(d + d')(m(d + d'))}{2} - \frac{d(md)}{2} \ .$$

Solving for d gives

Figure 5: Determining parallel edges

$$d = \frac{\tau}{md'} - \frac{d'}{2} \ .$$

Thus two additional probes are sufficient to determine the parallel edges. For non-parallel edges, the slopes (m_1, m_2) and possible intercepts $(b_1, -b_1, b_2, -b_2)$ define a total of 8 lines. We can determine which two are correct by probing along each of them. If p_1 and p_2 are the parallel probes which defined the section, the two correct lines l_1 and l_2 will result in probes of zero area and with p_1 and p_2 define a quadrilateral of exactly the observed area between p_1 and p_2. Thus eight additional probes suffice to determine the edge pair. In fact, only two additional probes are required, and we leave this to the reader to explore.

Lemma 6: $3n + 15$ probes are sufficient to determine the first edge pair and a point on the boundary of P.

Proof: The preceding discussion shows we spend five probes to initialize the search, $3n - 3$ additional parallel probes to define a section, two probes to bound the height of P, three additional origin probes (one of the parallel probes can be used as the fourth origin probe), and up to eight additional verification probes. Every point on l_1 or l_2 within the section is determined. □

2.4. Boundary Probes

A *boundary probe* is defined to be a probe through a point known to be on the boundary of P. We use boundary probes to develop a more efficient probing strategy based on the following observation:

Lemma 7: Three parallel probes through an edge pair are sufficient to determine the second edge, if one edge of the edge pair contained in line $l = mx + b$ is known.

Proof: Rotate P clockwise by $arctan(m)$ so that the known edge lies on the x-axis. Three parallel probes through the rotated edge define points M_1 and M_2 as described in the proof of Theorem 3 and subject to the inverse rotation define the other edge. □

2.5. Determining a Convex Polygon

After determining an edge pair, we have the situation in Figure 6. The edges contain known points p_1, p_2 and q_1, q_2. If the edge pair is not parallel, we conjecture the edges meet at v. To test this, we need a probe through q_1 and p_2. If it returns the area of triangle (q_1, p_2, v), we have verified vertex v, otherwise, there is at least one additional edge in the unexplored corner. Let v' be the point on l_1 such that the area returned by the probe through q_1 and p_2 equals $\alpha(q_1, q_{2,} p_2) + \alpha(p_{2,} q_2, v')$, where we define $\alpha(a, b, c)$ to be the area of the triangle defined by the three points.

Figure 6: Determining vertices of P

Thus edge $(p_1 p_2)$ cannot extend past v' without violating convexity. Probing parallel to $q_1 v'$ between q_1 and q_2 we intersect a new edge pair, one of which is $(q_1 q_2)$. Observe that this construction gives an appropriate v' even when the edge pair is parallel.

We parallel probe this section, and later we consider these probes as boundary probes when we have determined an edge pair with $(q_1 q_2)$ as the known line. More precisely, after two arbitrary probes of the same slope to the interior of the section, we aim the ith parallel probe between the $(i-1)$st and the $(i-2)$nd parallel probe until four of them verify an edge pair. By Lemma 7, these four probes also determine the other edge. If the first four parallel probes fail to verify an edge pair, it implies there is an unknown chain of edges in the section. In this case, the extra parallel probes with v' or q_2 define a new section containing an undetermined edge pair.

We probe through $(q_1 q_2)$, repeatedly determining the other edge in the rightmost unexplored section. Thus we can fill in all the edges between q_1 and p_1 (traversing counterclockwise). We determine the remaining edges by repeating this process through $(q_1 q_2)$, starting parallel to the last edge on this chain, which is initially $(p_1 p_2)$. For each remaining edge e, we have already implicitly computed the point v', since one of the origin or parallel probes that distinguished e's edge pair also returns the area to both sides of the probe and hence implicitly yields v' for edge e. Thus we need at most four additional probes to determine each remaining edge, which brings us to our main result:

Theorem 8: $7n + 7$ half-plane probes are sufficient to determine a convex n-gon.

Proof: By Lemma 6, $3n + 15$ probes are sufficient to determine the first two edges. From the preceding discussion, four probes are sufficient to determine each additional edge. Thus the total number of probes required is $(3n + 15) + 4(n - 2) = 7n + 7$. □

3. A Lower Bound for Half-Plane Probing

We can prove a lower bound on the number of half-plane probes necessary for determination through a dimensionality argument:

Theorem 9: At least $2n$ half-plane probes are necessary to determine a convex n-gon.

Proof: We can represent a convex polygon P as a point in $2n$-dimensional space. Every point in an ε neighborhood around P represents a slightly different convex polygon.

Lefschetz [13] proves that a region of a Euclidean m-space cannot be parameterized by less than m parameters. Since the result of a probe is an algebraic function of P it can be considered to be a parameter of P. Thus at least $2n$ probes are necessary to determine a convex polygon. □

It is counter-intuitive that a single probe is only one parameter, since in addition to the area returned by the probe it is specified by the line defining the half-plane, which requires two more real numbers. However, the half-planes are defined by the probing algorithm as a function of the previous probes, so no additional information is conveyed by them. The vector of probing results is in itself a specification of P. This lower bound argument does not hold for verification, because a specified polygon does provide information and the vector of probing outcomes does not determine P.

Note that a similar argument can be used to show that $2n$ x-ray probes are necessary for determination, improving the result in [7] of $(3n-1)/2$. A geometric argument which proves a weaker, but still linear lower bound for half-plane probing is given in the next section.

4. Bounds for Verification

Verification is the problem of proving that a particular object is indeed correctly described by the given representation of a polygon P. It is obvious that any lower bound to verification represents a lower bound to the determination problem, since it presupposes knowledge of the polygon.

Theorem 10: $n+1$ half-plane probes are sufficient to verify a convex n-gon.

Proof: For one of the edges of P, probe in both directions along the line containing the edge. For the remaining $n-1$ edges, probe once along the defining line. With each edge, we know P lies entirely within each of n half-planes. The intersection of these half-planes is P. Since the intersection of these half-planes has exactly the area of P, we have verified P. □

Note that fewer half-plane probes are necessary for verification than is the case for finger or x-ray probes [17].

Theorem 11: At least $2n/3$ half-plane probes are necessary to verify a convex n-gon.

Proof: We identify a collection of restrictions which a set of probes must meet for them to verify a given n-gon.

(1) If the relative interior of an edge is not intersected by a probe, both its vertices must be intersected.

(2) If a vertex is not intersected by a probe, then both its incident edges must be intersected in their relative interiors.

(3) No two consecutive edges (a, b) and (b, c) can be verified without at least one probe within the relative interior of either (a, b) or (b, c).

(4) No two consecutive vertices a and b from a chain of vertices (x, a, b, y) can be verified with single probes to the relative interior of (x, a), (a, b), and (b, y).

(5) No three consecutive edges (a, b), (b, c), and (c, d) may be verified with probes along edges (a, b) and (c, d), and a probe through the relative interior of (b, c).

Figure 7a illustrates restriction (1), which is a corollary of (2). For (2), see Figure 7b. We can shorten the edge intersected by a probe and use an additional vertex to raise a triangle on the unprobed edge to regain the area. Figure 7c demonstrates the necessity of

Figure 7: Forbidden cases for verifying probes.

(3). We can replace the center vertex with two other vertices without changing the result of any probe. For (4) see Figure 7d. We can replace the center edge and two incident vertices by a triangle without changing the result of any probe. Finally, for (5) see Figure 7e. As with (4), we can replace the center edge and two incident vertices by a triangle without changing the result of any probe.

We now walk around the boundary of the polygon and count the minimum number of sites which must be intersected to satisfy the four restrictions. Suppose no vertices are intersected. By restriction (2), the relative interior of each edge must be intersected at least once, and by restriction (4) every third edge must be intersected at least twice. Thus there are at least $4n/3$ intersections, no three of which are collinear, which requires at least $2n/3$ probes to verify.

Now suppose there are v vertices probed, no two of which are consecutive. For each of these vertices, by restriction (2) the interiors of both adjacent edges must be probed. By the previous discussion, $4(n-2v)/3$ incidences, no three of which are collinear, are necessary to verify the remaining $(n-2v)$ edges. Thus at least $(3v+4(n-2v)/3)/2 = (2n+v/2)/3$ probes are needed. This is minimized at $v=0$ yielding $2n/3$ probes.

Finally, suppose that a string of m consecutive vertices are probed. By restriction (3), the $m-1$ connected edges between them must contain at least $\lfloor (m-1)/2 \rfloor$ interior contact points. The two edges on either side of the chain must each have a probe incident upon it by restriction (2). Thus verifying this chain of $m+1$ edges requires at least $m+\lfloor (m-1)/2 \rfloor + 2$ contact points. The most efficient way to cover these contact points uses $\lceil m/2 \rceil$ probes, each of which passes through two vertices and an interior contact point, plus one probe to pass through the interiors of the two open edges. However, by restriction (5) this is not adequate, and at least $\lfloor m/4 \rfloor$ additional probes will be necessary to probe the relative interior of the unprobed edges, for a total of $\lceil m/2 \rceil + \lfloor m/4 \rfloor + 1$ probes per $m+1$ edges. This ratio is minimized for $m=2$, showing that at least $2n/3$ probes are necessary for verification. □

5. Open Problems and Extensions

We have given strategies for probing with half-planes. In particular, we have shown that complete information about a convex n-gon can be obtained with a linear number of carefully planned half-plane probes.

An obvious question concerns generalizing this problem to higher dimensions. For a three-dimensional polytope, one can consider either half-plane probes as discussed in this paper or half-space probes, which return the volume of intersection between a given half-space and a polytope. Reconstruction appears difficult for both these models, since the techniques described in this paper rely on isolating a section of P containing only two facets. Even the problem of determining a tetrahedra in a constant number of probes is open and appears difficult.

Another important generalization is from convex to simple polygons. Such a strategy will imply a solution to the problem of simple polygon reconstruction with Steiner symmetrals [712].

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