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GEOMETRIC PROBING

by
Steven Sol Skiena

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · URBANA, ILLINOIS

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GEOMETRIC PROBING

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We consider problems in *geometric probing*, the algorithmic study of determining a geometric structure or some aspect of that structure from the results of a mathematical or physical measuring device. A variety of problems from robotics, medical instrumentation, mathematical optimization, integral and computational geometry, graph theory, and other areas fit into this paradigm.

Finger probes return the first point of intersection between a directed line l and an object P. Chapter 2 presents results on finger probing convex polygons. We consider related problems in higher dimensions and with different classes of objects.

Hyperplane probes return the first hyperplane moving perpendicular to itself which is tangent to P. Chapter 3 discusses the duality relationship between finger and hyperplane probes. We establish the connection between hyperplane probes and certain algorithmic problems and consider the related silhouette and supporting line probe models.

X–ray probes return the length of intersection between P and l. Chapter 4 surveys the field of tomography and presents results for x–ray probes, which was inspired by it. We give linear bounds on determination and verification with x–ray probes in two and higher dimensions.

Half-space probes return the volume of intersection between a half-space h and P. Chapter 5 presents our linear determination and verification results for two dimensions and discused the difficulties of determination in higher dimensions.

Chapter 6 considers the power of infinite collections of these probes. We discuss Hammer's x-ray problem, presenting new proofs for convex polygons. Also, we discuss the combinatorial geometry problem of k-projections, which arises from aggregate probing. Finally, we consider other aggregate problems such as probing in rounds.

Chapter 7 extends probing to an object which is not usually considered geometric. *Cut-set probes* return the size of a cut-set of a graph. We present surprising results using these to reconstruct and thus represent graphs.

Each chapter concludes with relevant open problems.

ACKNOWLEDGEMENTS

I would like to thank Herbert Edelsbrunner for calling my attention to the problem of probing and his exceeding patient advice and teaching over the two and a half year gestation of this thesis. Without his careful guidance, this thesis would be a half–baked collection of questionable results – with it, these defects are much less notable. He has been a good friend as well as a perfect model of how research should be done.

Several people have contributed ideas and suggestions for this thesis. Discussions with S. Y. Bob Li have improved my understanding of lower bounds for probing and introduced new problems. Tomás Lozano–Pérez and W. Clem Karl provided me with references on tactile sensing which otherwise would not have come to my attention. Finally, I would like to thank my committee of Herbert Edelsbrunner, C. L. Liu, David Muller, Stephen Omohundro, and Edward Reingold for their suggestions and improvements.

Without question, the most interesting part of this thesis is the set of cartoons which appear at the beginning of Chapters 2 through 7. These were drawn by Susan Young of Baton Rouge, Louisiana. It should be noted that she was only thirteen years old at the time she did the drawings!

On a more personal note, my stay at Illinois was made much more bearable by many friends, too numerous to all be listed here. Specifically, I would like to thank a great pair of roommates, Luke Young and Arch Robison, for a happy home life over the last year. Bartlett Mel has been a very special friend, equally good for bouncing off ideas and personal troubles. A large set of theory graduate students including Nany Hasan, Arthur Goldstein, Scot Hornick, Arkady Kanevsky, Sanjeev Maddila, Ernst Mücke, Marsha Prastein, Harald

Rosenberger, Ioannis Tollis, and Jerry Trahan provided inspiration and support. Stephen Wolfram and the rest of the CCSR gang provided an intellectual excitement which kept me interested in other, more mercenary areas of computer science. I learned a lot teaching under Michael Faiman, who has been a great source of advice and help over the years. There is little chance I would have graduated if Barb Cicone had not known everything there was to know about the university and departmental rules which have governed my life over these last five years.

Last but not least, I thank my parents Morris and Ria and brothers Leonard and Robert for their help in making the first Dr. Skiena.

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CHAPTER 1.

INTRODUCTION

In this thesis we consider a variety of problems, differing in appearence but all of which are tied by a common theme. This theme is *geometric probing*, and the problems we consider involve determining a geometric structure or some aspect of that structure from the results of a mathematical or physical measuring device, a *probe*. We have considered a variety of problems from robotics, medical instrumentation, mathematical optimization, integral and computational geometry, graph theory and other areas which fit into this paradigm. These problems are interesting in themselves, but our results also have application to these and other fields.

Many problems in engineering and applied science can be fit into this format of reconstruction based upon measurement. A great deal of work in robotics [22,29,35–38,71,72,80] and computer vision [47,81] has concerned itself with providing machines with the means of sensing and understanding their environment. This work attempts to reproduce our senses of vision and touch by machines. Other sensing systems go beyond human capabilities, such as radar and sonar, which use reflected radio or sound waves to determine the size and structure of distant objects. Tomography [43,45,68,83,88] and similar technologies such as nuclear magnetic resonance [46] and ultrasound [33] are used for medical instrumentation, and reconstruct the geometry of the body from the amount of energy absorbed by tissues of different densities. In such fields as biology and geology, it is often necessary to determine the shape and size distributions of particles [4] from the cross–sections of samples. This has given rise to a field known as stereology [21,42,94].

Similar types of problems have long been studied within the mathematical community. Tomography was made possible by the study of Fourier and Radon transforms. Minkowski [60] proved that three–dimensional convex polytopes are defined by the areas of and normal vectors to each of its facets. These Gaussian images are now applied to problems in robot vision [16,47,57]. Alexandrov [2] considered a similar class of reconstruction problems. The study of integral geometry [79] provides tools for reconstructing convex sets. More recently, research in computational geometry has considered problems of probing [6,12,14,19,56,85] convex polygons with a variety of different devices.

Probing can be viewed as a discrete case of sampling problems encountered in signal processing. The *Nyquist rate* [65] specifies the amount of sampling needed to reconstruct a continuous waveform. Since our objects of interest have much more structure than continuous waveforms it is clear that tighter bounds can be obtained.

The goal of this thesis is to create a new area of study based on the problems, literature, and results of these disparate sources. Also, we aim to contribute a substantial collection of new problems and results to this field, *geometric probing.*

1.1. Taxonomy of Problems in Probing

There are a vast number of problems associated with probing, partially because we can take a "Chinese menu" approach to generating them. Choosing from column A, there are a wide variety of interesting models of sensors, with inspiration either from physical sensing devices or geometrical operations. Intuitive definitions of our main probing models are given below:

(1) *Finger Probes* – which measure the first point of intersection between a directed line and an object.

(2) *Hyperplane Probes* – which measure the first time when a hyperplane moving parallel to itself intersects an object.

(3) *X–ray Probes* – which measure the length of intersection between a line and an object.

(4) *Half–space Probes* – which return the area or volume of intersection between a half–space and an object.

(5) *Cut–set Probes* – which for a specified graph and partition of the vertices returns the size of the cut–set represented by the partition.

We can construct more sophisticated sensing devices by considering aggregates of probes sharing certain properties. For example, the set of all probes which are parallel to a given line or which pass through a given point. For x–ray probes, these models are analogous to x–ray photographs and point sources and have been the subject [23,24,26,27,77,92] of intense study. We can also consider the power gained by having access to more than one type of probe. How well do sensors work together to determine objects?

In column B of our menu, we have constraints on the type of object being probed. Most of our results hold only for convex polygons. For certain probing models, extensions to more general objects are impossible. Other objects of interest include collections of convex polygons, star–shaped and simple polygons, point sets, straight line graphs, polytopes in three or more dimensions, and continuous surfaces of specified degree. In all cases, we can also consider restricting the objects to come from a known finite class to create *model–based* problems. Interesting problems also arise when the disparity in dimension between object and probe increase beyond one.

Finally, in column C is the property which we are interested in optimizing or bounding:

(1) *Lower Bounds* – how many probes are needed to determine a particular object?

(2) *Upper Bounds* – what is the best strategy for using the particular probe on a particular type of object?

(3) *Verification* – given a reputed description of the object how many probes are necessary to test if the description is valid?

(4) *Computational Complexity* – assuming that a probe is a constant time operation, what is the computational cost of planning the probes to determine an object?

(5) *Simulation* – given a probe model and a representation of the object, how much time and space is necessary to simulate an actual probe?

(6) *Feature Determination* – how many probes are needed to determine some feature of the object, such as volume, orientation, or convexity?

In addition to those discussed above, other types of problems can be expressed in the paradigm of probing. For example, finding convex hulls of a point set can be considered as determining the object resulting from hyperplane probing the point set. Similarly, the additional constraints added to integer programs to make them linear [67] can also be considered as hyperplane probes. Insight into other subjects can result from considering them as probing problems. For example, the properties of cut–sets of graphs [86] can be studied under the guise of measuring the sizes of cut–sets as a probe.

Another example which fits into this class is the problem of reconstructing Gaussian images [16,47,57,58] of polytopes. Gaussian images have a connection to half-plane probing in E^3, since area probes along each face determine a Gaussian image.

1.2. The Literature of Probing

It is only recently that probing problems have been considered in a geometric sense, rather than as a problem in image processing [65] and hence the relevant literature is relatively small and spread across a variety of disciplines. Image processing is an applied field, concerned largely with working with images of variable fidelity. We will be more concerned with the geometrical issues of probing, and thus abstract away many of the important issues of image processing systems. In this section, we will survey only the papers to date which explicitly deal with problems in geometric probing as we have defined it. A glance at the references at the end of this thesis reveals the extent to which similar problems have been studied in other disciplines. These will be integrated within the appropriate sections of the thesis.

The problem of geometric probing was inspired by robotics [29,81] and first studied by Cole and Yap [12] who considered finger probing convex polygons in the plane and proved a tight bound of $3n$ finger probes as necessary and sufficient. Dobkin, Edelsbrunner, and Yap [14] extended the finger probe to three and higher dimensions, and introduced other models such as hyperplane and line probes. Greschak [34] and Li [56] both independently developed the notion of hyperplane probes, with Li extending the model to silhouette probes. Bernstein [6] considered the model-based problem of finger probing, where an object is identified from a given set. Natarajan [62] considered problems of determining the orientation of polytopes by simple sensors. Finally, we have considered the problems of x–ray [19], half–plane [85], and cut–set [86] probes.

1.3. An Overview of this Thesis

Since problems in probing are naturally ordered by probing model, this thesis has been structured accordingly. As this thesis represents a work of synthesis as well as a collection of new results, care has been taken to cite the original sources of all results. Thus, all otherwise unreferenced results can be attributed to the author.

Chapter 2 presents Cole and Yap's fundamental results on finger probing convex polygons. We consider a host of related problems – attempts to generalize the type of object being probed, extensions to higher dimensions, and probing when the number of vertices of the polygon is known. Finally, we consider model–based problems in tactile sensing. We present a brief overview of the literature in tactile sensing and consider the related problem of determining the orientation of polyhedra. Using our improved bounds for these problems, we solve the model–based determination problem for a general class of polygons.

Chapter 3 discusses the duality relationship between finger and hyperplane probes, which has interesting implications for both sensing models. We establish the connection between hyperplane probes and certain optimization and algorithmic problems. We also consider the related silhouette, line, and supporting line probing models and present results for them.

Chapter 4 begins with a survey of the practical and interesting field of tomography and discusses our model of the x–ray probe, which is inspired by it. We present our results on determination and verification with x–ray probes and extend them to higher dimensions. Most of these results are taken from our paper [19].

Half–space probes are covered in Chapter 5. We present our determination and verification results for two dimensions, which are based on our results for x–ray probes. We discuss the difficulties of determination in higher dimensions, and conclude with a discussion of the extended Gaussian images problem, which can be discussed in terms of verifying with half–space probes. Most of these results first appeared in Skiena [85].

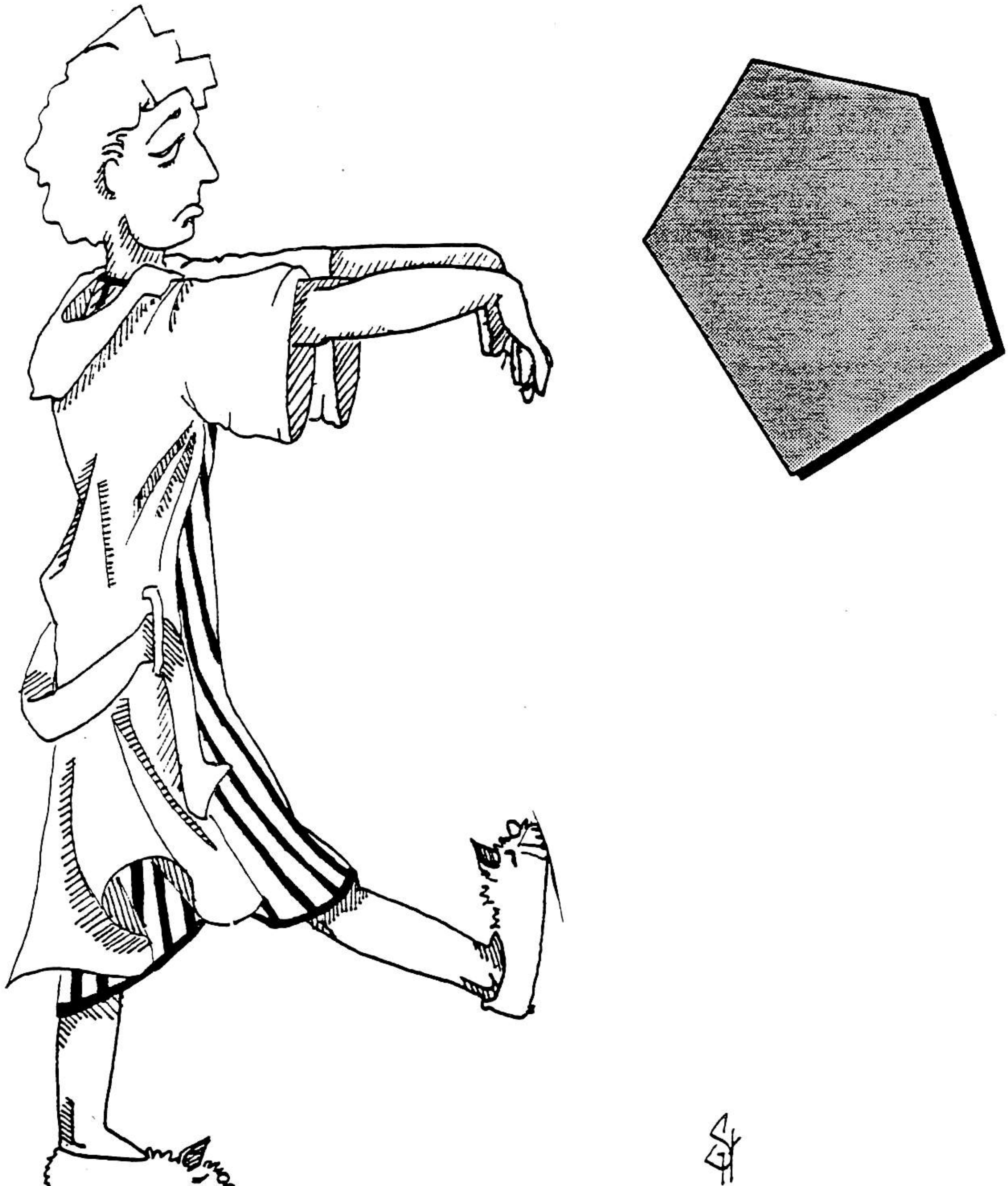
Chapter 6 considers the power of infinite collections of probes. We discuss Hammer's x–ray problem, which has established a substantial literature, and present new proofs for convex polygons. Also, we discuss the combinatorial geometry problem of k-projections, which arises from aggregate probing. Finally, we consider aggregate problems for other probing models such as probing in rounds. k-projections were introduced in Skiena[87].

Chapter 7 extends probing to an object which is not usually considered geometric. We introduce the notion of a cut–set probe, which measures the size of a cut–set of a graph, and present surprising results on using these to reconstruct and thus represent graphs. Our cut–set probing results first appeared in [86].

We suggest the implications and future directions of this work in our conclusions, Chapter 8. We have concluded each chapter with a collection of relevant open problems. It is hoped that these will shape and inspire further work in geometric probing.

CHAPTER 2.

FINGER PROBING



Tactile sensing is an important paradigm in robotics. Cole and Yap [12] developed the notion of a *finger probe* to model tactile sensors used in robotics. A finger probe is defined to be the first point of intersection p between a directed line l and an object P. The term "probe" will sometimes be used to refer to p and sometimes to l, and we rely on context to distinguish between these meanings.

The notion of finger probing has inspired work on a variety of problems and a growing body of literature [6,14,34], which includes this thesis. We present the main results of this literature, including new proofs for several theorems, and consider a collection of problems posed or inspired by it. Together, these problems show both the power and limitations of finger probing and thus of tactile sensing in robotics.

We assume that we are given O, a point within the interior of P. Without this information, we have no idea where P is located, and an unbounded number of probes can be required to find it. From each probe, in addition to the contact point we obtain a half–line defined by p and l which does not intersect P. This and the convexity restrictions on P can be used to identify points known to be within P or known to be outside of P. Together, these two sets represent the state of our knowledge about P. Let $inside(P)$ be the closed set of points which can be proved to be within P, specifically, they are on or within the convex hull of the set of contact points X. Let $outside(P)$ be the set of points which can be proven do not lie on or within P. Specifically, $z \in outside(P)$ if there exists a point $x \in X$ such that $x \in int(conv(\{z\} \cup X))$, where $conv(S)$ is the convex hull of a set S. Figure 2.1 shows $inside(P)$ and $outside(P)$ for a specific collection of probes. We say that P is *determined* when $inside(P) \cup outside(P) = E^d$, where d is number of dimensions of P.

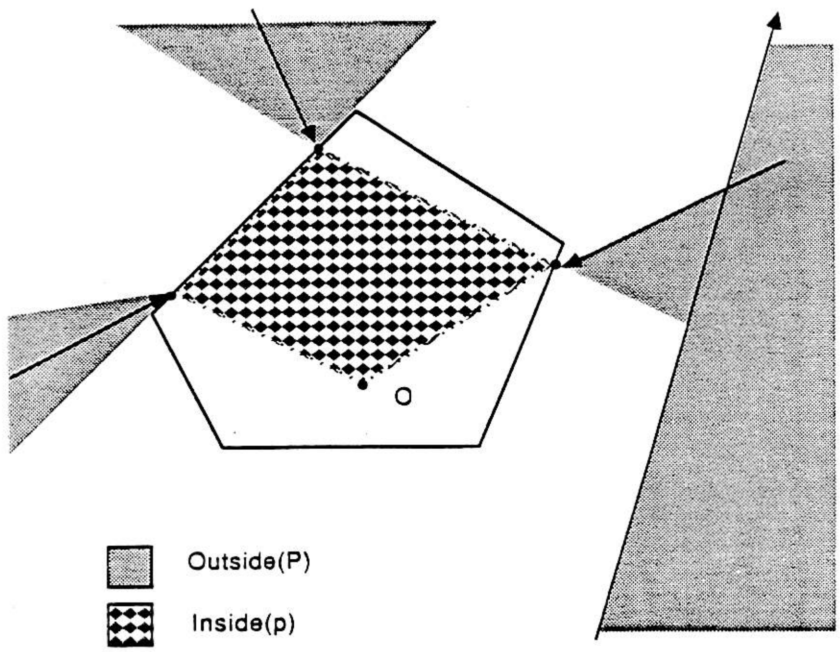


Figure 2.1: Information from Finger Probing.

We emphasize that we are interested in *absolute* determination, not an approximation to P. When a convex polygon is determined we have identified the exact coordinates of each of the n vertices. This notion of determination will be used throughout this thesis, even for probing models such as x–rays which do not return absolute information.

One final subtlety concerns tangent probes. We consider a finger probe along an edge of P as returning the first vertex it encounters, which seems reasonable and natural. However, it is less clear what should be returned when the probe line intersects only a vertex of P. Cole and Yap [12] make the assumption that such probes miss P entirely. The point is moot as far as this thesis is concerned, as we avoid the use of such probes.

This chapter is organized as follows. Section 2.1 discusses determination and verification problems in E^2. Section 2.2 shows that finger probing cannot be meaningfully extended beyond single convex polytopes. We consider higher dimensional problems in section 2.3. Better probing strategies result if additional information is known about P. Section 2.4 considers when the number of vertices n is known and section 2.5 when P is selected from a known, finite set. Finally, in section 2.6, we pose some problems in finger probing.

2.1. Lower and Upper Bounds for Two Dimensions

The fundamental problems in geometric probing are *determination* and *verification*. Determination counts the number of probes necessary to reconstruct an object P. We note that the sequence of probing outcomes for a determination provides a complete representation for P, because the probing algorithm can be used to generate the probe specifications. Thus determination strategies suggest alternate representations for geometric objects.

Verification counts the number of probes necessary to prove that P is indeed the object in question. This problem can be thought of as "non-deterministic" probing, since it is assumed that the algorithm always makes the right probe for any P. To do this, it is assumed that the algorithm has a description of P as its input. Verification problems are important because they provide a lower bound for determination, since P is an input for verification. For our first result, we present Cole and Yap's optimal strategy for verifying convex n-gons:

Theorem 2.1: $2n$ finger probes are necessary and sufficient to verify a convex n-gon.

Proof: To show necessity, note that each vertex and edge must be probed at least once, since an unprobed vertex can be truncated and an unprobed edge extended with another vertex. For sufficiency, we note that three collinear points must all lie on the same edge of P, by convexity. Thus probing each vertex and edge interior once determines the extent and position of each edge of P. $\square$

The deterministic version of this problem is more difficult. The following strategy is a variation on the algorithm in [12].

Theorem 2.2: $3n$ finger probes are sufficient to determine a convex n-gon.

Proof: Our strategy will consist of two phases. The first phase ends when there are three probes that are incident upon the same edge e_1, the second when P is determined.

All probes in the first phase will be directed through the origin O, and thus the ith probe can be specified by either its angle θ_i or contact point x_i. Send three probes through P, where $\pi < \theta_3 < \theta_2 < \theta_1 < 2\pi$. If x_1, x_2, and x_3 are collinear, we have completed the first phase. If not, note that x_1 and x_3 cannot be incident upon the same edge because of convexity constraints. We will insure that the next edge e_n in a counter-clockwise direction from e_1 has at most one probe incident upon it at the end of the first phase.

Let θ_m be the slope of the line defined by x_2 and x_3. If m is not between θ_1 and θ_2, as in Figure 2.2a, we will aim probe $i+1$ with angle $\theta_2 < \theta_{i+1} < \theta_i < \theta_1$. The three collinear points defining e_1 will be either x_{i-1}, x_i, x_1 or x_{i-2}, x_{i-1}, x_i. In the first case, convexity prevents (x_2, x_3) from being e_n, and in the second, we have shown that x_1 cannot be on the same edge as x_3. If θ_m is between θ_1 and θ_2, as in Figure 2.2b, we will aim probe $i+1$ with angle $\theta_2 < \theta_i < \theta_{i+1} < \theta_m$. Edge e_1 will again be defined by x_{i-1}, x_i, x_1 or x_{i-2}, x_{i-1}, x_i and the same arguments apply.

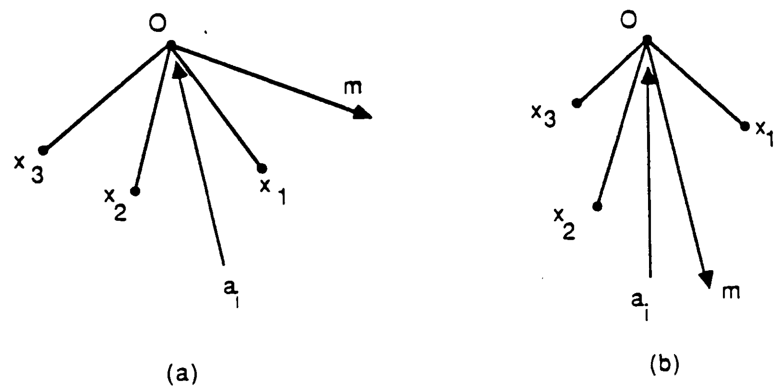


Figure 2.2: Determining the first edge with 2n Probes.

For the second phase of this strategy, we note that to determine our first vertex v_1, between e_1 and e_n, we can probe along the directed line from x_i to x_{i-1}. After determining v_1, we proceed in a *clockwise* direction around P. If e_i is the most recent verified edge, and x_a and x_b are the next two probe contact points around P, we aim the next probe along the line defined by p and O, where p is the intersection of e_i and the line through x_a and x_b as in Figure 2.3. If the contact point is p, then p is vertex v_i of P and a new edge is determined

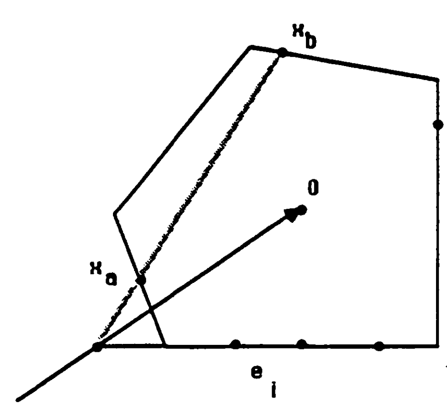


Figure 2.3: Determining the next edge of P.

by x_a and x_b. If not, the contact point lies on another edge. This process is repeated until v_a is v_1 and v_1, v_n and x_b are collinear, so P is completely determined. Since at most one probe was incident upon this last edge e_n from the first phase, and it is probed in the second phase only if necessary, it will be probed exactly once in its interior.

In summary, the interiors of $n-2$ edges are incident upon at most two probes each, the interior of e_1 is incident upon three probes, e_n by one probe, and all n vertices are verified once. Thus the number of probes used is at most $2(n-2)+3+1+n = 3n$. □

The probe directed along e_1 would be very unreliable for a real world application, since only a slight perturbation could cause the probe to miss P entirely. A more robust strategy would aim through the origin each time. In section 3.3, we prove that $3n+1$ finger probes are necessary and sufficient if all probes are directed towards the origin. Our strategy is more robust than the original one of Cole and Yap, since only one such probe is necessary.

Cole and Yap prove a lower bound of $3n-1$ for determination, which raises to $3n$ under the mild assumption that a probe which passes through one vertex of P and not its interior has a contact point at infinity. This proof is surprisingly complicated, involving substantial case analysis. See [12] for details. We give an alternate and simpler proof of a slightly weaker result.

Theorem 2.3: At least $3n-1$ finger probes are necessary to determine a convex n-gon.

Proof: We shall construct an adversary which will force at least $3n-1$ probes from any probing strategy. Our adversary will use the following strategy: for as long as possible, all contact points will be on a circle around O. Also, no probe will contact a vertex until it is unavoidable. To delay these exceptions, we may relabel the contact points until it leads to a contradiction.

First note that for any $2n-2$ points on a convex curve we can construct two convex n-gons such that each point is on the interior of an edge of each polygon, as is illustrated in Figure 2.4. Thus the $2n-1$th probe can be forced not to contact a vertex by suitable choice of the two possibilities.

To verify an n-gon, each of the n vertices must be probed. Thus at least $2n-1+n = 3n-1$ probes are necessary. □

It is tempting to try and push this lower bound argument further by constructing two distinct polygons on $2n$ interior points. While this can probably be done, such arguments become very subtle and great care is necessary to make to make the proof rigorous, as illustrated by the extensive case analysis in Cole and Yap's proof. The difficulties revolve around making sure that O is within the convex hull of the $2n$ points.

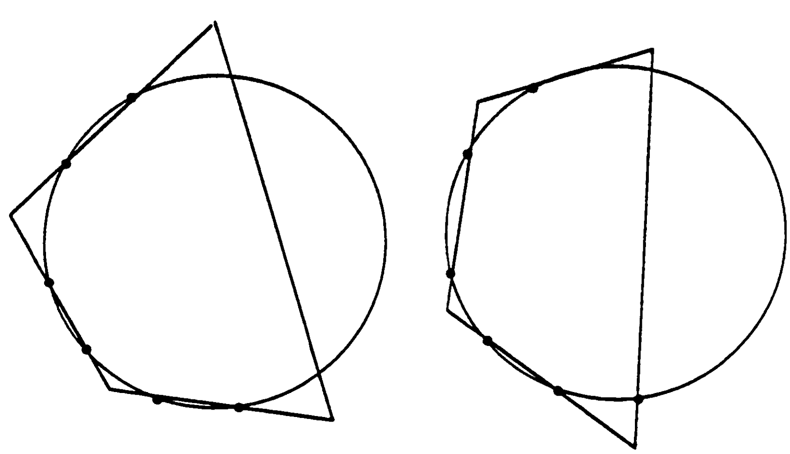


Figure 2.4: Two distinct n-gons defined by $2n-2$ points.

Nothing has previously been published on the question of the time complexity of probing strategies, perhaps because implementation of these strategies appear to have little practical value. By time complexity, we mean the time it takes to plan the probes necessary for determination. However, it is easy to demonstrate that the strategy of Theorem 2.2 is asymptoticly optimal, and an application will be developed in section 3.4. When we discuss time complexity, we ignore the actual time it takes to make the probe.

Theorem 2.4: The time complexity of a $3n$ finger probing strategy for determining convex polygons is $O(n)$.

Proof: The strategy of Theorem 2.2 strategy walks around the polygon, conjecturing an edge and then probing it. Either the probe indeed intersects the edge or else discloses the presence of another edge.

Let us maintain the points returned by probes in a circular list, with a pointer to maintain the conjectured vertex location. If the edge is verified, the point is added to the list and the pointer advanced one node. If not, the point is added and a new vertex is conjectured from the two nodes to the left and right of the pointer. Each calculation takes constant time, and only $3n$ are required, so the complexity follows. □

2.2. Wider Classes of Objects

Unfortunately, the powers of finger probing prove sharply limited when we attempt to generalize the objects beyond a single convex polygon. We prove that there does not exist a finite strategy for determining a star-shaped polygon or multiple convex polygons using finger probes. For multiple objects we make our standard assumption that we know the coordinates of a point O within each polygons, which provides the general location of each object.

Theorem 2.5: There does not exist a finite strategy to verify a star-shaped polygon with finger probes, even if the vertices are in general position.

Proof: Consider the following star-shaped 5-gon P. Let $v_0=(0,0)$, $v_1=(a,0)$, and $v_4=(0,b)$, $a,b>0$. The remaining vertices v_2 and v_3 are below the line $-bx/a+b$ and are defined in polar coordinates as $v_i=(\theta_i,d_i)$, where $0<\theta_2<\theta_3<\pi/2$. Clearly P is star-shaped, since v_0 must be in the kernal.

The difficulty is in verifying the edge $e=(v_2,v_3)$. No number of probes intersecting e is sufficient, since a polygon P' can be constructed for any such set of probes where the points of intersection are collinear despite each being incident upon different edges. Such a polygon is shown in Figure 2.5. Since v_1 and v_4 prevent probing directly along e, P is not distinguished

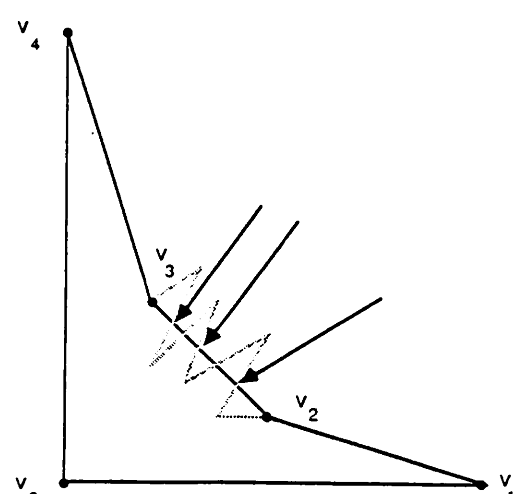


Figure 2.5: A star-shaped polygon which is unverifiable by finger probing.

from P'. □

As shown in Theorem 2.1, a single convex polygon can be verified and determined. However, this also breaks down when more than two convex polygons are considered.

Theorem 2.6: Two convex polygons P_1 and P_2 can be verified with $2n$ finger probes, where n is the total number of vertices of the polygons. A collection of m convex polygons $P_1, \cdots, P_m$ cannot be verified for $m > 2$.

Proof: For any two non-intersecting convex polygons, at least one edge of either P_1 or P_2 defines a separating line between them. Let a and b be the vertices incident upon this edge. The probes to verify a and b can be designed so that the probe paths partition the plane into two open regions, one containing P_1 and the other P_2. Using this partition and the origins of P_1 and P_2, any probe can now be identified as to which polygon it is incident upon. Each of the n vertices and points on the interior of each edge can be probed with the same verification strategy as for one polygon. With two convex polygons, there always exists a ray from any point p on the boundary of P_1 or P_2 to infinity without otherwise intersecting either polygon.

To show that three convex polygons cannot necessarily be verified, consider the configuration in Figure 2.6. No finger probe originating from infinity can intersect $e = (v_i, v_j)$, whose existence therefore cannot be verified. □

The situation is even worse for the determination of multiple polygons. A trivial argument on the impossibility of determining two convex polygons involves cutting a convex polygon into two convex pieces by removing an infinitesimally thin strip. A separating line between the two pieces cannot be found with a finite strategy. However, even knowledge of a

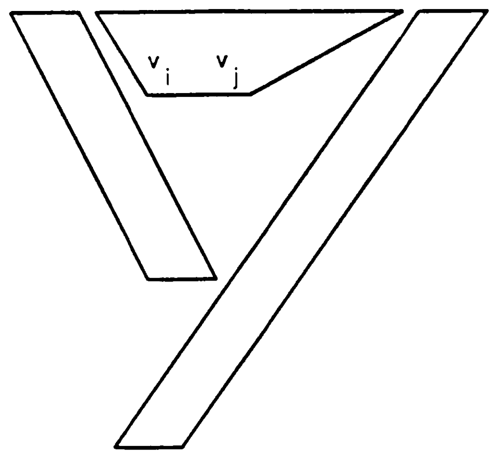


Figure 2.6: An unverifiable configuration of three convex polygons.

separating line is not sufficient for determination.

Theorem 2.7: There does not exist a finite strategy for determining two convex polygons with finger probes, even given points within each polygon and a separating line between them.

Proof: Consider the configuration of two polygons in Figure 2.7. Assume that the edge e_2 of polygon P_2 is very large relative to the separation between P_1 and P_2 and the edge e_1 of P_1. Further, assume that we have determined all information about the two polygons except for e_1 and the extent of the edges incident upon it.

Because of the size of e_2, any probe originating from infinity which intersects e_1 is sharply restricted as to its slope. An adversary argument can be used to show that any probe though the undetermined section of P_1 can either extend the known portion of the incident edges or miss P_1 entirely. In either case, it can take an unbounded number of probes until we finally intersect the interior of e_1. □

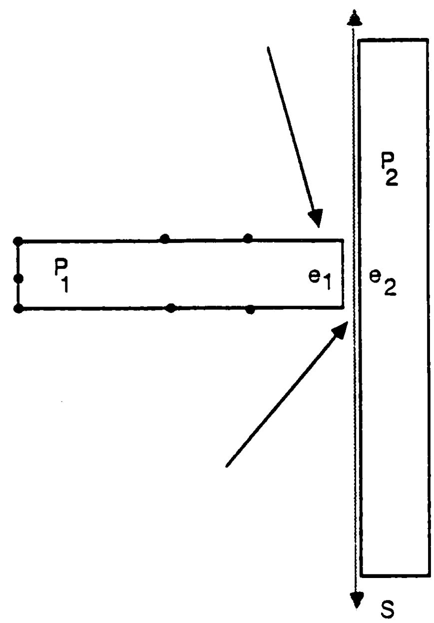


Figure 2.7: An indeterminable configuration of two convex polygons.

We note a wider class of objects may be determinable with a finite strategy if n, the number of sides of P, is known. Another generalization is to convex "splinegons" [15], where the edges are curves of a degree $d \geq 2$.

2.3. Higher Dimensions

Dobkin, Edelsbrunner, and Yap [14] consider probing convex polytopes in higher dimensions. Beyond two dimensions, the number of vertices is no longer identical to the number of facets, so upper and lower bounds must be expressed in terms of $f_i(P)$, which represents the number of i-dimensional faces of P, for $0 \leq i \leq d$. So $f_0(P)$ is the number of vertices, $f_1(P)$ the number of edges, and $f_{d-1}(P)$ the number of facets, the $(d-1)$-dimensional faces, of P. In this section, we prove bounds for finger probing in E^3.

Theorem 2.8: $f_0(P)+5f_2(P)$ finger probes are sufficient to determine a convex polytope in E^3.

Proof: We sketch the proof in Dobkin, Edelsbrunner, and Yap. For details, please refer to [14].

The basic strategy is similar to what we have seen in the plane; conjecture and verify vertices and facets of P. If we can insure that only a constant number of probes are incident upon each face, this yields a linear probe strategy. However, the generalization is not straightforward. In E^2, three collinear contact points verify an edge. Unfortunately, no number of co–planar contact points are sufficient to verify a facet in E^3, unless one point lies within the convex hull of the others, since these points can simply represent a cross–section of P.

Let H represent the convex hull of the contact points and A_H the cell complex generated by extending the facets of H to planes. We will insure that at most five probes are incident upon the relative interior of each facet of P. First send four probes to form a tetrahedra around O. In general, we attempt to confirm the facet with the most co–planar probes incident upon it, aiming through an unverified vertex (defined by the intersection of previously verified facets, if one exists) and avoiding any edges or vertices of A_H. This second restriction insures that each probe contacts only one unverified facet of P.

In [14], it is shown that the invariant that at most one unverified facet of P has four probes incident upon it is maintained through this strategy, and thus at most five probes are incident upon the relative interior of each facet. □

In [14], a lower bound of $f_0(P)+f_2(P)$ is given for verification, noting that each vertex and facet must be probed to ensure all vertices have been confirmed. We present a better lower bound for determination in E^3 based on the lower bound results for the plane.

Theorem 2.9: At least $f_0(P)+2f_2(P)$ finger probes in E^3 are necessary for determination.

Proof: Let P be a pyramid with $f_0(P)-1$ sides, plus a base. The adversary will ensure that the first two probes will be incident upon the base. Due to the lower bound of Theorem 2.3, at least $3(f_0(P)-1)$ probes will be necessary to determine the sides of P, since if they did not a more efficient probing strategy in E^2 results from simulating a pyramid and probing it. Including a verifying probe to the apex of P, at least $f_0(P)+2(f_2(P)-1)+2$ probes are made, yielding the result. □

We note that by an adversary argument it may be possible to raise the coefficient from two to three, noting each face might have three probes incident upon it.

These results are generalized in [14] to d dimensions, for a lower bound of $f_0(P)+f_{d-1}(P)$ and an upper bound of $f_0(P)+(d+2)f_{d-1}(P)$.

2.4. Finger Probing when n is Known

A natural question is how much knowing the number of vertices of P helps in probing it. Cole and Yap considered this problem, showing that 8 probes are sufficient to determine a triangle and giving a lower bound of $2n+1$ probes. In this section, we provide strategies for determination and verification which exploit the additional information.

Theorem 2.10: $3n-1$ probes are sufficient to determine a convex n-gon if n is known.

Proof: We will modify the stopping criteria of Theorem 2.2 to save one probe. The first phase of the strategy is the same as before, which determines an edge e_1 and insures that at most one probe is incident upon the adjacent edge e_n. By directing a probe along e_1 in the appropriate direction we determine the vertex v_1 between e_1 and e_n.

Throughout the second phase of the strategy, we maintain the situation that a chain of m consecutive vertices and edges are verified. There are some number of probes, no three of which are collinear, incident upon the unverified section of P. We observe that P is determined when there are $2(n-m)-1$ such probes. This follows since, including v_1, there is only one way to connect $2(n-m)$ points into a convex chain of $n-m$ edges.

Suppose the condition that there are $2(n-m)-1$ unverified probes has not occurred prior to the verification of vertex v_{n-1}. By definition, $m=n-1$. Edge e_n has either 0 or 1 probe incident upon it. If there is one, this contact point and v_1 defines the remaining edge. If not, probing along e_{n-1} returns v_{n-1}, which with v_1 defines the remaining edge. In either case, we have saved at least one probe over the $3n$ strategy, so $3n-1$ probes suffice. □

We have been unable to obtain a $3n-c$ lower bound for determination. The difficulty lies in quantifying when a polygon is determined when n is known. The necessary subtlety of a tight lower bound argument is illustrated by the dramatic consequences for verification of knowing n:

Theorem 2.11: $3\lceil n/2 \rceil$ probes are sufficient to verify a convex n-gon if n is known.

Proof: Label the edges as even or odd. For each even edge e, make three probes: along e in both directions and one incident to the center of e. These probes verify each even edge and prove that the endpoints are vertices. Clearly, knowledge of all n vertices determines P. □

2.5. Model Based Results

Since constructing high-level computer vision systems has proven to be a difficult problem, tactile sensing has been studied as an alternative for robotic assembly systems. The abstraction of such systems are very similar to finger probing. In a mechanical assembly problem, the geometry of the parts is known to the robot, which must use sensing to determine their type and orientation. Such problems are called *model-based* and differ from the determination problems we have discussed in that the objects come from a finite set.

In this section, we review the work that has been done in tactile sensing and robotics. This work is closely tied to real world applications. We then consider the problem of determining the orientation of a model convex polyhedron, and present improved results. Using these ideas, we solve the problem of model-based probing for both convex and more general polygons.

2.5.1. Tactile Sensing and Robotics

The problem of determining an object via tactile sensing can be studied from several different perspectives. Researchers at MIT [29,37] have considered using heuristics to reconstruct polyhedra from random or oblivious probes, instead of defining a strategy to plan the probes. Their probing model is somewhat more powerful than the finger probe, returning both a contact point and the surface normal at that point. From a small number of such tactile probes, they construct an *interpretation tree* consisting of the possible mappings between contact points and the faces of model polyhedra. By using local geometric constraints such as whether the distance between two probes is consistent with the edge labelings, they prune this interpretation tree for m edges and s probes from m^s nodes to, what is in practice, a small number of interpretations. This system works well even with non–convex polyhedra and for modeling sensing errors. It has been extended, with somewhat less success, to the bin searching problem of overlapping parts [38] by adding a "null face" to the interpretation tree, which accounts for probes which do not contact the intended object.

A brave attempt to explain the excellent average case performance of these oblivious tactile sensing strategies has been made by Grimson [35]. If p is the probability that two random probes represent a consistent set of interpretations, then it can be shown that the expected number of consistent interpretations I_{exp} after s probes is

$$I_{\text{exp}} = m^s p^{\binom{s}{2}}$$

where m is the number of model faces. From this can be calculated the expected number of probes where I_{exp} is largest, and when there remains only one consistent interpretation. One consistent interpretation can be expected when

$$s = 1 - \frac{2 \log m}{\log p}.$$

Of course, it is difficult if not impossible to determine p for a particular model. Regular convex polygons will never converge upon a single interpretation. Grimson computes p by assuming a uniform distribution of points in his *relative configuration space.* His results compare well with the results of simulations and are generalized to account for uncertainties in measurement.

Several researchers have proposed strategies for using these tactile sensors. Grimson [36] proposes the following scheme to select a probe which distinguishes between two or more possible orientations. Select a probing direction, ie. the slope but not the intercept. Project the visible vertices of the orientations onto the line normal to this direction. This divides the line into segments. The midpoint of the segment which distinguishes the largest number of orientations, subject to measurement uncertainty, defines the probe. An extensive analysis of the error terms are provided. The nicest feature of considering the uncertainty associated with probing is that curved objects can be represented as polyhedra.

Ellis, Riseman, and Hanson [22] describe a similar system which represents probes incident upon an edge as a trapezoid in a projective space and selects a probe which is represented in the intersection of a number of trapezoids. They consider the problem of stability, where a probe at too oblique an angle deflects off the polygon. Finally, Schnieter [80] considers problems of selecting probes when maneuverability of the sensor is a consideration.

2.5.2. Detecting the Orientation of Polyhedra

For industrial robot applications in manufacturing and assembly, either the parts must carefully positioned for the robot to manipulate or the robot must be able to detect the orientation of parts and adjust to them. Since the technical problems with computer vision systems remain very difficult, systems for the sensorless orientation of objects [61] or using simpler, more robust sensors to determine orientation have been studied. Natarajan [62] poses the problem of determining the orientation of a known polytope moving past simple ray sensors on a guide plane. We discuss this problem and present improved upper and lower bounds on the number of sensors required to determine orientation for both two and three dimensional objects.

Given a known convex polygon or polyhedron advancing, with one polygonal face on a guide plane and an edge resting against a lip, how many sensors are needed to determine its orientation? A sensor is a half–line in E^2 or E^3 which can detect whether it intersects the polyhedron or not. Natarajan proves that n sensors are sufficient and $n/2$ are necessary and extends the problem to three dimensions, with $6n$ sufficient and $n/4$ necessary.

We modify the problem to insist that all sensors are read at the same instant, rather than continuously, except for one sensor which determines when the others are read. This is to clear up an inconsistency in [62], since the polygon given in the lower bound construction has n edges, each of a unique size. Thus the orientation can be distinguished by one probe measuring the length of the edge on the guide plane. Also, we note that for polygons we only consider the number of possible orientations in a plane without "flipping" it over, which would require an additional degree of freedom. Thus there are only n possible orientations of P.

The requirement for convexity may be weakened, since this is a model based problem. We define two models as *distinguishable* if there exists a verification strategy which can distinguish between them. Figure 2.8 shows two distinct polygons which are not distinguishable. We now consider the problem where all models are pairwise distinguishable. Thus

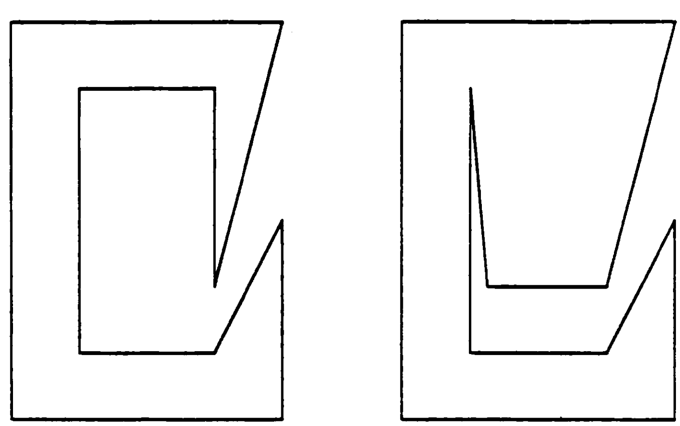


Figure 2.8: Two indistinguishable polygons.

more general objects, including star–shaped polygons may be considered.

First, we consider the two–dimensional case, tightening the results within an additive constant of optimality.

Theorem 2.12: n sensors are sufficient and $n-3$ sensors necessary to determine the orientation of a convex polygon.

Proof: We present an algorithm to prove the upper bound. We note that if two orientations of distinguishable polygons are distinct, then when the two orientations are aligned along the guide plane there exists at least one verifiable vertex of one orientation which is not overlapped with the other orientation. Thus one sensor is sufficient to distinguish between the two orientations.

To distinguish between n orientations, we note that one sensor placed as described above will partition the set of orientations into two subsets, depending upon the reading of the sensor. This partitioning process can be repeated with additional sensors for each subset containing two or more orientations, until each subset contains just one orientation. This can be modeled as a binary tree with n leaves, with each internal node representing a sensor. An n leaf binary tree contains $n-1$ internal nodes, and including a sensor to trip on the leading edge of the polygons gives the result.

To prove that $n-3$ sensors are necessary, consider a regular $(n-1)$–gon with sides of unit distance. Let $\theta = \pi(n-3)/(n-1)$ be the angle associated with each vertex. For one particular vertex, cut off the corner with a line parallel to the line defined by its two neighbors, where the area removed is less than $x\sqrt{1/4-x^2}$, $x = 1/(2\cos(\theta))$.

We shall restrict our consideration to the $n-2$ orientations where the forward vertex incident to the guide plane is the center of an uncut angle. No sensor can distinguish more than one orientation within the set. Thus $n-3$ sensors are needed to distinguish between these orientations. □

We note that this lower bound construction can be used to prove an $(n-1)/2$ lower bound when continuous reading of the sensors is considered. A more effective counting argument might eliminate the remaining gap between the upper and lower bounds. Also, an alternative criteria for when the model is in a position to be sensed, such as when the polygon's forward motion is stopped by a wall, would reduce both the upper and lower bound by one.

For polyhedra, we improve the lower bound from $n/4$ to $n-4$ sensors and the upper bound from $6n$ to $6n-12$, with a slightly tighter lower bound for small n.

Theorem 2.13: $6n-12$ sensors are sufficient and $n-4$ necessary to determine the orientation of a determinable polyhedron.

Proof: From Euler's formula for planar graphs, there are at most $6n-12$ distinct orientations of a polyhedron with n vertices. Each two distinct polyhedra must differ in at least one vertex, so extending the argument in the proof of Theorem 2.12 gives the result.

For the lower bound, consider a pyramid made by adding a vertex above the center-point of the modified regular $(n-2)$-gon of Theorem 2.12. If the pyramid was regular, there would be $n-2$ indistinguishable orientations of each of four types, depending upon which face and edge were incident to the guide plane and lip. If the wedge removed by adding a vertex is small enough, no two of these orientations can be distinguished by a single probe. Thus at least $n-4$ probes are necessary to determine the orientation for each of the four types of orientations. □

An argument based on considering the four distinct types of orientations of the pyramid could push this bound to $4n-9$. Unfortunately, it is difficult to account for the degree of overlap between between the different types of orientation. Also, a better lower bound might be based on almost regular polyhedra such as used in geodesic domes. It is unfortunate that there do not exist regular polyhedra beyond the five platonic solids. For example, at least 56 sensors are necessary to determine the orientation of a convex polyhedron with 13 vertices. Consider an icosahedron, which is a regular polyhedron with 20 equilateral triangles for faces and contains twelve vertices. One of these faces can be raised into a tetrahedra by adding a new vertex v a distance ϵ above the face. Position v nearest to one vertex of the face and equidistant from the other two vertices.

If ϵ is small enough, it will require one sensor scanning each of the 19 faces not resting on the guide plane to determine which face has been raised. Further, in general two more sensors are needed for each face to determine which vertex v is nearest to. Even if we assume that we can detect when one of the raised faces rests on the guide plane, $19\times3-1$ probes are necessary.

Studying such "almost–regular polyhedra" is an interesting and important open problem.

2.5.3. Model–Based Determination Results

Two distinct model–based determination problems will be considered, the first when all the models are convex and the second with much more general objects. For both problems, we assume there are m distinct models of at most n sides each.

Bernstein [6] gives a solution to the problem for convex polygons. Surprisingly, his result is independent of m. This proof improves on the upper bound by an additive constant.

Theorem 2.14: $2n+1$ probes are sufficent and $2n$ necessary to determine a convex n–gon P selected from a finite set ρ.

Proof: The intuition behind these strategies is that if the set of candidate polygons is known, it is possible to aim probes close enough to each other to insure that they are incident upon the same edge. In particular, we need to pre–calculate two quantities from the set of polygons ρ.

For any point s in polygon P, define θ_s^P as the smallest angle spanned by any edge of P by a point s. Further, let $\theta_{\min}^P = \min\{\theta_s^P, s \in P\}$. In a convex polygon, the point $s_{\min}$ which gives rise to the minimum angle must be on a vertex or edge of P. Any point s within the interior of P will be with a triangle defined by an edge e of P and a point p on another edge, and hence define a larger angle with e than p, as illustrated in Figure 2.9. Finally, define $\theta_{\min}^{\rho} = \min\{\theta_{\min}^P \,|\, P \in \rho\}$. .

For any convex polygon P with vertices $V = v_1, \cdots, v_n$, let $d_{\min}^P = \min\{d(\overline{v_i v_{i+1}}, v) \,|\, v \neq v_i, v_{i+1}$ and $v, v_i, v_{i+1} \in V\}$, where $d(l, v)$ represents the minimum distance from line l to point v. Thus any probe sent parallel to a known edge e of P within $d_{\min}^P$ of e is guaranteed to hit the next edge, as in Figure 2.10. Finally, let

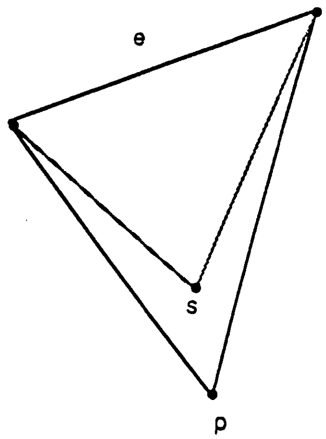


Figure 2.9: The point defining the minimum angle is on the boundary of P.

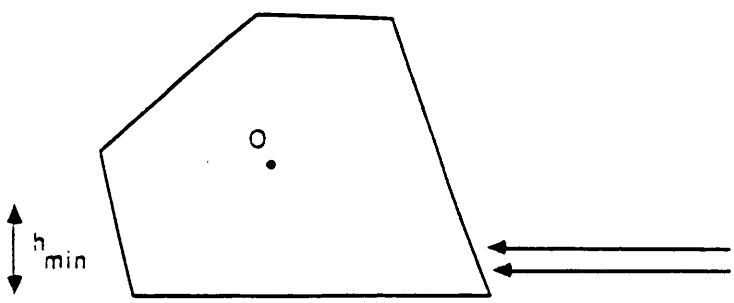


Figure 2.10: Determining the next edge of model polygon P.

$d^{\rho}_{\min} = \min\{d^{P}_{\min} \mid P \in \rho\}$.

With these two quantities pre–calculated for ρ, our strategy is to send probes towards O within an angular sector of $\theta^{\rho}_{\min}$ until three are collinear. This must happen after at most five probes regardless of which model P is, with up to two probes contacting a neighboring edge. We now walk around P, aiming probes parallel to the previous edge of P within $d^{\rho}_{\min}$ of it, so these two probes are guaranteed to contact the next edge. These two points determine the new edge and we can resume walking around P in this manner until P is determined.

Recycling the up to two extra probes means at most $2n+1$ probes are necessary. The $2n$ lower bound follows from verification, since the model set can be selected to contain as many perturbations of P as desired. □

Things get much more difficult when the models move beyond convex polygons. For example, $\theta^{P}_{\min}$ is no longer necessarily >0, since a point within P can lie on a line defining one of its edges. Unfortunately, as we move beyond convex polygons, we lose the independence on m.

Theorem 2.15: $O(mn)$ probes are both necessary and sufficient to determine an n-gon P selected from a set ρ of m star-shaped models if O is within the kernal of P.

Proof: Sufficiency can be shown using the following strategy. From the model set, the minimum angle range $\theta'^{\rho}_{\min}$ spanned by a pair of edges as visible from any point in the kernal can be calculated as above. The point generating this angle will lie on the kernal boundary for some P in ρ. If we send probes towards O in an angle range of $\theta'^{\rho}_{\min}$, they can contact at most four different edges. Thus at most four point images can be collinear without being incident upon the same edge. By the pigeonhole principle, five collinear point images determining an edge will occur after at most 17 probes.

Probing along this edge provides the first point of contact in this direction. By considering the determined line as a guide plane and the contact point as a reference, we can now consider the problem as distinguishing the possible mn orientations. Clearly $mn-1$ additional probes are sufficient to determine the correct orientation out of the mn possibilities, for a total of $O(mn)$ probes.

Necessity is shown by constructing m models of regular $(n-3)$–gons each with a small triangle cut from one of the faces. The base of the triangle $b < l/2m$, where l is the length of the repeated side of the $(n-3)$–gon. These triangles are all mutually disjoint over all orientations, as shown in Figure 2.11. To distinguish a particular orientation requires a probe incident upon the triangular cut. Since no two orientations share a cut, and there are $(n-3)m$ total orientations, at least $mn-3m-1$ probes are required. $\square$

We note that this strategy improves upon Bernstein's result when $m=1$, which is essentially Natarajan's original problem. In this case, $n+5$ finger probes are sufficient and $n-1$ necessary to determine the orientation of a known convex n–gon. It is an open problem to eliminate the gap in lower order terms between the upper and lower bounds and generalize the strategy to distinguishable polygons.

This dependence on m proves it is difficult in the worst case to distinguish between multiple non–convex models. Thus, it is understandable that heuristics have been used for this problem, particularly in light of Grimson's [36] average case results.

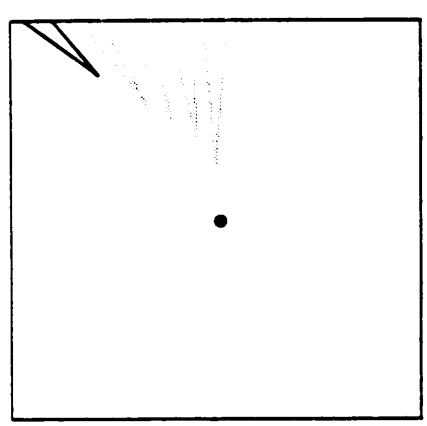


Figure 2.11: A worst case for model–based determination.

It would be desirable to find a strategy for model–based probing where all probes pass through the origin. In addition to greater robustness, such probes have an important property to be discussed in the next chapter, namely duality with hyperplane probes.

2.6. Conclusions and Open Problems

We have presented a variety of results for finger probing in two and three dimensions. The intuitive nature of finger probes makes it easy to define interesting problems.

Dobkin, Edelsbrunner, and Yap [14] consider the problem of probing with uncertainty. This is clearly of practical importance, since any real world sensing device would have some degree of measurement error, and the strategies discussed in this chapter are not robust. For example, we aim probes directly at vertices, which is not a stable operation. One way to formalize uncertainty is to assume a lower bound l on the size of any edge of P and that the returned contact point on all probes is within a distance ϵ of the actual contact point. If ϵ is a function of l, meaningful results can be expressed in terms of ϵ and n. Particularly interesting will be the relationships between the uncertainty of the probing device ϵ, the desired accuracy of reconstruction δ, and the number of probes needed to achieve it. Obviously, there are many alternative ways to formulate probing with uncertainty.

Other problems arise from a more traditional algorithmic approach. For example, for a given collection of contact points and probe paths, does there exist a simple polygon with the contact points as vertices which does not intersect any probe path? Alevizos, Boissonnat, and Yvinec [1] give an optimal $O(n \log n)$ algorithm for constructing the simple polygon, if one exists. A collection of probes on a non–convex object may well not yield such a polygon, since the vertices are limited to the contact points.

We have demonstrated some of the limitations of Cole and Yap's finger probing model. Some of these limitations will disappear if we permit the probes to originate from arbitrary points rather than exclusively from infinity. A similar issue arises with supporting line probes, to be discussed in section 3.1.

Specific open problems associated with finger probes include:

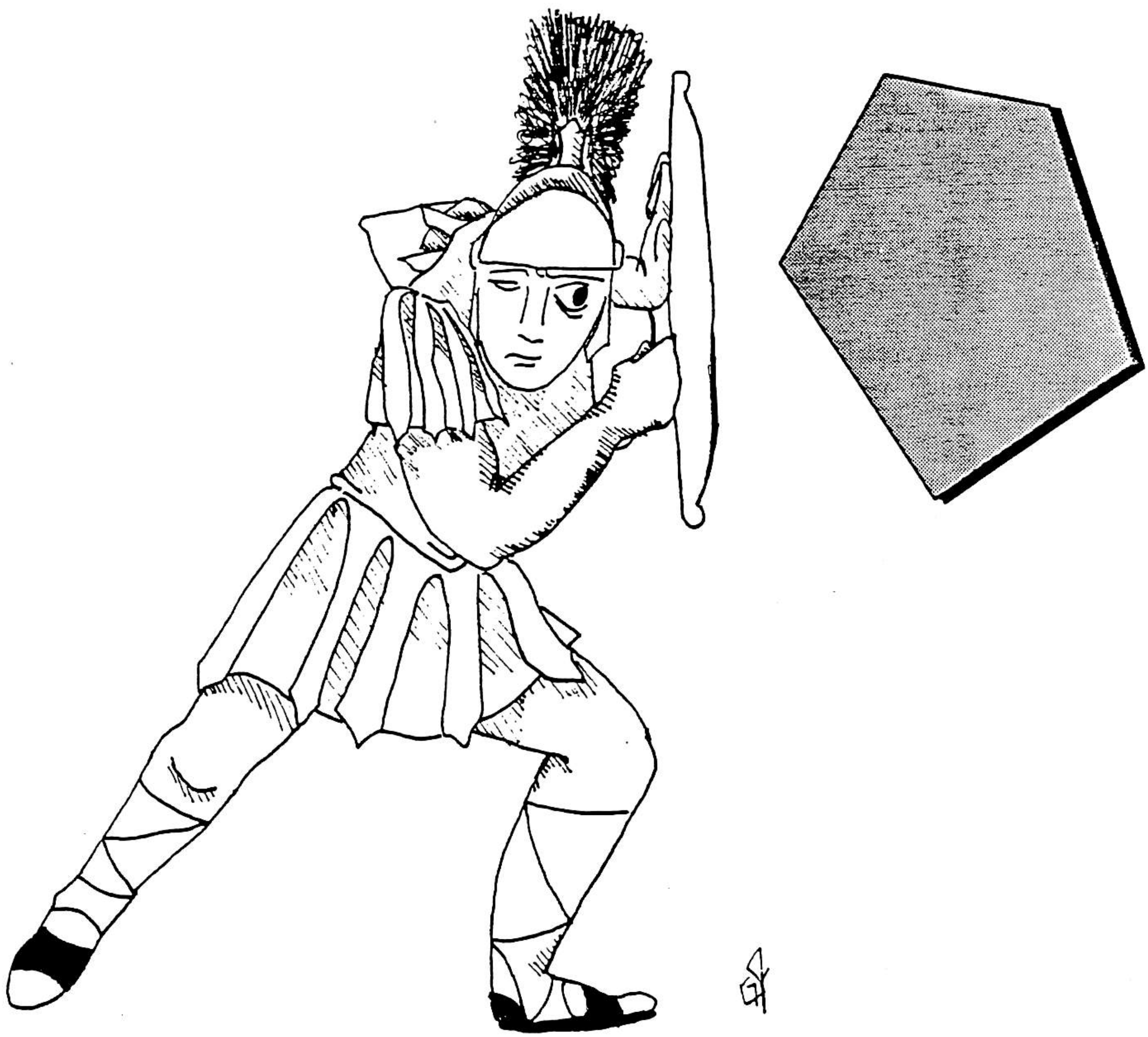
(2.1) Tighten the gap between the lower and upper bounds for determination in higher dimensions.

(2.2) Consider a convex polytope P of n vertices or k faces, each uniquely labeled. What is the largest number of orientations of such a P which are identical if the labeling is ignored? In other words, what is the maximum size of the automorphism group for a convex polytope?

(2.3) Consider a convex polygon P containing origin O and enclosed within a circle of radius r. A *billiard ball probe* is directed along a line and returns the point of intersection between the circle and the path of the ball as it is reflected off the unknown object. To make the problem well defined, let the angle of incidence equal the angle of reflection off edges of P and let the ball be absorbed if it hits a vertex of P. This seems to be analogous to problems in particle physics. Billiard ball paths [8] can be related to problems of illuminating every point within a mirrored polygon with a single light source [18].

CHAPTER 3.

LINE AND HYPERPLANE PROBING



At an early age, we are taught that it is not polite to point. The finger probes of Chapter 2 are defined in terms of a moving point, like a finger. In this chapter we introduce the notion of the hyperplane probe, which is defined as a moving hyperplane which stops when it is tangent to the polytope. The palm of a hand which hits an object is an example of such a probe. As we shall see, this anthropomorphic description is not the only connection between finger and hyperplane probes.

The terminology of this chapter has the potential to become confusing, as the same concept has been referred to in the literature as line probes [56], support planes [34], and hyperplane probes [14]. We will use the following terminology. A ***hyperplane probe*** will be a $(d-1)$–dimensional hyperplane, typically aimed at a d–dimensional object. A one–dimensional hyperplane probe will be called a ***line probe***. A two–dimensional hyperplane probe will be a ***plane probe***. Like finger probes, hyperplane probes originate from infinity.

Section 3.1 discusses the duality relationship between finger, hyperplane, and supporting hyperplane probes. Applications of duality are provided in the next several sections. In section 3.2 we discuss the dual problems of determining convex regions which contain P and are contained within P. In section 3.3 we consider silhouette probes, which return the shadow cast by P and is the dual of a probe which returns a cross–sectional slice of P. Hyperplane probes have a close connection to optimization problems, which is explored in section 3.4. Interesting problems occur when we consider probing objects whose dimensionalities differs by more than one from the probe. In section 3.5, we give a strategy for determining convex polytopes using line probes. Finally, we conclude in section 3.6 with some open problems.

3.1. Duality between Finger and Hyperplane Probes

Duality [11] is one of the fundamental ideas of computational geometry and occurs frequently in many areas of mathematics. Two problems P_1 and P_2 are duals if there is a transform which relates each instance of P_1 to a unique instance of P_2 and each instance of P_2 to a unique instance of P_1. Thus any algorithm which solves one of the problems can also be used to solve the other, since any instance of P_1 can be transformed to one of P_2 and solved with the known algorithm. Duality is important because it provides alternative representations for problems which lead to solutions which otherwise might not become apparent.

We will demonstrate the power of duality by proving that finger and hyperplane probing are really identical problems. This duality relationship was independently discovered by Dobkin, Edelsbrunner, and Yap [14] and Greschak [34].

Let P be a convex polytope and O be a point within P. Each point $p \neq O$ defines a vector in E^d. Let the dual of p, $d(p)$, be the closed half-space containing O, defined by the hyperplane normal to the vector p containing the point $p/|p|^2$. The *dual* of the polytope P is defined

$$D(P) = \bigcap_{p \in P} d(p)$$

Figure 3.1 shows the effect of the duality transform on a polygon. Each vertex v of P is replaced by an edge e of $D(P)$ which is normal to (O,v). Likewise, the normal to each edge of P defines a vertex of $D(P)$. The dual of any hyperplane probe which contacts v of P corresponds to a finger probe which intersects e. If we consider a finger probe moving towards the origin, it dualizes to a hyperplane probe moving normal to it away from the ori-

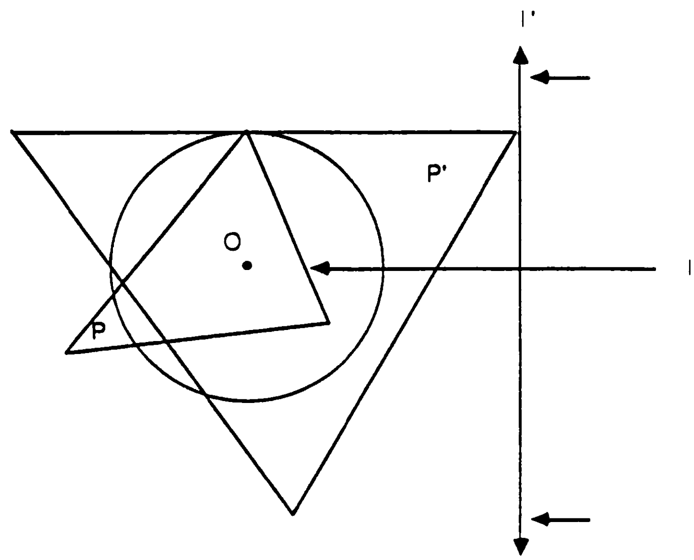


Figure 3.1: The dual of a polygon.

gin. The finger probe contacts P at the last point in time when the hyperplane probe intersects $D(P)$. We can thus view the hyperplane as moving towards $D(P)$ and stopping when it touches $D(P)$ – this gives the same result and is more intuitive than the hyperplane that moves away from the origin.

Thus, any strategy for finger probing where all probes are aimed through O defines a strategy for hyperplane probes. Dualizable results include the $2n$ bound for verification of Theorem 2.1 and the determination lower bound of Theorem 2.3. Unfortunately, the optimal $3n$ determination strategy of Theorem 2.2 and the model–based strategy of Theorem 2.14 rely on at least one non–origin probe and do not dualize to line probes. Li [56] proves the following tight bound on determination with line probes.

Theorem 3.1: $3n+1$ line probes are sufficient to determine a convex n-gon.

Proof: This strategy is a simplified dual of the finger probing strategy of Theorem 2.2. We will use three line probes to define a triangular *envelope* around P. We continue to send line probes until three are incident upon the same vertex. From this vertified vertex v_i, we proceed counter-clockwise around P, directing the probe at the line defined by a verified vertex and the second unverified vertex v_{i+2} of the envelope. Either this probe verifies unverified vertex v_{i+1}, creates a new unverified vertex, or verifies the edge between v_i and v_{i+2}.

Since the third and verifying probe for each vertex except v_1 also verifies an edge, the procedure stops when n probes have verified edges. v_1 is incident upon three other probes and the remaining $n-1$ vertices are incident on two other probes each, for a total of $n+2(n-1)+3 = 3n+1$. □

Theorem 3.2: $3n+1$ line probes are necessary to determine a convex n-gon.

Proof: We will create an adversary which will ensure that after $2n+1$ line probes only one vertex will be verified. If the probing strategy defines a closed envelope, the adversary will make it grow to $2n$ vertices without verifying a vertex. There are at least two distinct inscribed n-gons which can define it, which alternate vertices of the envelope. The $2n+1$st probe will verify the first vertex. From this it follows that $3n+1$ is a lower bound since each of the n edges must be verified.

If the envelope is kept open, the adversary will wait until the $2n$th probe to yield its first vertex. However, the adversary will ensure that the first probe to close the gap does not verify an edge. Thus $3n+1$ probes are necessary in this case as well.

We note that less care is needed to prove a lower bound for line probes than for finger probes since line probes have only one degree of freedom, their slope. □

If we dualize a finger probe f which is not directed through O, it sweeps out a region between two hyperplanes. The first hyperplane is defined by its normal f and O and the second hyperplane intersects $D(P)$ in the appropriate point. Thus any finger probing strategy is equivalent to a strategy of hyperplane probes and probes which rotate hyperplanes around a $(d-2)$–dimensional flat, which we call ***supporting hyperplane probes.*** The $(d-2)$–dimensional axis of rotation is a dual of the line defining the finger probe. The direction of rotation is specified by the direction traveled along f.

Supporting hyperplane probes occur naturally in various robotics problems. In E^2, we can consider supporting line probes. Specified by a starting point p, angle, and direction of rotation, they return the angle of the line through p which first intersects P. By considering two such probes with opposing directions of rotation, we obtain an angle range over which P lies. This is exactly the information obtained by gripping the object between two fingers of an endless hand. An alternate and more realistic problem involves a finite sized claw, such as the section of an object one can grab with a hand. Okada and Tsuchiya [64] discuss a system which distinguishes between a ball, cylinder, and various prisms using the contact points of finger positions while grasping the object.

This probing model also relates to certain questions of robot navigation. Consider [84] a room with hot pink walls and more conservatively clad obstacles. A robot with a simple sensor can determine the angle ranges over which obstacles lie and plot a course avoiding the obstacles and mapping the room. Only incomplete information is obtainable with multiple objects. Other problems [49] arise when the robot must also determine its orientation in the room.

3.2. Bounded Regions and Bounding Regions

A simple strategy suffices for using $d+1$ hyperplane probes to define a simplex in E^d which contains P. Similarly, it is obvious that $d+1$ finger probes suffice to determine a d-dimensional simplex which is contained within P, since the region within the convex hull of the contact points must be within P. What isn't clear is how many hyperplane probes are sufficient to identify a region within P or how many finger probes are necessary to determine a finite region which encloses P.

These problems of determining bounded regions and bounding regions for P are duals for finger and hyperplane probes. More specifically, the problem of determining a bounding simplex with finger probes is dual to determining a bounded region within P containing O using line probes, since the duality transform assumes that O is within P and thus P'. Greschak [34] shows that at least 5 points in two dimensions and at least 8 in three dimensions are necessary to specify a finite convex region. This serves as a lower bound for our problem, but unfortunately it cannot be achieved.

In this section, we will see that this nice duality relationship does not actually help us, since a constant probe strategy for determining bounding polygons with finger probes cannot exist. However, we salvage the result for line probes, showing that seven line probes are sufficient to determine a bounded region which may not contain O.

To see the difficulty in constructing a bounding region, it is useful to construct a map of three finger probing outcomes. Aiming three probes aimed through O with angles $2\pi/3$ radians apart from each other gives three points which define a triangle within P. These points define three lines, and these slopes and the third point define three more lines. These lines divide the plane into 16 regions as shown in Figure 3.2.

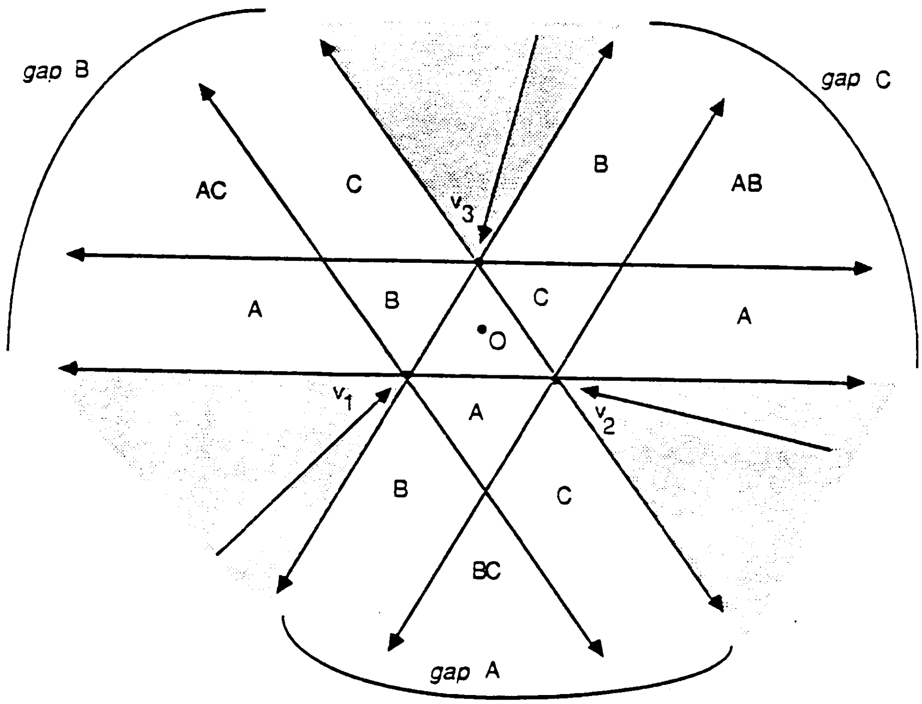


Figure 3.2: Mapping a bounding polygon.

The shaded regions of Figure 3.2 represent points which cannot be within P because of convexity constraints. They are the shadows behind each contact point. To determine a bounding polygon for P, we must close off the open gaps A, B, and C between the contact points by such shadows and take the convex hull of the boundary. The regions on the map have been labeled to show which gaps will be closed off if a contact point occurs within the region. Note that every point in the plane closes at least one gap except for the shadows and points on the lines defining triangle (x,y,z). However, note that three of the four regions in each gap close off only *neighboring* gaps and they surround the region which closes the local gap.

Suppose we direct probes through O and x, y, and z. Each of these three probes (O,i) will intersect P in one of four places, an outside region which closes two gaps, the point i, an inside region, or on a line defined by two of the first three probes. Further, assume that probes y and z close off gaps A and C respectively and that probe x contacts in region AC. Thus gap B has been split into two gaps. An adversary can construct a polygon which repeatedly prevents access to this region to close the remaining gap, until a linear number of probes have been expended. Thus no constant probe strategy for our dual problem exists.

However, the situation is better for line probes.

Theorem 3.3: Seven line probes are sufficient to determine a bounded region within P, and six are sufficient if an interior point O is given.

Proof: Send five line probes with directions defined by a regular pentagon. These will define a convex envelope containing P. We perform a case analysis on the number of vertices of this envelope.

Suppose there are five vertices. Then the inscribed pentagon as shown in Figure 3.3 must be within P, since the P must be tangent upon each line probe. If we label the vertices

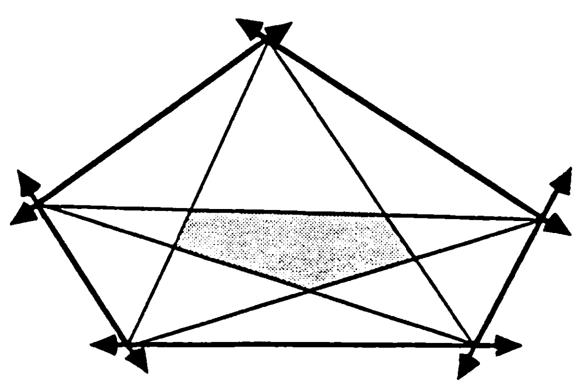


Figure 3.3: The inscribed pentagon must be within P.

of this inscribed pentagon $v_1, \cdots, v_5$ around O, the intersection of the half-planes defined by (v_i, v_{i+2}) and O must be within P. If not, no point of P can exist to contact one of the line probes without violating convexity.

Suppose the envelope contains three vertices. Two vertices of the envelope are tangent upon three line probes and are thus vertices of P. The edge between them is an edge e of P. Sending a sixth probe parallel to e will either determine the third vertex of P or create a trapezoidal envelope, and the intersection of the appropriate half-planes defined by e and the two diagonals of the envelope determine a bounded region within P. This sixth probe is unnecessary if O is known, since the convex hull of O and the two vertices is within P.

Finally, suppose that the envelope contains four vertices, one of which must be incident upon three tangent lines and belong to P. Aiming a sixth probe in the direction defined by this and a neighboring vertex will yield an envelope of three or five vertices, which has already been discussed and require at most one additional probe. □

We note that if O is not within the bounded region, the convex hull of the union of this region and O is within P. However, O is not *within* this new region – which would be necessary to dualize to a finger probe bounding strategy. It is surprising that such a slight distinction means the difference between a constant and linear probe strategy.

Similar techniques can be used to extend these results to higher dimensions. However, it is difficult to visualize the analogous cell complex in E^3 and extend the case analysis.

3.3. Silhouette Probes

One problem researchers in computer vision must contend with is that photographic images often provide too much information. It is difficult for a program to associate regions of different shadings as portions of the same object. Thus they often deal with *silhouettes*, thresholded binary images when the object is dark and the rest of the image is light, like a shadow where the illumination originates from infinity. Projecting a silhouette defines an infinite cylinder in which the object must lie. Intersecting several silhouettes refines our knowledge of the object's shape and has formed the basis of many vision and solid modeling systems. For example, Wang, Magee, and Aggarwal [93] describe a system which performs model based recognition of different types of cars from their silhouettes. See Martin and Aggarwal [59] for more extensive references on the applications of silhouettes.

There is a duality relationship between silhouette probes and the equally natural notion of cross–sectional slices. The cylinder defined by a silhouette with direction $\bar{v}$ represents the intersection of half–spaces containing O and defined by all hyperplane probes perpendicular to $\bar{v}$. The duals of these hyperplanes are the complete set of finger probes along a slice through O perpendicular to $\bar{v}$.

In two dimensions, a silhouette probe represents two line probes with identical slope originating from $\pm\infty$. Each such probe dualizes to the pair of finger probes defined by a line through O, the cross–sectional interpretation. Li [56] terms two–dimensional silhouette probes *projection probes* and proves the following tight bounds for determining convex polygons with them.

Theorem 3.4: $3n-2$ projection probes are sufficient to determine a convex n-gon, $n \geq 3$.

Proof: We will specify probing directions using essentially the same algorithm as presented in Theorem 3.1. To achieve this bound, we must save three probes over the previous analysis. After the first two probes, we have determined a rectangular envelope around P. The third projection will either result in an envelope of six vertices, an envelope of one verified vertex and four unverified vertices, or verify two vertices of the previous rectangular envelope.

The first two possibilities represent configurations which are achievable in six line probes by the strategy of Theorem 3.1. Since we reached it in three, we have saved three probes and can continue to run the line probing algorithm, eventually stopping after $3n-2$ probes.

However, the two non-consecutive verified vertices of the third case cannot occur with our line probing algorithm. If we pick one of the verified vertices and continue with our algorithm, the fourth direction will be defined by the two verified vertices. This probe can either verify the remaining two vertices, verify one and introduce an extra unverified one, or introduce two more unverified vertices. In the first case, the quadrilateral was determined in four probes, and $4 \leq 3(4)-2$. In the second case, we have three consecutive confirmed vertices and two unconfirmed ones, which might take eight line probes, so four probes have been saved. Finally, in the third case, we have two confirmed vertices and four unconfirmed ones. Since one confirmed and five unconfirmed vertices can take seven line probes, we have saved three, giving the result. The special case of $n=3$ falls out of the same strategy. □

Theorem 3.5: $3n-2$ projection probes are necessary to determine a convex n-gon.

Proof: Let P be a convex polygon such that both the interior angles incident upon an edge e

are acute. Thus at least one half of any projection probe contacts e.

For such a P, an adversary can ensure that after at least four probes e and both its vertices can be determined, along with two unverified vertices on the envelope. The first two probes define a box around P, and for the third probe, the adversary will verify two opposing vertices of this box. Any fourth probe which is not defined by the two determined vertices can be made to contact these vertices again, wasting a probe. Thus the fourth probe must determine e and the envelope as described above.

From here, the adversary can ensure that $2n-5$ line probes are made leading to an envelope with $2n-3$ unvertified vertices on it. For such an envelope, there exists a convex n-gon such that only e is common with the envelope. Since $n-1$ edges must still be verified, at least $4+(2n-5)+(n-1) = 3n-2$ probes are necessary for determination. □

It is interesting that doubling the number of probes only buys us an improvement of three in the time it takes to determine polygons. We shall see in section 6.4 that more dramatic improvements are possible when the two probes per iteration are not restricted to the same line.

Li observes that just knowing the projection image, ie. the distance between the two lines for θ independent of position, is not sufficient for reconstruction. The *constant-width* or *Rouleaux triangle* of Figure 3.4, returns identical images for each projection as a circle, although the images are not identically located.

Silhouette probes also provide surprisingly little power in higher dimensions. Dobkin, Edelsbrunner, and Yap [14] proved the following bounds which are optimal within a multiplicative constant, since $3f_0(P)-6 \leq f_2(P)$ by Euler's formula. Recall that silhouette probes

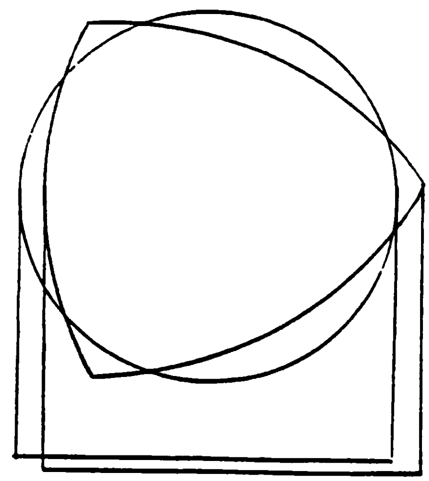


Figure 3.4: A Rouleaux or constant–width triangle.

are dual to cross–sectional slices.

Theorem 3.6: $f_0(P)/2$ cross–section probes ($f_2(P)/2$ silhouette probes) are necessary and $f_0(P)+5f_2(P)$ ($5f_0(P)+f_2(P)$ silhouettes) are sufficient to determine a convex polyhedron in E^3.

Proof: A simple verification argument suffices to show the lower bound. Let P have no three vertices be co–planar with O. Since every vertex must be the vertex of a cross–section to be verified, at least $f_0(P)/2$ probes are necessary.

For sufficiency, we note that a finger probe can be wastefully simulated by a cross–section probe perpendicular to the desired probing direction. The result follows from the complexity of the higher dimensional finger probing strategy of Theorem 2.8. For silhouettes, we simulate a hyperplane probe. □

We leave it as an open problem to tighten these results, noting that Chazelle, Edelsbrunner, and Guibas [10] prove there exist polytopes such that each silhouette has at most $O(\log n / \log\log n)$ edges.

3.4. Optimization Problems and Algorithmic Paradigms

Hyperplane probes have a close connection to optimization problems, since the constraints for mathematical programs are typically represented by hyperplanes. In this section, we shall look more closely at this connection, as well as at hyperplane probes as a paradigm for solving algorithmic problems.

The results of a hyperplane probe can be simulated by a linear program, assuming each facet is described by the equation of the hyperplane which contains it. Let d be the number of dimensions of P. Each facet i is represented by one constraint:

$$c_{1i}x_1 + c_{2i}x_2 + \cdots + c_{di}x_d \leq c_{(d+1)i}$$

The objective function $F(x)$ describes the moving hyperplane, which is specified by the vector $(a_1, a_2, \cdots, a_d)$, is

$$F(x)\colon\ a_1x_1 + a_2x_2 + \cdots + a_d\ .$$

The result of the probe is determined by maximizing or minimizing $F(x)$, depending upon whether the probe originates from $+\infty$ or $-\infty$. This value is the intercept which with $F(x)$ completely specifies the hyperplane.

These simulated probes are actually more powerful than the previously defined hyperplane probes, since the contact vertex is specified by the optimal vector x. Rajan [75] has considered probing strategies for these, which we whimsically call "linear probegrams".

Theorem 3.7: $f_0(P)+f_{d-1}(P)$ linear probegrams are necessary and sufficient to determine P.

Proof: Necessity is shown by a verification argument – each probe can only verify one vertex or facet and each must eventually be verified.

For determination, we shall use the following strategy. With the first $d+1$ probes, we aim to determine a d–simplex of vertices. To determine the ith probe, construct a hyperplane which contains the first $i-1$ vertices. Probing in the opposite direction will result in a new vertex.

Once the simplex is identified, each facet must be probed. Each probe will either return a new vertex or verify the facet. In either case, each vertex and facet is probed exactly once, giving the result. □

A problem of interest in combinatorial optimization can be solved using these linear probegrams. Given a convex polytope P containing the origin O, what vertex of P, $v_{\max}$, is furthest from O? This can be easily formulated as a quadratic program, with the previous set of constraints and maximizing the objective function

$$x_1^2+x_2^2+\cdots+x_d^2 \ .$$

Unfortunately, the problem of solving quadratic programs is NP–hard [28] and Rajan has considered using linear probegrams to determine $v_{\max}$. If all vertices of P are equidistant from O, P will need be determined as in Theorem 3.7. Since the number of vertices can be exponential in the number of constraints this is not efficient, but it is reasonable to consider heuristics to select the probes.

A closely related problem is how close an approximation to $v_{\max}$ can be obtained using k probes. In higher dimensions the problem gets difficult, although there is an elementary result in E^2.

Lemma 3.8: If r is the maximum distance from O to any vertex v_i in a convex polygon P, a distance x can be identified in k line probes such that

$$x \geq r \sin(\frac{(k-2)\pi}{2k}) \ .$$

Proof: If the k probes are spaced at regular angular intervals, we define a (possibly degenerate) convex k-gon which contains P. The largest distance r_k from O to the envelope defined by these probes must be to a vertex v of this k-gon. Clearly, $r_k \geq r$. If the envelope is degenerate, we have determined the actual distance to any degenerate vertex, so we need only consider the case of a real convex k-gon.

As in Figure 3.5, let a and b be the perpendicular bisectors through O of the lines which define v. Clearly, a and b represent the minimum distances the contact point of the

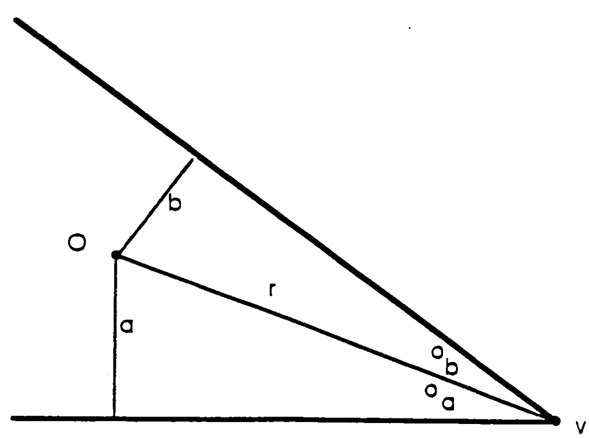


Figure 3.5: Approximating the maximum distance from O.

two probes can be from O. Without loss of generality, let $a \geq b$. We are interested in the ratio between a and r_k.

Since the probes are regular angular intervals apart from each other, the angle θ between the two probes defining v is $(k-2)\pi/k$. Let θ_a and θ_b be the sectors of the angle partitioned by the line (O,v). Since $a \geq b$, $\theta_a \geq \theta_b$, so $\theta_a \geq (k-2)\pi/2k$. By definition $\sin(\theta_a) = a/r_k$, which can be rewritten to give the result. □

Determining an approximation to the volume of an object has been considered for a slightly different computational model. Bárány and Füredi [3] consider an oracle which for a point x tests whether $x \in P$, where P is a convex body and seek to determine upper and lower bounds to the volume of P. They prove that no polynomial time algorithm exists such that the ratio of these quantities will be less than $(d/\log d)^d$ in E^d. Elekes [20] proves the combinatorial result that the volume of the convex hull of n points on a unit sphere in E^d is at most $n/2^d$, which leads directly to results for finger and hyperplane probes.

If the coefficients and variables in a linear program are restricted to integer values, the result is an integer program. Although the problem of solving an integer program is NP-complete, its great practical importance has lead to the development of a variety of techniques for solving integer programs. One class of techniques, the *cutting-plane algorithms* [67], relaxes the problem by considering the equivalent linear program and repeatedly adds constraints or cutting planes which prunes the feasible region without removing any integer lattice points. The algorithm terminates when the optimal point of the revised linear program has integer coordinates. These cutting-planes are in fact hyperplane probes, and an integer programming algorithm results from using simulated hyperplane probes to determine the convex hull of lattice points beneath the feasible region. It is left as an open problem

In order to origin probe the edge pair between the section defined by two vertical probes, we must have an upper bound U on the distance the edges within this section are from the x-axis. Without this knowledge, we cannot design origin probes which we can be certain will intersect the boundary of P within the section, and thus whether the origin probes all hit the same pair of edges. For the situation where exactly one of the two initial vertical probes l_1, l_2 intersect P, we know that the x-axis intersects P throughout the resulting section. In this case we can bound U because it is clear that the distance of a point from the x-axis is no larger than the height of the associated histogram at the vertical line through the point.

This argument fails in the case of Figure 4.6, since we cannot be certain on which side P contains the x-axis. We will use a convexity argument to put a bound on U. Let us arbitrarily select the section to the right of O. If this section contains the x-axis, we know a bound on U. If not, we know that the other section contains the x-axis within P for a distance $\lambda/2$. The slope of the upper edge of P that intersects the y-axis is greatest if it is the only upper edge of the left section, $l_1 = l((-\lambda/2,0),\pi/2)$ intersects P entirely below the axis, and $l_O = l(O,\pi/2)$ intersects P entirely above the axis. By convexity, no edge in the other section can increase faster than this line, which passes through the points $(0,X(P,l_O))$ and $(-\lambda/2,0)$. Reflecting this situation along the x-axis bounds the lower edges, and together provides the information we need to origin probe.

A further complication occurs when the edge pair is parallel, meaning b_1 and b_2 are undefined. If another potential edge pair exists in this section, that is, it required more than five parallel probes within the section to locate the parallel edge pair, this non-parallel edge pair can be uncovered by a total of $2n$ parallel probes, since now two edge pairs can have three probes each.

If the first edge pair is parallel and another edge pair is not known, we must now repeat the sectioning process parallel to the original pair. Clearly, a line l_p through O parallel to the first edge pair intersects P between the edge pair. By a process of binary search, we can enlarge this strip of P known to lie between the edge pair as much as desired. If δ_1 is the distance between the two edges, a probe parallel to l_p $\delta_1/2$ below l_p widens this strip by $\delta_1/2$. If this probe intersects P, the strip is between l_p and the last probe. Otherwise, the strip is on the other side of l_p. Similarly, we can widen this known strip to $3\delta_1/4$ with a probe parallel to l_p $\delta_1/4$ on either side of the known strip. We can continue to widen this strip by this method, although for our purposes two of these probes will suffice. This strip will serve to define a section for the next set of probes. Note that there is no reason to actually probe l_p and that at this point we do not know how long the edges of the first parallel pair are.

Since these probes are parallel with our previously encountered edge pair, they will intersect at least two edges different from the parallel edge pair. We will make five of these, one at each side of the boundary of our $3\delta_1/4$ region, two more within this region $\delta_1/2+\epsilon$ apart for $0<\epsilon<\delta_1/4$, and one between these final two probes. Note that it may be possible to reuse the binary search probes, but only if they intersected P. If the center three of these probes are not all of the same magnitude, they do not all intersect a parallel pair of edges. Thus with up to $2n-3$ more parallel probes we can locate a non–parallel edge pair, which can be origin probed to determine the edge pair. Unfortunately, as in Figure 4.7, the center three probes can instead intersect a parallel edge pair. This parallel edge pair must be greater than $\delta_1/2$ in length.

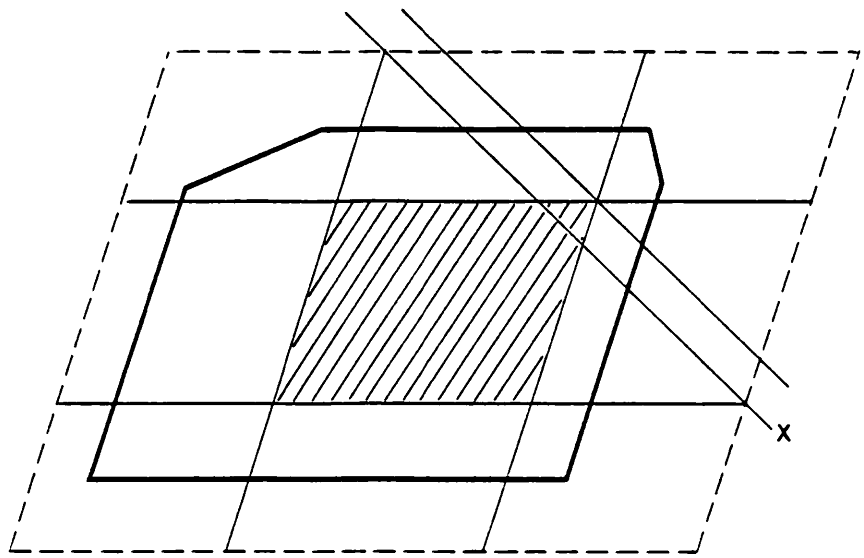


Figure 4.7: Handling parallel edge pairs.

If δ_2 is the distance between the second pair of parallel edges, we can define a strip $3\delta_2/4$ wide within P through the binary search strategy using probes parallel to the second edge pair. Two parallel probes within this strip $\delta_2/2+\epsilon$ apart for $0<\epsilon<\delta_2/4$ can confirm that the first edge pair is greater than $\delta_2/2$ in length. If this is not the case, we can find a non-parallel edge pair between the offending probe and the appropriate end of the $3\delta_2/4$ region. Otherwise, the intersection of the two strips defines a rhombus Q which must lie within the interior of P. We note that the remainder of P must lie in strips less than $\delta_1/2$ and less than $\delta_2/2$ wide around Q. Extending these boundaries for each of the two edge pairs surrounds Q by a skewed grid of eight regions, which together contain all of the edges of P. None of these regions can contain parallel edges, since P is convex. Further, no three neighboring regions around Q including only one corner region contain any parallel edges.

Referring to Figure 4.7, it is clear that a probe X from the upper left corner of the top–central region to the bottom right corner of the right–central region cannot intersect a parallel pair of edges. Because of the size and position of the central region, this probe must intersect Q, meaning it intersects a non–parallel edge pair. Along with a probe parallel to X that intersects the upper right–hand corner of Q this defines a section which can only contain non–parallel edge pairs, and thus can be parallel probed until three are collinear on the histogram. Using the earlier counting argument with $n-2$ edges, since the other two parallel edges cannot be within the section, shows finding an edge pair can require up to $2n-5$ additional probes.

Finally, one confirmation probe of the non–parallel edge pair will distinguish the edges on P from the pair on $-P$. Note that there is no ambiguity between P and $-P$ for the parallel edge pairs.

Lemma 4.7: With restriction to origin and parallel probes, $2n+23$ x–ray probes are sufficient to identify the lines that contain the first pair of edges and a point on one of the two edges.

Proof: The above strategy will identify the lines that contain a pair of edges and a point on one of the two edges. The final accounting of probes is as follows. Four probes were used initially to define a section to probe, at least two of which can serve as parallel probes. Three more parallel probes can identify an edge pair, with the center three probes incident on a parallel pair of edges. Three origin probes found the slopes of these lines. Two parallel probes will widen the strip to $3\delta_1/4$. Up to five probes between the two edges will identify that this edge pair is also parallel, and three more are necessary to origin probe it. Two more probes widen the new strip, two more enlarge Q, and two diagonal probes select a non–parallel section. $2n-5$ parallel probes in this section will locate a non–parallel edge pair, the first two of which were the diagonal probes. Three more origin probes find the equations of these lines. Finally, there is the confirmation probe. Thus two edges can be determined in a total of $4+3+3+2+5+3+2+2+2+(2n-5-2)+3+1 = 2n+23$ probes. Any point on these two edges within the appropriate section is on the boundary of P. □

A complete probing strategy for P could perhaps be constructed along these lines by repeating the process for each pair of edges. However, since $O(n)$ edge pairs can be parallel this would lead to a quadratic number of probes. A simpler strategy can be developed once we know a point on the boundary of P.

4.3.4. Boundary Probes

The power of the finger probe is that it returns a point on the boundary of the polygon. To get a similar effect, we define the *boundary probe*, which relies on the observation that sending an x–ray line probe through a known point on the boundary of a convex polygon identifies another point on the boundary of the polygon. This means that once we have identified the coordinates of any point p on the boundary of the polygon, any x–ray probe through p determines another boundary point. If we also are given a boundary point we can formulate our first probing algorithm.

Theorem 4.8: With restriction to origin, boundary, and parallel probes, $5n+19$ probes suffice to determine a convex n-gon P.

Proof: By Lemma 4.7, $2n+23$ probes suffice to find a boundary point and *semi-verify* two edges. The dual of Theorem 3.1 gives a strategy using $3n+1$ finger probes to determine convex polygons which can be modified to use boundary probes in place of finger probes.

Starting from one of the semi-verified edges, we will walk along the polygon, conjecturing vertices based on the intersection of the semi-verified edge and the line defined by the next two known points. Each of the n vertices will eventually be probed, and each of the $n-2$ other edges will have at most two interior points probed, for a total of $5n+19$. □

Being clever about re-interpreting the parallel probes may reduce the total by $O(n)$ more probes since once the edge one of them passed through is determined, a probe on an unverified edge can be recorded. No doubt, the additive constant of Lemma 4.7 can be lowered by more careful arguments. The total time-overhead of planning these probes is clearly $O(n)$ as well.

Note that the optimal $3n$ strategy of Theorem 2.2 cannot be adapted to x-ray probes since they probe along a semi-verified edge to obtain a vertex, which will not work with x-ray probes unless the location of the other vertex is known.

4.3.5. Close Probes

If the measurements we have been using were performed on real sensing devices, there will be some uncertainty with respect to accuracy. Thus for us to completely determine an n-gon we must know that no edge has length less than this uncertainty, or else we could never find the edge. Knowing a lower bound on the length of all the edges ϵ gives us extra information about the polygon. We can exploit this with a collection of *close probes*, where each probe depends on intersecting a point on the boundary within some fraction of ϵ of another close probe. By close probes, we mean a collection of probes made with ϵ in mind and cannot give a formal definition as with origin and parallel probes.

Close probes present a problem because they suggest strategies that are somehow "unfair" as they require additional information. However, this information is available in model based problems. Certainly in any physical implementation they would be extremely non–robust. The virtue of close probes is that they enable us to find a boundary point in a constant number of x–ray probes, as opposed to the linear probing strategy described above:

Lemma 4.9: Two lines that contain a pair of edges of a convex polygon P and a point on one of the two edges can be determined in 37 x–ray probes including close probes.

Proof: Our strategy is similar to that used in Lemma 4.7, but modified to take advantage of close probes. It should be noted that there is no fraction α such that parallel probes within $\alpha\epsilon$ of each other are guaranteed to intersect the same edge pair. The reason is that the angle between an edge and the probing direction can be arbitrarily close to π, so even a long edge can slip between two seemingly close probes. A sequence of such edges can slip between two close probes if the angles they define are sufficiently close to π. Thus the largest angle between edges will have to be bounded to take advantage of close probes.

We will replace the linear strategy of searching a bounded section of P for an edge pair by the following constant one. Within the bounded section of P, send two more parallel probes, giving four probes intersecting P labeled from left to right a, b, c, and d. By the argument in the proof of Lemma 4.7, a and b determine the steepest increasing slope possible between b and c, and probes c and d determine the steepest decreasing slope. We define $\theta_v < \pi/2$ as the greater of the two angles formed by these steepest increasing and decreasing slopes with the horizontal, so θ_v represents the angle nearest to vertical which can occur within the section without violating convexity. An edge pair will be found within seven close probes spaced $\epsilon\cos(\theta_v)/8$ apart between b and c. Seven are required because up to two vertices, one each from the upper and lower vertex chains of P, may slip between the close probes. We have thus guaranteed that at least three probes hit a common edge pair.

Thus we can use the strategy of Lemma 4.7, substituting the two parallel and seven close probes for the linear edge pair search. Using the counting argument of Lemma 4.7 with this change, we can determine the first edge pair in 37 probes. □

With the ability to find and identify an edge pair in a constant number of probes we can improve the result of Theorem 4.8:

Theorem 4.10: $3n+33$ x–ray probes (including close probes) are sufficient to determine a convex polygon P given a point O within P.

Proof: Lemma 4.9 enables us to find an edge pair in 37 probes instead of the $2n+23$ of Lemma 4.7. Substituting the new strategy for the old improves Theorem 4.8 by $2n-14$ probes, for a total of $3n+33$. □

It is certainly possible that these constants can be reduced. Other strategies involving close probes are no doubt possible.

It would be nice to find an efficient x–ray probe simulation of a finger probe. It is possible by modifying the above strategy making one of the parallel close probes along the desired probing line and then if it hits on a parallel edge pair, perform boundary probes through the located point to finish the description of the edge pair and calculate what the finger probe returned. However, since this constant will be over twenty it is unlikely the simulation can prove useful in any context.

4.4. Bounds for Verification

A lower bound on the number of probes required to determine an object can be based on a comparison to the *verification* problem. Suppose we are given the representation of a polygon P, how many probes will be necessary to test whether P correctly describes a particular object. It is obvious that any lower bound to verification represents a lower bound to the determination problem, since it presupposes knowledge of the polygon. Although this method leads to reasonable lower bounds for some probing problems, the lower bound in Theorem 4.2 is too strong to allow an improvement along these lines. We consider the verification problem in its own right.

For verification, clearly each vertex and edge must be confirmed. Otherwise, P could have a triangle on any unconfirmed edge or be truncated before any unconfirmed vertex. Since an x–ray probe passes through members of the set of edges and vertices, and there are at least $2n$ points of interest, the trivial lower bound is for n probes. It can be easily shown that three probes do not suffice to verify a triangle, since no matter how the three probes are taken the object would be indistinguishable with one of four or more sides. We conjecture that the actual bound for verification is $(3n/2)+k$ for some constant k. This is based on the observation that although $n/2$ probes are sufficient to verify the edges given the vertices or verify the vertices given the edges, it appears at least n probes are necessary to verify either the vertices or the edges independently. This conjectured lower bound is sufficient:

Theorem 4.11: $(3n/2)+6$ x–ray probes are sufficient to verify a convex n–gon P.

Proof: Given the polygon to verify, three parallel probes are sufficient to verify the existence of a non–parallel edge pair and three origin probes are enough to define the hyperbola of it.

One final probe to verify that P is not reflected through O identifies a boundary point.

From this boundary point, n boundary probes can verify the vertices. The remaining $n-2$ edges can be verified with $(n-2)/2$ probes, each bisecting a different pair of edges. Since P is the convex hull of its vertices, none of these probes can have length other than expected without violating convexity unless there exists another vertex. □

Note that since $2n$ finger probes are required to verify polygons, this is a case where x–ray probes are more powerful than finger probes.

4.5. Higher Dimensions

In two dimensions, our strategy for determination is based on finger probes once the first edge pair has been discovered. It is natural to hunt for higher dimensional strategies based on Theorem 2.8.

Theorem 4.12: At most $f_0(P)+4f_2(P)+(d+2)f_{d-1}(P)+46$ x–ray probes are required to determine a convex polytope P in E^d.

Proof: Our strategy will have two phases. The first will determine four points of P defining a tetrahedra. Then we will adapt the strategy of Theorem 2.8 by boundary probing through known points.

Probing in a plane through O, we use Lemma 4.7 and at most $2f_2(P)+23$ probes to determine the first edge pair. From this edge pair can be selected three non–collinear points which define a triangle within P. Repeating this in an orthogonal plane determines a fourth point giving the tetrahedra within $2f_2(P)+23$ additional probes.

The strategy of Theorem 2.8 aims probes at conjectured vertices, avoiding edges of the cell complex A_H defined by the verified facets. For boundary probes, we must also aim through a verified piece of a facet. Since this piece is two–dimensional, we have the degree of freedom necessary to avoid the one–dimensional edges of A_H while passing through the conjectured vertex. These probes do not have to pass through O. The problem of avoiding A_H does not occur until after a facet has been verified, so we initially aim at one of the tetrahedra points.

In total $4f_2(P)+46$ probes suffice to define the tetrahedra and $f_0(P)+(d+2)f_2(P)$ more in the second phase, giving the result. □

Another linear strategy can be based on determining cross–sectional arcs in various planes through P, and then adapting the strategy for line probes in E^3 of Theorem 3.9. This approach does not generalize beyond E^3 and has higher multiplicative constants than the strategy discussed above.

A linear lower bound in E^3 and higher dimensions can be obtained from the lower bound for finger probing and the fact that two finger probes can simulate an x–ray probe. Tighter upper and lower bounds will follow from more careful analysis.

4.6. Open Problems and Extensions

We have presented strategies for probing convex polygons with x–rays. In particular, we have shown that complete information about a convex n–gon can be obtained with a linear number of carefully planned x–ray probes. Still, the power of x–ray probes is not well understood. For example, no algorithm is known that decides whether or not a given collection of x–ray probes (and answers) determines the probed object.

A different type of probe to consider would measure the area or volume of intersection with a half–plane or half–space instead of a line. Such an "Archemedian" probe in two dimensions would have as its derivative an x–ray probe. In three dimensions, its derivative is a cross sectional area probe. We prove linear upper and lower bounds for determination with half–plane probes in the next chapter.

Open problems associated with x–ray probes include:

(4.1) Tighten the gap between our lower and upper bounds for determination. We conjecture that $3n + c$ probes are necessary and sufficient, since up to $2n$ probes in our strategy are "wasted" locating the first edge pair. Perhaps these can be reused in the second phase of our algorithm, although we do not see how this can be done.

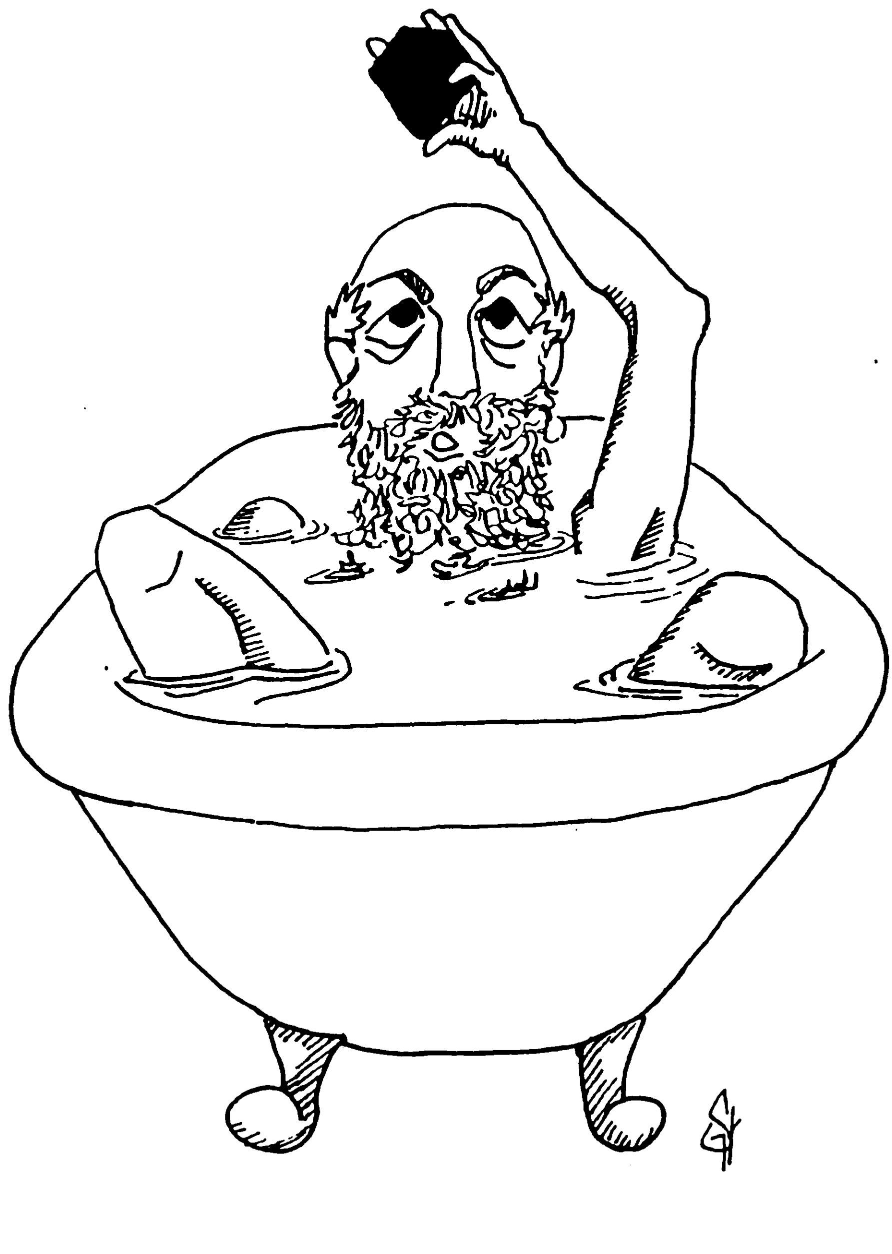
(4.2) Tighten the gap between our upper and lower bounds for verification.

(4.3) Can x–ray determination results be extended to star–shaped polygons? It appears that it may be necessary to have a lower bound on the size of an edge to do so, since a little crack in a long edge can be detected by probing along the edge, but a second probe passing through the crack would be difficult to locate.

(4.4) Can the techniques of this chapter be applied to real tomographic systems? Specifically, how effective are algebraic reconstruction techniques when probing directions can be interactively selected?

CHAPTER 5.

HALF-SPACE PROBING



One of the most famous stories in the history of science is that of Archimedes figuring out how to test if the King's crown was made of solid gold as ordered or if the goldsmith cheated and added silver to the metal. The problem revolved around how to measure the volume of the irregularly shaped crown. Archimedes had the inspiration, while taking a bath, of measuring how much water the crown would displace and comparing it to the volume of water displaced an equal weight of gold. On making this discovery, he was so excited he ran naked through the streets of Syracuse yelling "Eureka". In addition to creating a cottage industry of children's books about Archimedes [25,50,55], this tale provided the inspiration for half-space probes and this chapter of this thesis.

We define a half-plane probe to be the area of intersection between a closed half-plane h and a polygon P. Let $h(l)$ be defined as the area of intersection between P and the closed half-plane to the left of the directed line l. This notion can be generalized to half-space probes in higher dimensions, where each probe returns the volume of P which intersects with the half-space. There is a close relationship between x-ray and half-plane probes which we shall exploit to develop a linear half-plane probing strategy. Our strategy for half-planes is similar to the strategy for x-ray probes discussed in section 4.3, but requires different and more interesting geometric arguments to prove its correctness.

As mentioned above, the original inspiration for studying half-plane probes was the famous story of Archimedes measuring the volume of water the crown displaced. Such dunks in the tub are really half-space probes. More importantly, half-plane probing problems have application to tomography [43] and remote sensing, such as the lunar occultation observations used to map astrostellar radio sources [90]. The instruments for measuring such radio sources have a lower resolution than desired, so each measurement represents the total amount of energy over an area. By waiting until the moon passes over a portion of the region and measuring how much the energy is reduced, detailed maps of the source can be produced. This is very close to our notion of a half-plane probe.

As always, we assume that we are given a point O within the interior of P. A collection of half-plane probes through an object provides us with a great deal of information about it but not directly with the coordinates of a point on the surface. Half-plane probes have the advantage over other types of probes that they in some sense reflect the entire structure of the polygon in every probe. Thus they provide the possibility of extending probing results to simple polygons, since unlike with finger and x-ray probes concave edges are potentially verifiable.

Section 5.1 gives our main result, linear upper and lower bounds for half-plane probing convex polygons. Section 5.2 presents linear bounds on verification with half-planes. We consider higher dimensions in section 5.3, finding it difficult to extend our results. The related problem of extended Gaussian images is discussed in section 5.4. Finally, we conclude in section 5.5 with some open problems.

5.1. Upper and Lower Bounds for Two Dimensions

To obtain absolute information about P from half-plane probes, it is necessary to think in terms of groups of probes which work together. This section considers different classes of probes, what powers and limitations they possess and how they interact to lead to probing strategies. These classes are designed to reflect the complementary goals of recognizing and determining edge pairs.

5.1.1. Origin Probes

The first class of probes are ***origin probes***, a set of half-plane probes bounded by lines all aimed through a common point O within the object. Any half-plane probe which intersects a convex polygon and avoids its vertices will go through exactly two edges of the object. As shown in Theorem 4.3 the largest number of such edge pairs is n.

Each half-plane is defined by a directed line. We can therefore consider the complete set of origin probes through a point $O=(0,0)$ as defined by $\ell_t: y=tx$, where $t=\tan(\theta)$, $2\pi \leq \theta \leq 0$. These define a function $f(t)=h(\ell_t)-h(\ell_0)$. This function will contain enough information to determine the edges probed through, except for special cases. Here, we consider $f(t)$ for a wedge defined by two lines and containing the origin.

Lemma 5.1: Let l_1: $y=m_1x+b_1$ and l_2: $y=m_2x+b_2$ be two distinct lines, $m_1,m_2,b_1,b_2 \neq 0$, P be the unbounded wedge between l_1 and l_2 containing the origin, and let $y=f(t)$ be defined as above. Then

$$Ayt^2+Byt+Cy+Dt^2+Et=0\ ,$$

where $A=2m_1m_2$, $B=-2(m_1{}^2m_2+m_1m_2{}^2)$, $C=2m_1{}^2m_2{}^2$, $D=b_2{}^2m_1-b_1{}^2m_2$, and $E=m_2{}^2b_1{}^2-m_1{}^2b_2{}^2$.

Proof: Consider the situation in Figure 5.1, where both edges intersect the x-axis. This involves no loss of generality, since a rotation of the axes can always be performed. Hence, we need not consider the case where either slope is 0. For any t, the area swept out between $y=0$ and $y=tx$ is the sum of the areas of the two triangles defined by $y=0$, $y=tx$, and either l_1 or l_2. The value of $y=f(t)$ is defined to be the difference in area between the two triangles, $y=A_1-A_2$. More formally,

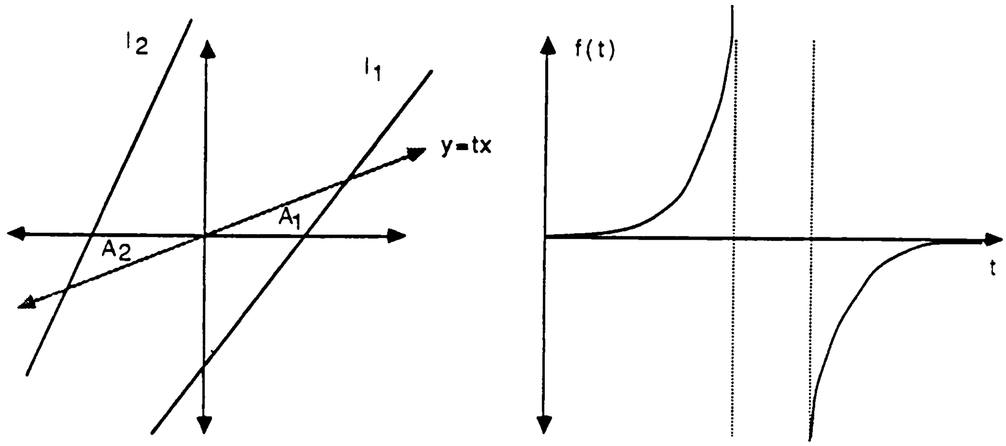


Figure 5.1: Defining f(t), the probes through an edge pair.

$$y = \frac{tb_1^{\,2}}{2m_1(m_1-t)} - \frac{tb_2^{\,2}}{2m_2(m_2-t)} .$$

Multiplying through by the denominators and simplifying gives the result. □

We note that $f(t)$ is infinite and hence ill-defined when t is between m_1 and m_2. For example, when $t = m_1$ or $t = m_2$, $f(t)$ reduces to $m_1 = m_2$. This complication does not occur when probing polygons since additional edges occur in this range. We will use Lemma 5.1 to determine the equations of the lines that contain edge pairs. If we have some number of origin probes through a common edge pair, then we can determine the $f(t)$ through the associated points. From $f(t)$, we then deduce the equations of the lines.

The function $f(t)$ is determined by five constants: A, B, C, D, and E. It follows that, in general, five probes through a pair of edges are enough to determine the function. Since all five constants are functions of the four line parameters they cannot all be independent. Indeed, $C = A^2/2$. Given A, B, D, E, we can solve for the parameters of the equations:

$$m_1 = m_2 = \frac{-B \pm \sqrt{B^2 - 2A^3}}{2A}$$

$$b_1^2 = \frac{E + m_1 D}{m_2(m_2 - m_1)}, \quad \text{and} \quad b_2^2 = \frac{E + m_2 D}{m_1(m_2 - m_1)}.$$

From these equations several limitations on our ability to reconstruct the edges become apparent. Since b_1 and b_2 are squared, we obtain no information on the sign of the intercepts. Neither m_1 or m_2 are distinguished from each other, meaning we cannot associate which intercept belongs to which line. More serious is that b_1 and b_2 are undefined when $m_1 = m_2$. Thus any probing strategy using origin probes must take special action on parallel edges.

However, to exploit Lemma 5.1 we must insure our probes intersect the same edge pair. Unfortunately, extra probes lying on $f(t)$ are not sufficient to verify $f(t)$.

Lemma 5.2: There is no constant k such that k half-plane probes lying on $f(t)$ implies that the probes pass through the same pair of edges of P.

Proof: Consider a regular $2k$-gon with center at O. All probes through O give $f(t)=0$, regardless of whether they intersect the same edge pair. □

It would be nice to generalize the proof of Lemma 5.2 to non-parallel edge pairs. Verifying edge pairs is the motivation for parallel probes, discussed below.

5.1.2. Parallel Probes

Parallel probes are a set of half-plane probes defined by lines of identical slope and direction. A complete collection of parallel probes of a given slope θ results in a cumulative area histogram $C(P,\theta)$ of the area of the object. The derivative of C at any point gives the value of the x–ray probe defined by the probing line. The complete derivative of C gives the result of a parallel x–ray aggregate probe (section 6.1) as shown in Figure 5.2. By analogy with x–ray probes, they provide a mechanism for verifying edge pairs.

Theorem 5.3: Four parallel half–plane probes through an edge pair are sufficient to verify the edge pair.

Proof: Without loss of generality, let us consider four parallel half–plane probes defined perpendicular to the line y=0. Label these X_1, X_2, X_3, X_4 in order of increasing x coordinate where p_i is the x–intercept of the line containing each probe. Let the area to the left of each probe be defined as $h(X_i)$. We can then define three points which we assert are collinear if and only if X_1, X_2, X_3, X_4 are all incident on the same edge pair:

$$M_1 = \left(\frac{p_1+p_2}{2}, \frac{h(X_2)-h(X_1)}{p_2-p_1}\right)$$

$$M_2 = \left(\frac{p_2+p_3}{2}, \frac{h(X_3)-h(X_2)}{p_3-p_2}\right)$$

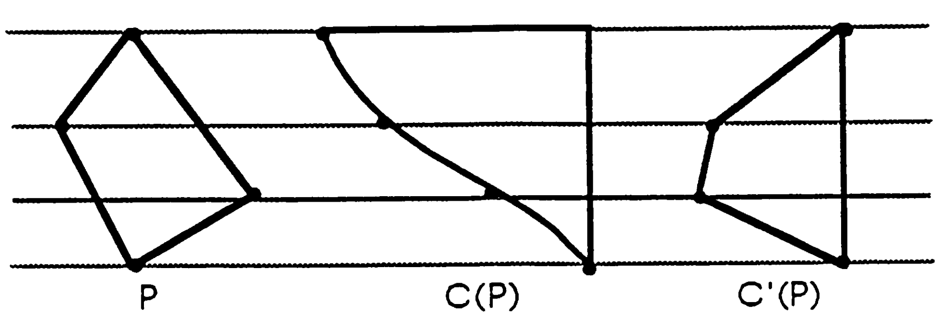


Figure 5.2: A polygon with $C(P,\theta)$ and its derivative.

$$M_3 = \left(\frac{p_3+p_4}{2}, \frac{h(X_4)-h(X_3)}{p_4-p_3}\right)$$

First, we show that if X_1, X_2, X_3, X_4 all intersect the same edge pair, then M_1, M_2, and M_3 are collinear. By considering the derivative histogram of P, without loss of generality we can treat one of the edges of the pair as lying on the x-axis, and the probes as perpendicular to this edge, so each probe can be described by its x-coordinate. Let $L(x)=mx+b$ contain the other edge of the pair. By the trapezoidal rule,

$$h(X_2)-h(X_1) = \left(\frac{L(p_2)+L(p_1)}{2}\right)(p_2-p_1) .$$

Using the above and the definition of M_1, we get

$$M_1 = \left(\frac{p_1+p_2}{2}, \frac{m(p_1+p_2)}{2}+b\right) .$$

This implies $M_1 \in L$. Since a similar proof can be given for M_2 and M_3, the three points are collinear if the probes intersect the same edge pair.

Now we show that if the four probes do not intersect the same edge pair, then M_1, M_2, and M_3 cannot be collinear. Considering the derivative histogram again, we note that the values of x-ray probes through P represent points on a concave curve. We observe that the midpoints M_i between two such probes must lie on or below this concave curve c. As Figure 5.3 shows M_2 must be below the line through the intersections between c and X_2, X_3 in order maintain concavity. But then, p_3-p_2 times the y-coordinate of M_2 is the area of the trapezoid bounded by $y=0$, X_2, X_3 and the line through M_1 and M_3. This is a contradiction since this trapezoid is properly contained between the x-axis and curve c. □

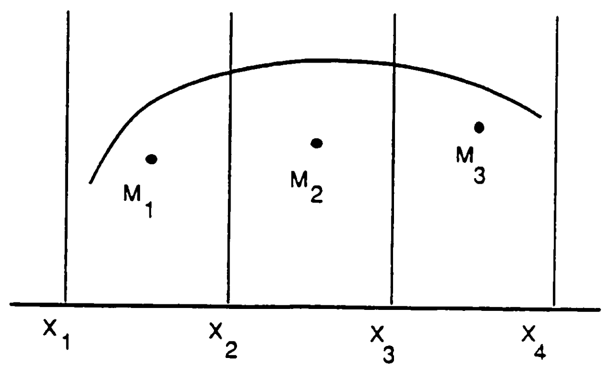


Figure 5.3: Non–edge pairs violating concavity or collinearity.

5.1.3. Determining a Boundary Point

Since we know how to verify and determine edge pairs, we can now proceed to locate a point on the boundary of P.

First, we must identify a section of P through which we can parallel probe. We assume knowledge of a point O within P; to parallel probe we must find another such point to insure all our probes intersect P. We start by sending two horizontal probes through O, one directed to the left and one directed right. Together, they total the area A of P. By using this information, we can now consider each additional probe as returning the area on both sides of the probe.

Three additional probes will be sufficient to identify a section to probe. Assume that the area A_0 above y=0 is concentrated in a square centered on the y–axis and resting on the x–axis. At least one of the probes $x = \pm\sqrt{A_0}/2$ or $y=\sqrt{A_0}$ must intersect P and with either x=0 or y=0 determines a section to parallel probe.

Four parallel probes on an edge pair is sufficient for verification. By the pigeonhole principle, after $3(n-2)+4=3n-2$ parallel probes we are guaranteed to have them, since there are a total of at most $n-1$ such edge pairs. At least one of the preliminary probes can also be used as a parallel probe, meaning at most $3n-3$ additional probes are needed.

5.1.3.1. Bounding the Extent of the Polygon

To determine the edges, we must origin probe the edge pair. To origin probe we must insure that our probes intersect within the section we have defined. This implies knowing a bound on where the edges actually are within the section, so that the angle of the probes can be selected accordingly.

By sending two probes g and h perpendicular to the parallel probing direction, we can partition the area of P into three sections, the area between the two probes α, above the probes β, and below the probes γ. Up to two of the areas α, β and γ may be 0, depending on whether g and h intersect P. By choosing g and h to lie on opposite sides of O, we insure that $\alpha>0$. If β and γ are 0, g and h represent bounds on the edges of P. If not, we are interested in the convex object which intersects g and h and maximizes its height subject to the area constraints.

Lemma 5.4: There exists a triangle with base on h with area β above g and α between g and h.

Proof: Let the distance between g and h be 1. If η is the height of the triangle, then the part above g is a similar triangle with height $\eta-1$. Since the two triangles are similar, the ratio of their areas is

$$\frac{(\eta-1)^2}{\eta^2} = \frac{\beta}{\alpha+\beta}$$

If we choose $\eta = \dfrac{\alpha+\beta+\sqrt{(\alpha+\beta)\beta}}{\alpha}$ and the base of the triangle equal to $2(\alpha+\beta)-2\sqrt{(\alpha+\beta)\beta}$ then the triangle fulfills the two conditions. □

Lemma 5.5: If there exists a convex figure different from a triangle with areas α and β as in Lemma 5.4, then there is a triangle that is higher and also fulfills the requirements.

Proof: Notice that this triangle has height η defined above, which implies the height of any other convex figure is less than η, if the assertion is true.

Let t be the topmost point of the figure. Construct a triangle Δ such that (a) its base lies in h and t is its topmost point, (b) it is similar to a triangle of height η and area $\alpha+\beta$, and (c) the part of Δ above line g is contained in the part of the convex figure above g.

Triangle Δ exists since we can choose the sides such that they intersect interval (a,b) (see Figure 5.4) which is the intersection between g and the convex figure. If the two sides go

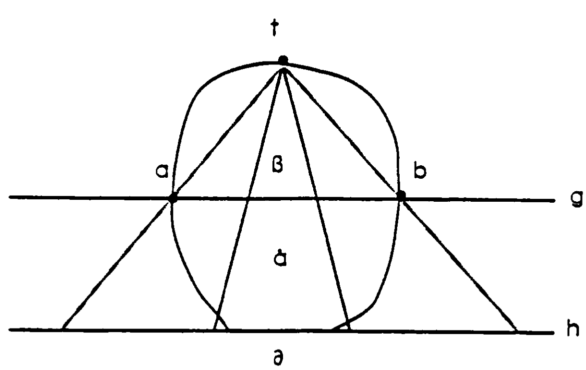


Figure 5.4: Constructing a similar triangle Δ to the height bound.

through a and b then its upper part is contained in the figure's upper part, but then its lower part contains the figure's lower part. Thus this triangle contains another triangle with the right proportions, that is, the intersection between this new triangle and g must be within the interval (a,b).

The upper part of Δ is smaller than β which shows that Δ has to be increased in order to realize total area $\alpha+\beta$. Thus η is greater than the height of the figure which is the same of the height of Δ. □

By similarly considering the areas α and γ, the other side of P can be bounded and the origin probes aimed.

5.1.3.2. Parallel Edges

For x–ray probes, parallel edges proved to be a tremendous nuisance, since the intercepts were undefined and could not be determined by additional probes through the edge pair, since the length of intersection is a function only of the separation between the edge pair and the angle of the probe, see section 4.3.3. We now give a procedure for determining the intercepts of parallel edges, using half–plane probes.

In the case of parallel edges, from the degenerate $f(t)$ we can determine the slopes and with the probes defining the section, the distance between two parallel edges. To complete our knowledge, we must determine the intercepts. By performing a rotation on P so that the parallel edges of P are perpendicular to the x-axis, we can obtain the situation in Figure 5.5. Let p be a point known to be outside, determined via the techniques of the previous section. Both edges can be determined from d, the distance from p to l_1. We define d' to be distance

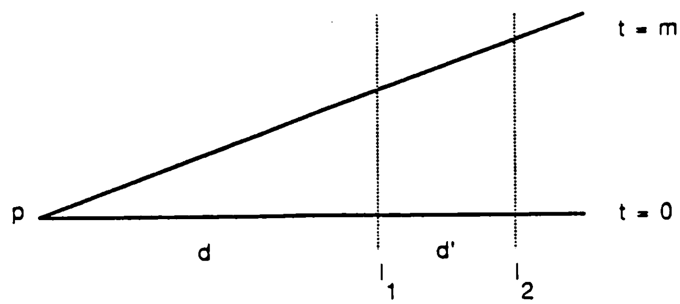


Figure 5.5: Determining parallel edges

between l_1 and l_2, which is determined by the parallel probes through l_1 and l_2. Aiming two probes with slopes 0 and m through p and l_1, l_2 determines the area of the trapezoid τ. This trapezoid represents the difference between two similar triangles so:

$$\tau = \frac{(d+d')(m(d+d'))}{2} - \frac{d(md)}{2}.$$

Solving for d gives

$$d = \frac{\tau}{md'} - \frac{d'}{2}.$$

Thus two additional probes are sufficient to determine the parallel edges. Ironically, this special case requires less probes than determining which constants belong to which lines. For non-parallel edges, the slopes (m_1, m_2) and possible intercepts $(b_1, -b_1, b_2, -b_2)$ define a total of 8 lines. We can determine which two are correct by probing along each of them. If p_1 and p_2 are the parallel probes which defined the section, the correct two lines l_1 and l_2 will result in probes of zero area and with p_1 and p_2 define a quadrilateral of exactly the observed area between p_1 and p_2. Thus at most eight additional probes will actually determine the edge pair.

Lemma 5.6: $3n+16$ probes are sufficient to determine the first edge pair and a point on the boundary of P.

Proof: From the previous discussions, we spend five probes initializing the search, $3n-3$ additional parallel probes defining a section, two probes bounding the height of P, four additional origin probes, and up to eight additional verification probes. Any point on l_1 or l_2 within the section is determined. □

5.1.4. Boundary Probes

Probes through known points on the boundary of P are defined as *boundary probes.* We can use boundary probes to develop a more efficient probing strategy through the following observation:

Lemma 5.7: Three parallel probes through an edge pair are sufficient to determine the second edge, if one edge is known to be contained in $l_1{:}mx+b$.

Proof: Rotate P clockwise by $arctan(m)$ so that the known edge lies on the x-axis. Three parallel probes through the rotated edge define points M_1 and M_2 described previously and subject to the inverse rotation define the other edge. □

5.1.5. Determining a Convex Polygon

After determining an edge pair, we have the situation in Figure 5.6. The edges contain known points p_1, p_2 and q_1, q_2. We conjecture the edges meet at v. To test this, we need a probe through q_1 and p_2. If it returns the area of triangle (q_1, p_2, v), we have verified vertex v, otherwise, there is at least one additional edge in the unexplored corner. Let v' be the

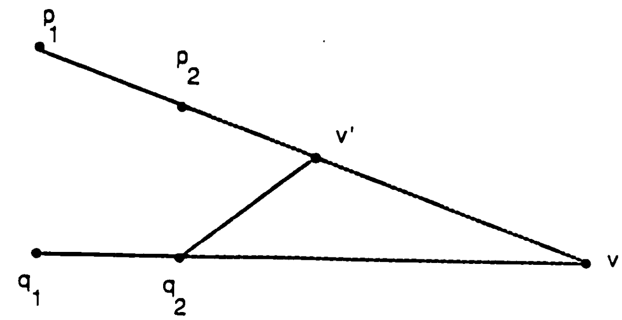


Figure 5.6: Determining vertices of P

point on l_1 such that the result of this probe equals $\alpha(q_1,p_2,v)-\alpha(q_2,v',v)$, where we define $\alpha(a,b,c)$ to be the area of the triangle defined by the three points.

Thus edge (p_1p_2) cannot extend past v' without violating convexity. Probing parallel to q_1v' between q_1 and q_2 we intersect a new edge pair, one of which is (q_1q_2).

We can parallel probe this section, and then consider these probes as boundary probes once we have determined an edge pair with (q_1q_2) as the known line. Aim the ith parallel probe between the $(i-1)$st and the $(i-2)$nd parallel probe. If it takes more than five parallel probes to verify an edge pair, we have identified another section to parallel probe.

We shall pivot around edge (q_1q_2), repeatedly determining the edge in the rightmost unexplored section. Since the initial probe to verify the vertex corresponds to one of the origin or parallel probes used previously, we only need five additional probes each to determine the rest of the edges. This brings us to our main result:

Theorem 5.8: $8n+6$ half-plane probes are sufficient to determine a convex n-gon.

Proof: By Lemma 5.6, $3n+16$ probes are sufficient to determine the first two edges. From the preceding discussion, 5 probes are sufficient to determine each additional edge. Thus the total number of probes required is $(3n+16)+5(n-2) = 8n+6$. □

For x–ray and finger probing models, a linear lower bound on the number of probes required can be based on the need for every vertex to be probed to verify that it exists. Since a half–plane probe measures the entire area on one side of a directed line, if we have determined the two edges incident on the suspected vertex we can verify a vertex by probing through the two known edges. If the area returned by the probe is the area A of the triangle defined by the vertex, edges, and probing line l, the vertex is verified since P is convex. This technique was used in the proof of Theorem 5.8.

A lower bound of $2n$ half–plane probes can be based on the same dimensionality argument used in the proof of Theorem 4.2. The next section contains a geometric argument which also proves a linear, although weaker lower bound for half–plane probing.

5.2. Bounds for Verification

A lower bound on the number of probes required to determine an object can be based on a comparison to the *verification* problem. Suppose we are given the representation of a polygon P, how many probes will be necessary to test whether P correctly describes a particular object. It is obvious that any lower bound to verification represents a lower bound to the determination problem, since it presupposes knowledge of the polygon.

Theorem 5.9: $n+1$ half–plane probes are sufficient to verify a convex n–gon.

Proof: For one of the edges of P, probe in both directions of the line containing the edge. For the remaining $n-1$ edges, probe once along the defining line. With each edge, we know P entirely lies within each of n half–planes. The intersection of these half–planes is P. Since the intersection of these half-planes has exactly the area of P, we have verified P. □

Note that fewer half-plane probes are sufficient for verification than for finger (Theorem 2.1) or for x-ray probes (Theorem 4.11).

Theorem 5.10: At least $2n/3$ half-plane probes are necessary to verify a convex n-gon.

Proof: We identify a collection of restrictions which a set of probes must meet for them to verify a given n-gon.

(1) If the relative interior of an edge is not intersected by the line bounding a half-plane, both vertices must be intersected.

(2) If a vertex is not intersected by a probe, then both of its incident edges must be intersected in their relative interiors.

(3) No two consecutive edges (a,b) and (b,c) can be verified without at least one probe within the relative interior of either (a,b) or (b,c).

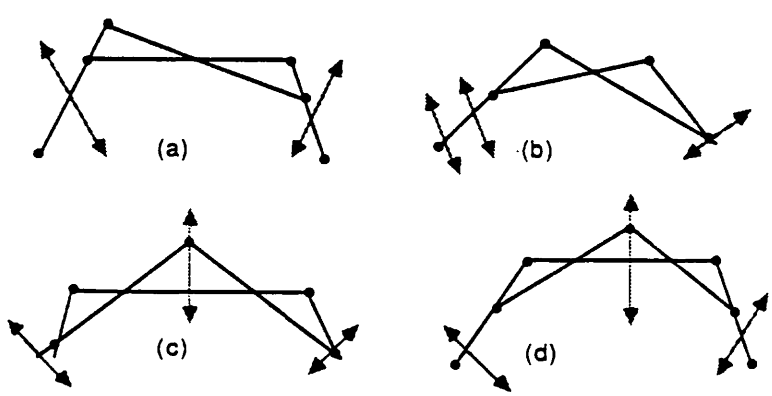


Figure 5.7: Forbidden cases for verifying probes.

(4) No two consecutive vertices a and b can be verified with single probes to the relative interiors of (x,a), (a,b), and (b,y) and no probes through a and b.

To show that (1) is forbidden, see Figure 5.7a. If not, we can move the two endpoints collinear to the two adjacent edges without changing the result of any probe. For (2), see Figure 5.7b. We can shorten the edge intersected by a probe and use an additional vertex to raise a triangle on the unprobed edge to regain the area. Figure 5.7c demonstrates the necessity of (3). We can replace vertex b with two other vertices without changing the result of any probe. Finally, for (4) see Figure 5.7d. We can replace the center edge and two incident vertices by a triangle without changing the result of any probe.

We now walk around the boundary of the polygon and count the minimum number of sites which must be intersected to satisfy the four restrictions. Suppose no vertices are intersected. By restriction (2), the relative interior of each edge must be intersected at least once, and by restriction (4) every third edge must be intersected at least twice. Thus there are at least $4n/3$ intersections, which requires at least $2n/3$ probes to verify.

Now suppose there are v vertices probed, no two of which are consecutive. By restriction (1) both adjacent edges must be probed in their relative interiors. Any edge that is not adjacent to a probed vertex is the middle edge of a chain of three edges to which restriction (4) applies. Thus there are at least

$$v + n + \frac{n-2v}{3} \geq \frac{4n}{3}$$

intersections, where the $(n-2v)/3$ term comes from the fact that at least one third of a consecutive chain of edges not adjacent to any probed vertex must be probed at least twice.

Finally, consider the general case and decompose the boundary of P into maximal chains of edges so that either no edge is adjacent to any probed vertex or it is a chain that remains after removing all chains of the first kind. Let n_1 (n_2) be the number of edges that belong to chains of the first (second) kind and c_1 (c_2) be the number of chains of the first (second) kind. By restrictions (1) and (3) at least $\lfloor \frac{k_i+2}{2} \rfloor$ of the k_i edges of the ith chain of the second kind must be intersected in their relative interiors. Thus, the number of intersections is at least the number of probed vertices $n_2 - c_2$, plus the number of edges of the first kind n_1, plus the number of edges of the second kind $\sum_{i=1}^{c_2} \lfloor \frac{k_i+2}{2} \rfloor$, plus $n_1/3$ to satisfy (4) for all edges of the first kind. The total

$$\frac{4n_1}{3} - n_2 - c_2 + \sum_{i=1}^{c_2} \lfloor \frac{k_i+2}{2} \rfloor$$

can be simplified by pulling c_2 out of the summation and using $n = n_1 + n_2$ to

$$n + \frac{n_1}{3} + \sum_{i=1}^{c_2} \lfloor \frac{k_i}{2} \rfloor .$$

The remaining summation is at least $n_2/3$, being smallest when each chain has exactly 3 edges, which gives the result. □

This result can probably be improved by identifying more restricted situations. However, we note that $3n/4$ probes are sufficient for each vertex and the relative interior of each edge to be probed as follows. Use $n/2$ probes along every other edge to intersect all n vertices and $n/2$ of the edges and $n/4$ additional probes, each of which intersects the interior of two unprobed edges. Thus a tight n lower bound for verification will not follow from this argument.

5.3. Higher Dimensions

To a limited extent, these results can be extended to higher dimensions. If we assume a half-plane probe as described, we can consider collections of half-planes within a common plane to determine a particular cross-section of P. By combining these cross-sections as in the proofs of Theorems 3.9 and 4.12 a polytope in E^3 can be reconstructed in a linear number of probes. However, two more interesting generalizations prove to be much more dificult to analyze.

Consider a *cross-sectional area* probe which, for a given plane in E^3 returns the area of intersection with P. This differs from the half-plane probe described above in that the line which defines the appropriate half-plane is at ∞. The results in this chapter do not appear to help with cross-sectional area probes, since they rely on isolating a section of P containing only two facets, which is not possible with a full cross-section of P. Even the problem of determining a tetrahedron in a constant number of probes is open and appears dificult.

The other interesting generalization would be to half-space probes in E^3, which for a specified half-space returns the volume of intersection. The simpler problem of determining tetrahedra is also open for half-space probes.

5.4. Extended Gaussian Images

The *extended Gaussian image* (EGI) of a convex polytope in three dimensions represents each facet of P as a normal vector proportional in length to its area. In 1897, Minkowski [60] proved that every convex polytope is uniquely determined (independent of translation) by its extended Gaussian image. Further, an extended Gaussian image is realizable by a convex polytope if and only if the sum of its vectors is zero. An English version of this analytic proof appears in Lyusternik [58].

Gaussian images have been used in robotics [47] as models of objects, since the areas and normals can be calculated from images of objects. Unfortunately, there is no known algorithm to invert a Gaussian image. Little [57] provides an iterative algorithm based on Minkowski's proof which eventually converges on the correct polytope. One dificulty with inversion in E^3 is that the incidence graph of the faces is not uniquely determined by the normals as they are in E^2. Dolan and Weiss [16] claim an $O(n \log n)$ algorithm for determining the incidence graph based on a notion of a weighted Voronoi diagram on the sphere. Unfortunately, the problem of inverting an EGI given the incidence graph is still open.

Minkowski's theorem has an interesting implication for the verification of convex polytopes in E^3 with cross-sectional area probes. $f_2(P)$ such probes are sufficient to verify a convex polytope P with the special property that no plane h intersects P only in vertices and edges unless h supports a facet. These probes determine an extended Gaussian image, which inverted verifies P, the cell in the arrangement of planes containing O. The restriction on P is necessary, since a pyramid might be raised on one or more of the faces, which would also result in an extended Gaussian image which vector sums to 0.

A similar argument can be given in E^2 for verifying convex polygons with x-rays, although a triangle raised on any edge would be undetected. This does provide an upper bound of n x-ray probes for verification when n is known, since the EGI can be inverted but no additional triangles raised without increasing n.

5.5. Conclusions and Open Problems

We have given strategies for probing with half-planes. In particular, we have shown that complete information about a convex n-gon can be obtained with a linear number of carefully planned half-plane probes. Open problems include:

(5.1) Tighten determination and verification bounds for half-planes.

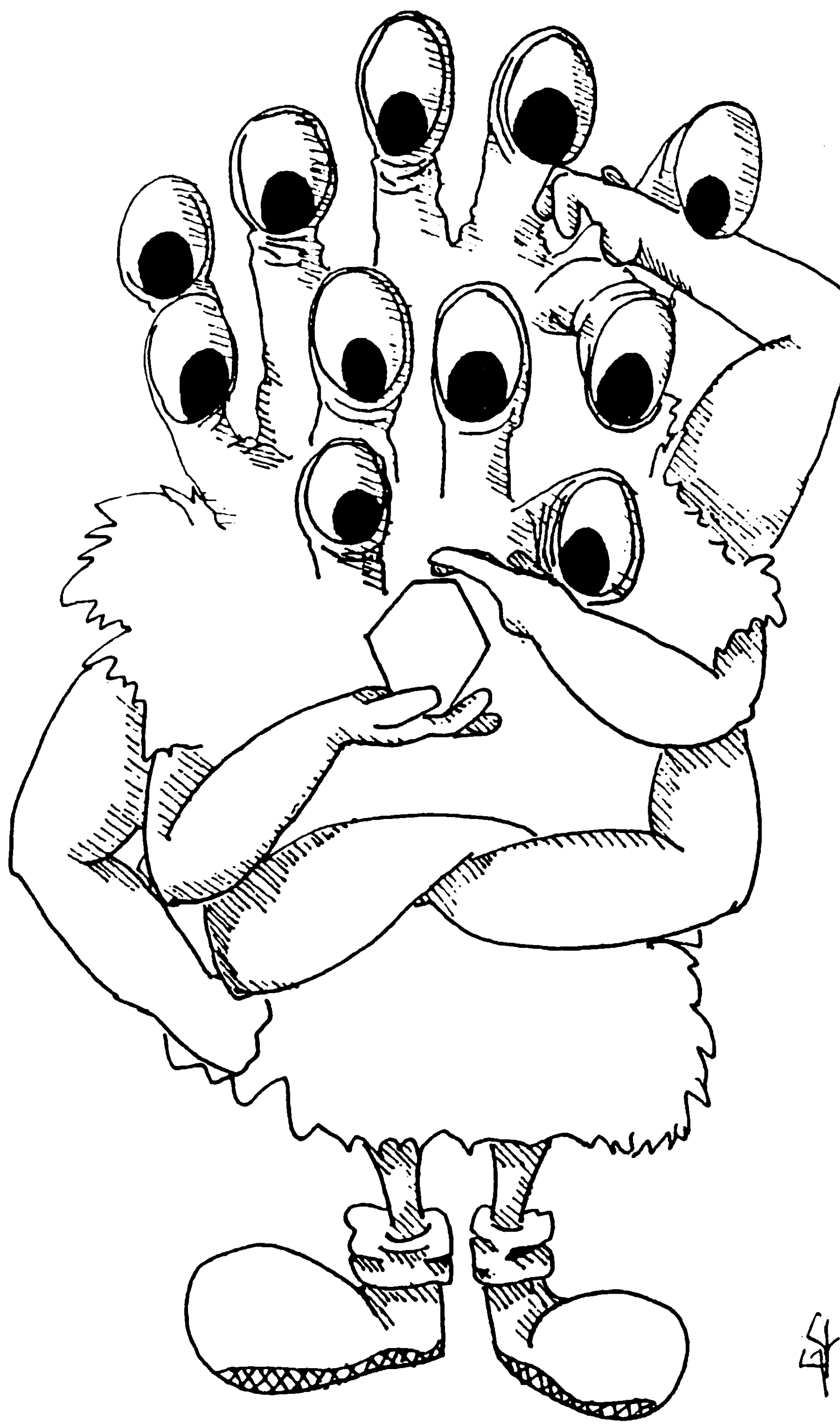
(5.2) Is there a finite strategy for reconstructing convex polyhedra from half-space or cross-sectional area probes?

(5.3) Give an algorithm for reconstructing Gaussian images given the incidence graph of the polytope.

(5.4) Does there exist a finite probing strategy for reconstructing star-shaped polygons from half-plane probes?

CHAPTER 6.

AGGREGATE PROBES



It has been said that there is safety in numbers. This is true with respect to probing, for entirely new problems arise when more than one probe can be made at a time. This chapter is particularly interesting because these problems touch upon a wide range of subjects in mathematics and computer science. X–ray aggregate probes accurately model sensing devices used in medical imaging, and the study of them leads us to an important problem in integral geometry. This in turn will lead us to an interesting problem in combinatorial geometry. The problem of probing in rounds introduces the notion of parallelism and demonstrates its limitations.

Section 6.1 presents results for x–ray aggregate probes. These include solving Hammer's x–ray problem for both parallel and origin probes. Difficulties in generalizing Hammer's problem to non–convex polygons leads to the combinatorial problem of k–projections, which is analyzed in section 6.2. Attempts to aggregatize finger and other probes are documented in section 6.3. If we permit more than one probe to be made at a time, we can obtain some speedup on the number of rounds of probes required to determine an object. These problems are treated in section 6.4. Section 6.5 concludes as usual with some open problems.

6.1. Hammer's X–ray Problem

P. C. Hammer [39] posed the following problems in 1963: How many x–ray pictures must be taken to permit exact reconstruction of a convex body if the x–rays issue from a finite point source? How many are needed if the x–rays are assumed to be parallel? These problems have since generated a substantial literature [23,24,26,27,30,77,92] which is based on integral geometry. The distinction between the two problems is exactly the distinction between origin and parallel probing [19] models as discussed in Chapter 4.

Here we survey the results of this literature, phrased in the language of probing. Giering [30] proved that three ***photograph probes*** are sufficient to verify any convex set, where a photograph probe consists of the set of all x–ray probes parallel to a given direction. Gardner [26] shows that three photograph probes are sufficient to determine and necessary to verify a convex set. Gardner and McMullen [27] showed that any four photograph probes are sufficient to determine a convex set, so long as their directions are not a subset of the directions of diagonals of a regular polygon. There also exist a body of results for *point sources*, the complete set of x–ray probes originating from a point O. This is a more powerful probe than the complete set of x–ray probes passing through O (an ***origin probe***), since the second probe would be unable to distinguish between a convex set K and the same set rotated π radians around O. Falconer [24] proved that two point sources p_1 and p_2, which lie on a line through the interior of K are sufficient to reconstruct K. Volčič [92] proves that three non–collinear point sources are sufficient for determination provided all points are outside K. Also, four points, no three of which are collinear, are sufficient to determine K. Except for Falconer's, these results only demonstrate the uniqueness of K and are thus non–constructive.

These theorems have been derived for convex sets, not the more restricted set of convex polygons. In this thesis, we are only concerned with polygons. However, since a convex set can be approximated arbitrarily closely by a convex polygon, it is not clear just how much weaker our results are. In this section, we present discrete and therefore simpler proofs for many of the results above. First, we prove a lower bound on the number of photograph probes for determination.

Theorem 6.1: Two photograph probes are not sufficient to determine a convex polygon.

Proof: We will use an adversary argument to show that there will be at least two convex polygons P which satisfy the results of two photograph probes, regardless of how they are selected. Let the first probe return the image of a trapezoid τ_1 of height h_1, top t_1, and base $b_1 > t_1$, which is symmetric along an axis l perpendicular to b_1. The second probe, aimed at an angle θ with respect to l, returns a similar trapezoid τ_2 of height h_2, top t_2, and base $b_2 > t_2$.

Figure 6.1 shows the construction of two quadrilaterals which both give rise to images τ_1 and τ_2. To prove this, we show that $h_1 = \overline{OM} = \overline{LN}$. Clearly, triangle ΔQLM is similar to triangle ΔQRS. Thus $a = \overline{LN}(\frac{x-a}{x-2a})$ or $\overline{LN} = \frac{a(x-2a)}{(x-a)}$. Also, note ΔTUV is similar to ΔTON, so $a = \overline{ON}(\frac{x-a}{x-2a})$ or $\overline{ON} = \frac{a(x-2a)}{(x-a)}$. Thus we have

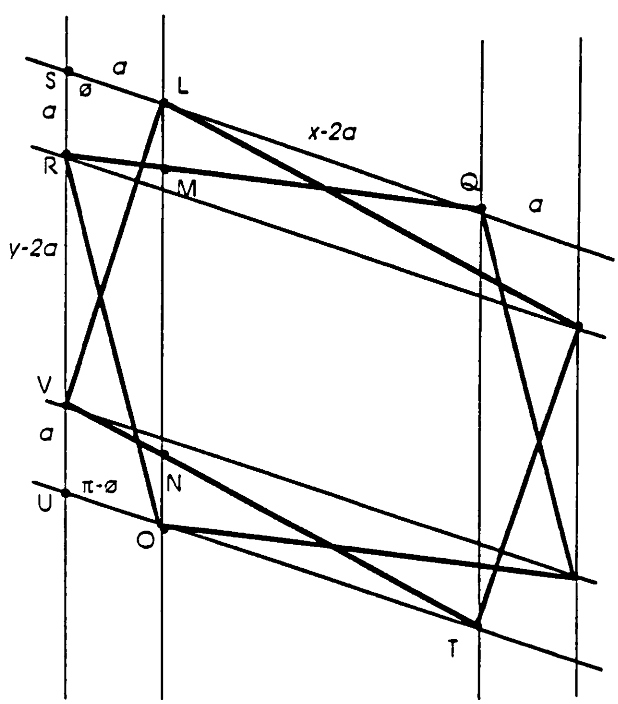


Figure 6.1: Two photograph probes are not sufficient for determination.

$h_1 = y - a(x - 2x)/(x - a) = |\overline{MO}| = |\overline{LN}|$. Note that $b_1 = x\sin\theta$. Repeating this argument for r_2 shows $h_2 = x - a(y - 2a)/(y - a)$ and $b_2 = y\sin\theta$.

Given values for b_1, h_1, and $a\sin\theta$ as returned from the first probe, for any value of θ an x and y can be selected to complete this construction. Thus two photograph probes do not suffice to determine P. □

Edelsbrunner and Skiena [19] show that three photograph probes are sufficient to determine a convex polygon. We show here that three photograph probes can reconstruct a convex polygon in $O(n)$ time, an improvement over our previous quadratic result.

Theorem 6.2: A convex n-gon can be determined in $O(n)$ time using 3 photograph probes.

Proof: Consider two orthogonal probes. From the first, we will obtain the complete set of x-coordinates of vertices in P. From these we can determine $x_{\min}$, the smallest distance between distinct x-coordinates. Note that up to two vertices may lie on any line of the form $x = c$. From the second probe we will determine the complete set of y-coordinates of vertices and thus $y_{\max}$, an upper bound on the length of intersection between any line $y = c$ and P.

Aiming the third probe with angle α, $\pi/2 - arctan(x_{\min}/y_{\max}) < \alpha < \pi/2$ will insure that no two vertices will be incident upon the same histogram line. This is steep enough so that no line with angle α will contain two of the old intersections. A linear sweep through the histogram vertices from the first and third probe will permit the intersections to be computed in $O(n)$ time. □

For the probing models discussed earlier in this thesis, the ability to interactively select probing directions is what made finite probing strategies possible. Aggregate probes are powerful enough to wave this restriction. However, as Gardner [26] showed, problems arise when the probing directions are chosen in the directions of a convex n-gon. Let P be a $2n$-gon with equal sized isoscelese-triangular "bites" taken out of every other corner. Two distinct orientations of P exist where each direction is perpendicular to the base of a cut, and as Figure 6.2 shows for both of these every probe image is identical.

Origin probes present a different set of problems, which to solve we will need some earlier results from Chapter 4. Recall Theorem 4.4 in section 4.3.1, where we showed that the complete set of x-ray probes through O determines a "spider-web" $S_O(P)$, which can be inverted such that any non-parallel edge pair of P is determined up to rotation by π. It is this ability to determine edge pairs which makes origin probes more powerful than photograph probes.

Theorem 6.3: Two origin probes are necessary and sufficient to determine a convex polygon.

Proof: Let O_1 be the origin of our first origin probe. From the previous discussion, it is clear that if the resulting spider web indicates no parallel edges (ie. a degenerate linear segment of $S_{O_1}(P)$), that P is determined up to rotation and that a second origin probe can

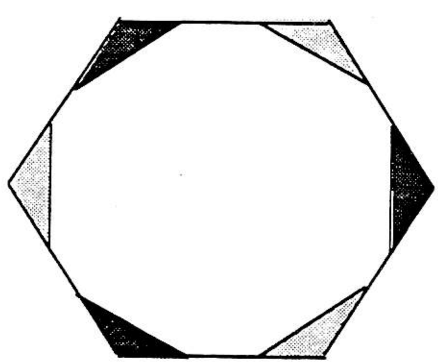


Figure 6.2: Directions from regular polygons do not suffice for determination.

easily be selected to distinguish between the possibilities P and P'.

Further, P is determined up to central reflection, or equivalently rotation through π, if there is even one non-parallel edge pair in $S_{O_1}(P)$ since this pair of edges is determined. The neighboring parallel edge pair has their slope m and the distance between them determined by the inversion formula. A vertex of this edge is determined from the known edge pair and with m gives the next edge. Walking around in this way determines P.

The only cases remaining are illustrated in Figure 6.3. Either O is within P or it is outside P. If O is within P, all edge pairs are parallel which implies that P is centrally symmetric with center O. However, for such a polygon $P = P'$ and the polygon is determined. In the second case, O is outside P and P may be located anywhere in the angle sectors defined by P. By convexity, there is at most one parallel edge pair of P, and the other edges are defined by directions through O. With the slopes of all four edges known, a second point can be selected to yield a non-parallel edge–pair and determine P. □

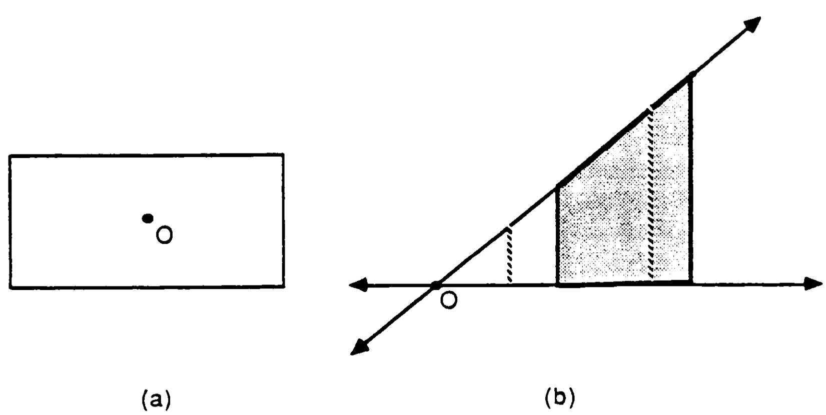


Figure 6.3: The two cases with all parallel edge pairs.

Aggregate probes make possible the determination of a larger class of polygons than for single probes, since aspects of the entire polygon are recorded in each probe. However, one important property of photograph probes for convex polygons does not hold for star-shaped polygons, namely that each vertex of P lies on a line determined by each probe. Figure 6.4 shows how vertices in star-shaped polygons can be *invisible* to photograph probes.

This problem of invisible vertices leads to the combinatorial problem of k-projections, discussed in the next section. The hope is that further study of k-projections will provide insight into how many invisible vertices can possibly remain after m probes.

6.2. Counting the Number of k-Projections in a Point Set

An *orthogonal projection* of a point set onto a line l maps each point to a point on l such that the original and projected points define a line perpendicular to l. Since points that lie on a common line perpendicular to l get mapped to the same point, we can consider the number of points $k \leq n$ in a particular projection. A *k-projection* is an orthogonal projection that yields at most k point images. In this section, we bound the number of distinct k-

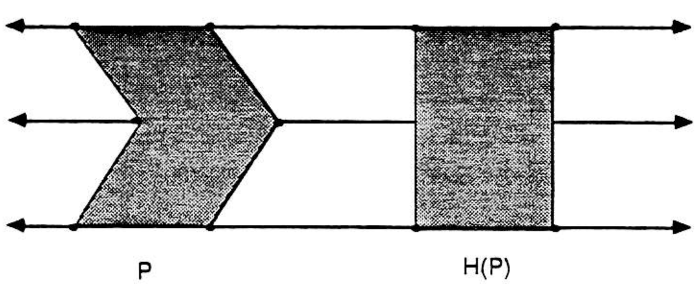


Figure 6.4: Invisible vertices in star-shaped polygons.

projections of a point set in the plane. These results first appeared in Skiena [87]. Note that we are interested in distinct projections. Since all parallel lines define identical projections, we can assume that l contains the origin.

These results provide insight into the structure of degenerate point sets, where more than two points define the same *direction* or line. A potential application is in image processing, to count the possible sizes of a set of indistinguishable objects given a number of views of the set. For example, given a number of simultaneous views of a flock of birds, how many birds can there be in the flock?

Each direction of a point set determines a k-projection for some $k < n$. A related problem, that of minimizing the total number of directions in a point set, was solved at $2\lfloor n/2 \rfloor$ by Unger [91].

We can define a function $v_k(n)$ which specifies the largest number of k-projections in any configuration of n points N. The following observations concern special cases of $v_k(n)$

$$v_k(k-i) = \infty,\ i \geq 0 \tag{a}$$

$$v_k(k+1) = \binom{k+1}{2} \tag{b}$$

$$v_k(k^2+i) = 1,\ i \geq 1 \tag{c}$$

$$v_k(k^2-i) = 2,\ 0 \leq i < \lfloor k^2/4 \rfloor . \tag{d}$$

The first statement is evident since every line defines a k-projection on a set of $n \leq k$ points and the second since every direction in a set of $k+1$ points in general position is a k-projection. Observation (c) follows from the total of k^2 intersections on a $k \times k$ grid which limits the size of any point set generating two distinct k-projections to k^2. For (d), note that the first two k-projections define a grid of k^2 points. The third direction defining the minimum number of lines, $2k-1$, must be a diagonal of the grid. There are two lines containing each of from 1 to $k-1$ points and one other contains k points. Selecting the lines with the highest number of points leaves

$$i = 2\sum_{j=1}^{(k-1)/2} j = \frac{k^2-1}{4}$$

points from the grid if k is odd and

$$i = \frac{k}{2} + 2\sum_{j=1}^{(k-2)/2} j = \frac{k^2}{4}$$

if k is even. Considering any direction other than a diagonal leaves even more of the grid points uncovered. Therefore, at most $k^2 - \lfloor k^2/4 \rfloor$ points allow three different k-projections.

The function $v_k(n)$ observes the following monotonicities

$$v_k(n+1) \le v_k(n) \tag{e}$$

$$v_k(n) \le v_{k+1}(n+1)\,. \tag{f}$$

The first monotonicity follows from the deletion of any point in a configuration with $v_k(n+1)$ k-projections. To obtain the second, consider an arrangement of n points with $v_k(n)$ k-projections. Adding another point to the arrangement which is not on one of the at most $k \cdot v_k(n)$ projection lines defines a configuration with at least $v_k(n)$ $(k+1)$-projections. Together, they provide the following order on values of this two parameter function:

$$v_k(n+1) \le v_k(n) \le v_{k+1}(n+1) \le v_{k+1}(n)\,.$$

We now present a variety of upper and lower bound results for $v_k(n)$ which are tight over different values of k, $1 \le k \le n-1$. Our upper bound results on $v_k(n)$ rest on the maximum size of the collinearity graph representing the lines involved in all the k-projections of N. Its nodes are the points of N and it connects two points by an edge if the corresponding two points lie on a line parallel to a k-projection.

Theorem 6.4: For integers $a \geq 1$ and $k > b \geq 0$, we have

$$v_k(ak+b) \leq \frac{(ak+b)(ak+b-1)}{a(ak-k+2b)}.$$

Proof: When m vertices are collinear on a projection line, they account for $\binom{m}{2}$ edges in the collinearity graph. Distributing the points uniformly among the projection lines for each k-projection minimizes the total number of edges added to the graph from any projection. The most uniform distribution of $ak+b$ points, assuming $b<k$, puts $a+1$ points on each of b lines, and a points on the remaining $k-b$ lines. Since the total number of edges in collinearity graph cannot exceed $\binom{ak+b}{2}$, we have

$$v_k(ak+b)\,\left((k-b)\binom{a}{2}+b\binom{a+1}{2}\right) \leq \binom{ak+b}{2}$$

which leads to the result. □

Two special cases of Theorem 6.4 are when $n=k+b$ or $n=ak$. The first case is applicable when n is only slightly larger than k and is tight when $k=n-1$. The second is a generalization of observation (b) and applies when n is considerably larger than k. The resulting bounds are given below.

Corollaries: For integers $k<n$, we have

$$v_k(n) \leq \frac{n(n-1)}{2(n-k)}, \text{ and} \tag{1}$$

$$v_k(n) \leq \frac{n-1}{n/k-1}. \tag{2}$$

All of the above results are combinatorial rather than geometric, and thus the bounds do not fully reflect the relationships between points in the plane and lines incident upon them. The next two theorems give a tight bound over a large range of n and k.

Theorem 6.5: There is a positive constant c such that $v_k(n) \leq ck^2/n$ if $c_0 k \leq n \leq k^2/c_0$ where c_0 is a sufficiently large constant.

Proof: Let N be a set of n points that realizes $v_k(n)$ k-projections. Szemerédi and Trotter [89] prove that the number of incidences i between n points and t lines is bounded by $i \leq c_1 n^{2/3} t^{2/3}$ with the restriction that $\sqrt{n} \leq t \leq \binom{n}{2}$. The number of lines over all k-projections of N is $t \leq kv_k(n)$, with the inequality coming from projections of less than k lines. Each point in N is incident upon exactly $v_k(n)$ lines, so $i = n \cdot v_k(n)$. Thus:

$$v_k(n) \leq c_1 n^{-1/3} t^{2/3} .$$

Substituting $t \leq k \cdot v_k(n)$ and rearranging terms gives the result. The range of validity on this formula follows from the range where Szemerédi and Trotter's result is valid. □

Theorem 6.6: $v_k(n) = \Omega(k^2/n)$ if $ck \leq n \leq k^2/c$, where c is a sufficiently large constant.

Proof: We use the example of a $\sqrt{n} \times \sqrt{n}$ grid of lattice points G. We shall consider only the directions with slope $0 \leq x/y \leq 1$, where $x \leq y \leq \sqrt{n}$ and x and y are relatively prime. The complete set of such rational numbers are known as the *Farey sequence of order* $\sqrt{n}$ [63]. The Farey sequence of order $\sqrt{n}$, negated and inverted, accounts for all the directions in G.

The fractions in a Farey sequence with denominator d are exactly those $x < d$ where x is relatively prime to d. Thus the number of fractions in a Farey sequence of order m is $\sum_{i=1}^{m} \phi(i)$, where $\phi(i)$ is the Euler totient function, the number of positive integers less than or equal to i which are relatively prime to i. Hardy and Wright [41] prove that $\sum_{i=1}^{m} \phi(i) = 3m^2/\pi^2 + O(m \log m)$.

To determine the size k of the k-projection of G onto a line perpendicular to direction x/y, we note that each point image in the projection is represented by a point in the $y \times x$ lower corner of the grid, as shown in Figure 6.5. Counting the points in this L-shaped region gives $k = \sqrt{n}(x+y) - xy$.

Consider the directions within a $n^a \times n^a$ portion of G, $0 \le a \le 1/2$. By the previous analysis, this square defines the $\Theta(n^{2a})$ slopes of a Farey sequence of order a. For any of these directions, $x \le y \le n^a$, so the size of such a projection is $O(n^{(2a+1)/2})$, which gives the result. □

Our lower bound results over other ranges are by construction. The following is tight when $n = k+1$ and applicable when n is only slightly bigger than k.

Theorem 6.7: $v_k(n) \ge \binom{\lfloor n/(n-k) \rfloor}{2}$.

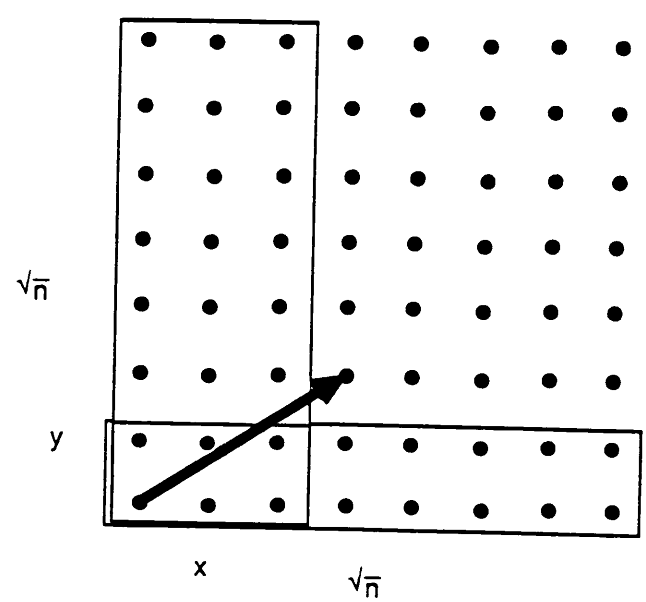


Figure 6.5: The size of the projection with slope x/y of G.

Proof: Arrange $\lfloor n/(n-k)\rfloor$ points so that no three are collinear or two pairs of points lie on parallel lines. Replicate this arrangement $n-k$ times, so that the orientation of the point set is unchanged and no points from different copies of the arrangement are collinear along a projection line. If $\lfloor n/(n-k)\rfloor < n/(n-k)$, then we choose the appropriate number of points of an additional copy so that n is the total number of points. Each direction in the original arrangement represents a projection of at most k since there are at least $n-k$ disjoint pairs of points that lie on lines parallel to the projection. □

The following lower bound applies to $v_k(n)$ for smaller k, but also represents an interesting special case.

Theorem 6.8: $2k-1 \geq v_k(2k) \geq k$.

Proof: The upper bound follows from Theorem 6.4. For the lower bound, consider a regular $2k$-gon. Any projection parallel to an edge is a k-projection, and each edge is parallel to exactly one other in the polygon. In passing we mention that if the vertices are labeled in order, the direction defined by vertices v_i and v_{i+2} represents a $(k+1)$-projection, so $v_{k+1}(2k) \geq 2k$.

An alternate construction nests two regular k-gons as shown in Figure 6.6. The inside k-gon vertices are at the intersections of the lines defined by the two neighbors of each vertex of the outside k-gon. Each of these k directions defines a k-projection with one line incident upon four points and two upon singleton points. □

We can also consider the number of k-projections for point sets subject to a restriction on the number of points which can be collinear. Let $v_k^a(n)$ be the maximum number of k-projections on n points with at most a points collinear, that is, on a common line.

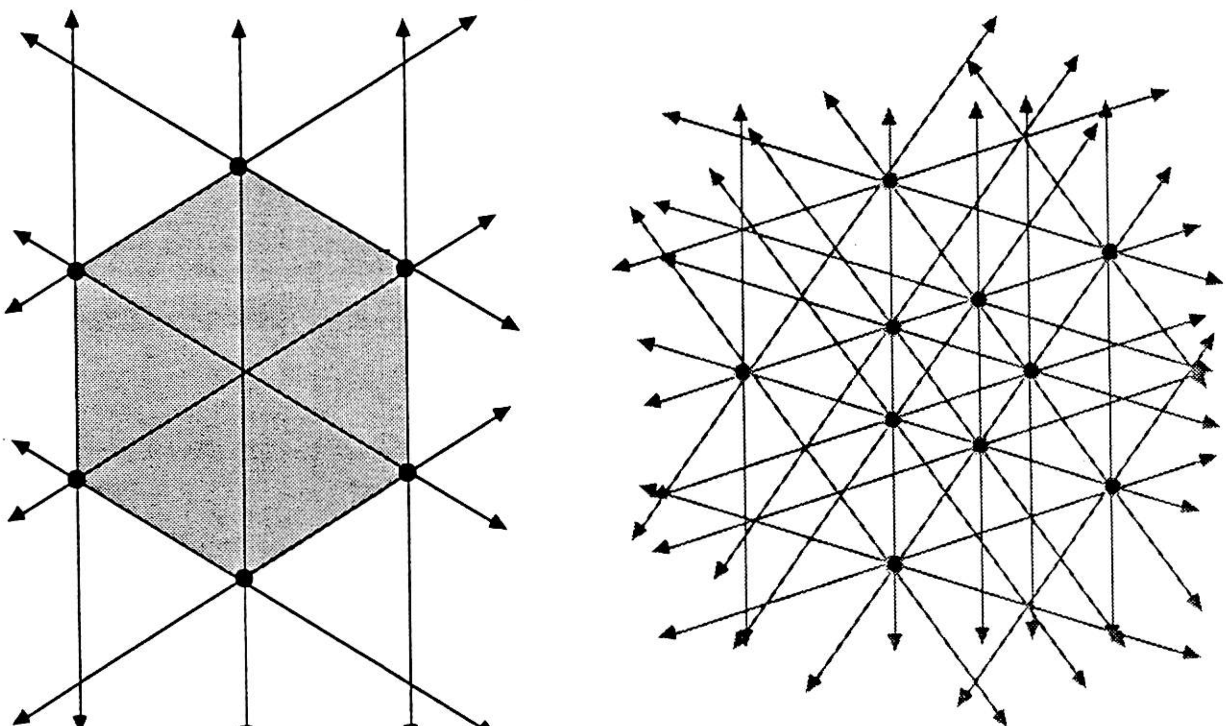


Figure 6.6: Point sets of size 2k maximizing the number of k-projections.

Theorem 6.9: $v_k^a(ak) \leq k$.

Proof: Note that a k-projection on ak points with at most a points collinear implies that every projection line contains exactly a points. For any point p in N, the possible k-projections are defined by the directions in N through p. Each direction through p partitions N into two subsets, both of which must contain a multiple of a points if the direction defines a k-projection. At most k of the up to $ak-1$ directions can, giving the result. □

We note that Theorems 6.8 and 6.9 together imply that $v_k^2(2k) = k$.

We have given several upper and lower bounds on the function $v_k(n)$. Figure 6.7 shows how $v_k(n)$ varies with increasing k. Further study can be expected to further improve and

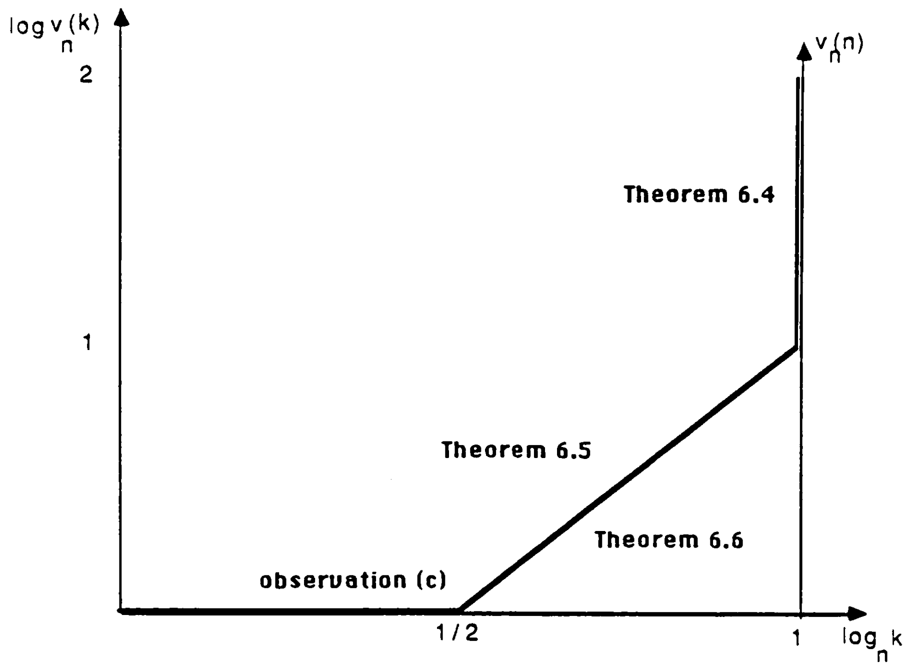


Figure 6.7: Upper and lower bounds on $v_k(n)$.

hopefully unify these results.

We note that several of our results also hold for point sets in E^3. Specifically, the upper bounds of Theorems 6.4 and 6.5 and the lower bound of Theorem 6.7 immediately generalize to three dimensions. An alternate problem in E^3 considers the number of planes parallel to a direction sufficient to contain all the points and generalizes $v_k(n)$ accordingly.

Another generalization of v, $V_k(n)$, maximizes the number of projections of exactly k point images. Clearly, $V_k(n) \leq v_k(n)$, but we conjecture equality. Finally, we can consider the sizes of central projections as we have considered the size of parallel projections.

6.3. Other Aggregate Probes

We can also consider aggregate probes based on probing models other than x–rays, such as finger, line, and half–plane probes. These problems are interesting, because in many cases we have seen them before in other guises.

For example, consider a parallel half–plane probe, which returns the cumulative area of P as a function of a half–plane sweeping from left to right over it. Due to the integral/differential relationship between x–rays and half–planes, each such probe is equivalent to a photograph probe and thus Theorems 6.1 and 6.2 hold for half–planes. On the other hand, no such relationship exists between x–ray and half–plane origin probes. As shown in Figure 6.8, all regular $2k$–gons of area A yield identical results when probed through their center, which does not happen with x–ray probes. This perhaps provides moral

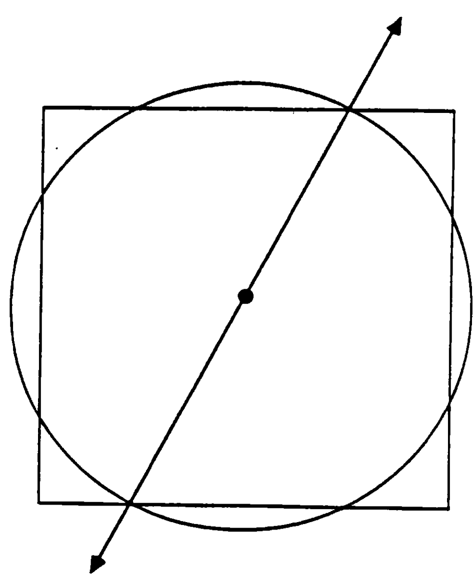


Figure 6.8: Regular 2k–gons of area A yield identical half–plane origin probes.

justification to those cranks who have spent thousands of years trying to square the circle, since success would give two figures with identical half–plane origin probes!

It proves more profitable to consider aggregate finger probes. Let a *parallel finger probe* be the complete set of finger probes with slope m and direction ($\pm\infty$) d. Clearly, two opposing parallel finger probes are sufficient to determine a convex polygon, since each probe determines the convex chain spanning exactly half of P. The problem gets more difficult when we consider star–shaped polygons.

Theorem 6.10: $\lfloor n/3 \rfloor$ parallel finger probes are necessary and $\lceil n/2 \rceil$ are sufficient to verify a star–shaped n–gon.

Proof: Figure 6.9 gives a star–shaped polygon that requires $\lfloor n/3 \rfloor$ parallel finger–probes to verify, since each of the cracks is small enough that no two finger probes with the same slope will verify vertices at the bottom of two different cracks.

The upper bound argument follows from probing perpendicular to the line defined by every other vertex of P and O, a point in the kernal of P. Since every point in P is visible from O, no probe destined for either of the two edges incident to the vertex can be obstructed. □

The set of polygons verifiable with a linear number of parallel finger probes cannot be extended too far past star–shaped polygons. A necessary condition is that every such polygon must be externally visible, for if not there exist points which cannot be reached by finger probes. However, consider the eight–sided polygon in Figure 6.10. The gap to the inner chamber is not wide enough to permit the verification of more than a small piece of an internal edge with one probe.

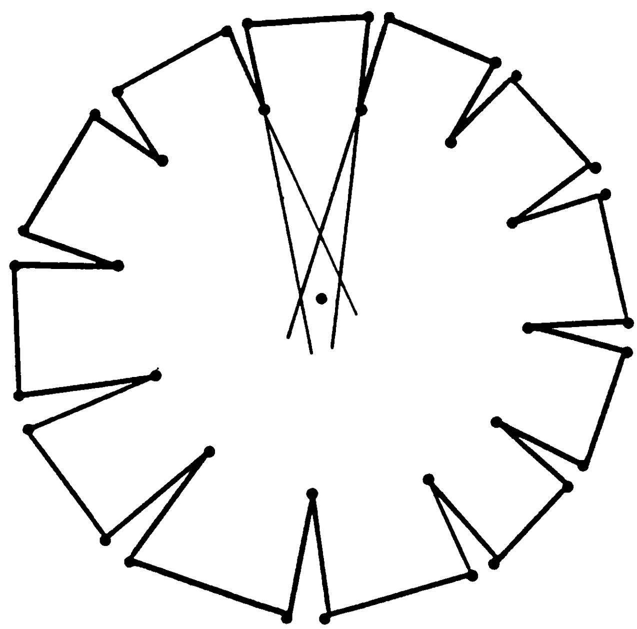


Figure 6.9: n/3 parallel finger probes are necessary to verify an n–gon.

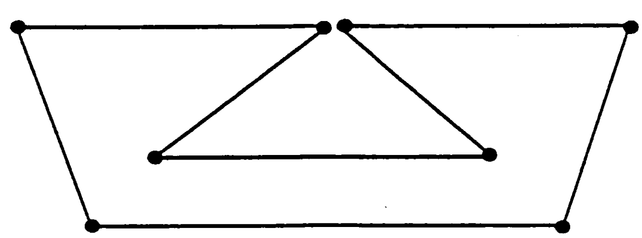


Figure 6.10: A polygon which cannot be verified by parallel finger probes.

$\lceil n/2 \rceil$ parallel finger probes appears sufficient to determine a star–shaped polygon, but the argument is complicated and will not be discussed.

Let an *origin finger probe* be the complete set of finger probes emanating from point p. A different class of problems result when p is inside the polygon. The contact point for each probe is a point visible from p, thus such problems have a close connection to "art gallery" problems, which seek the minimum number of point guards which can see every point in a polygonal art gallery. The monograph by O'Rourke [66] provides an excellent summary of what is known about art gallery problems, including open problems.

Aggregate finger probes dualize to aggregate line probes. Specifically, a parallel finger probe dualizes to a line probe which "rolls" around P after contacting it. Depending upon how you generalize the notion of aggregate finger probes to higher dimensions, this can represent a cross–section of P, which is itself the dual of silhouette probe, as discussed in section 3.3.

6.4. Probing in Rounds

The aggregate probing models in this chapter have introduced a form of parallelism, by considering all probes defined by a certain characteristic as a more powerful model. The main problem associated with parallelism is the degree of achievable time speedup for a problem of size n given k processors. Clearly, in the best case, the job can be completed k times faster. However, the structure of most problems makes it impossible to realize this. This is true in real life as well as theory, as anyone who has ever been on a committee can attest.

In this section we consider a problem, proposed by Raghavan [74]. To what extent can we speed up the number of "rounds" it takes to determine an object with the ability to make up to k probes per round. This is similar in flavor to the "sorting in rounds" problem [69] which has been extensively studied. We will limit ourselves to finger probes and convex polygons. We show that significant speedup can be obtained with $k=2$ probes per round, which is surprising since Theorem 3.5 showed that permitting two opposing finger probes per round reduces the number of rounds only by a small additive constant.

Theorem 6.11: $8n/3$ rounds of two finger probes per round are sufficient to determine a convex n-gon.

Proof: The goal of our strategy will be to ensure that the relative interior of each edge is probed at most three times. We will modify the strategy of Theorem 2.2 to account for our ability to make up to two probes per round.

For the first phase, send two probes per round directed to O until three outcomes are collinear. The m previous outcomes define m angle ranges around P. So long as we send the two probes to different angle ranges we are certain that no four contact points will be collinear.

Once we have determined our first edge, we can walk around P in clockwise order, conjecturing vertices from the intersection of two lines defined by four successive contact points. If we probe the first two conjectured vertices we encounter, we can be assured that these probes will not be incident on the relative interior of an edge with two other points. If there is only one possible conjectured vertex, we make only one probe that round, since an adversarial argument shows that making an arbitrary probe can be incident on a previously determined edge and does not move us closer to our goal.

At most $4n$ probes are made before the polygon is determined, $3n$ to edges of P and n for the vertices. We note that when only one probe is made, either this probe verifies a vertex or else defines two conjectured vertices for the next round. Thus any round of one probe is followed by at least one of two probes. When a vertex is verified in a one probe round, P is determined. Thus at least $3r/2$ probes are made per r rounds, which given at most $4n$ probes yields the result. □

It appears that this analysis is not tight and in fact fewer rounds are necessary for determination. Tighter bounds hinge upon a combinatorial analysis of how often we can be left with only one probing opportunity. That there exists a probing one–cycle of four rounds for this algorithm is illustrated in Figure 6.11. If this is the shortest one–cycle, $16n/7$ rounds

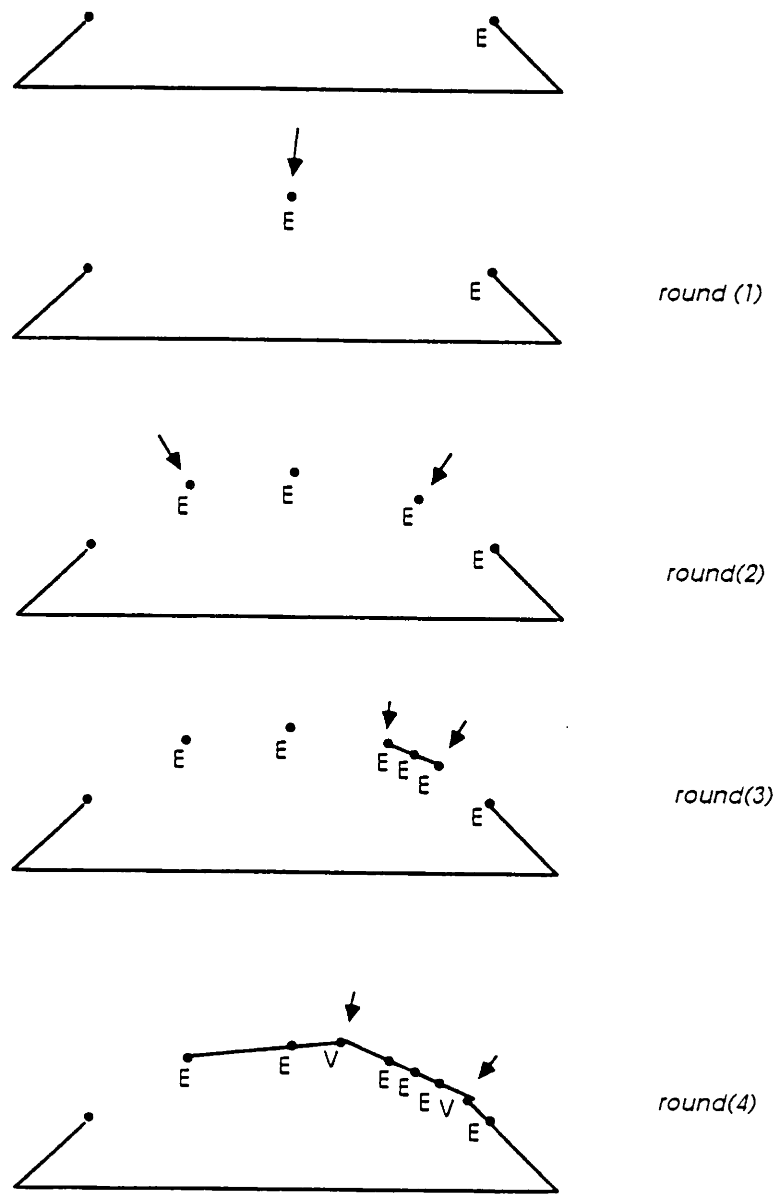


Figure 6.11: A probing one–cycle of length four.

are sufficient by the previous argument. However, this does not necessarily mean $16n/7$ is a lower bound, since alternate probing strategies might increase the length of the shortest cycle or ensure that less than three probes are incident on the interior of each edge.

This and the problem of generalizing to $k > 2$ are left as an open problems. We conjecture that $2n$ rounds might be achievable for some k. We note that a trivial lower bound for probing with rounds of $3n/k$ follows from Theorem 2.3.

6.5. Conclusions and Open Problems

In this chapter we have considered a variety of problems for aggregate probes. For almost none of these do the lower and upper bounds match, so they suggest topics for further work. The most interesting of these problems are stated below:

(6.1) How many x–ray photograph probes are necessary to determine a star–shaped n–gon? This generalizes Hammer's problems beyond convex polygons.

(6.2) Tighten the bounds on $v_k(2k)$. We have shown that $2k-1 \geq v_k(2k) \geq k$ and $v_{k+1}(2k) \geq 2k$.

(6.3) How many rounds of k finger probes per round are necessary to determine a convex n–gon? We have no interesting results beyond $k=2$.

We can also generalize probing strategies beyond homogeneous probes. Interesting problems result when we have access to more than one type of probing device. The following problem dates back to Greschak's [34] thesis.

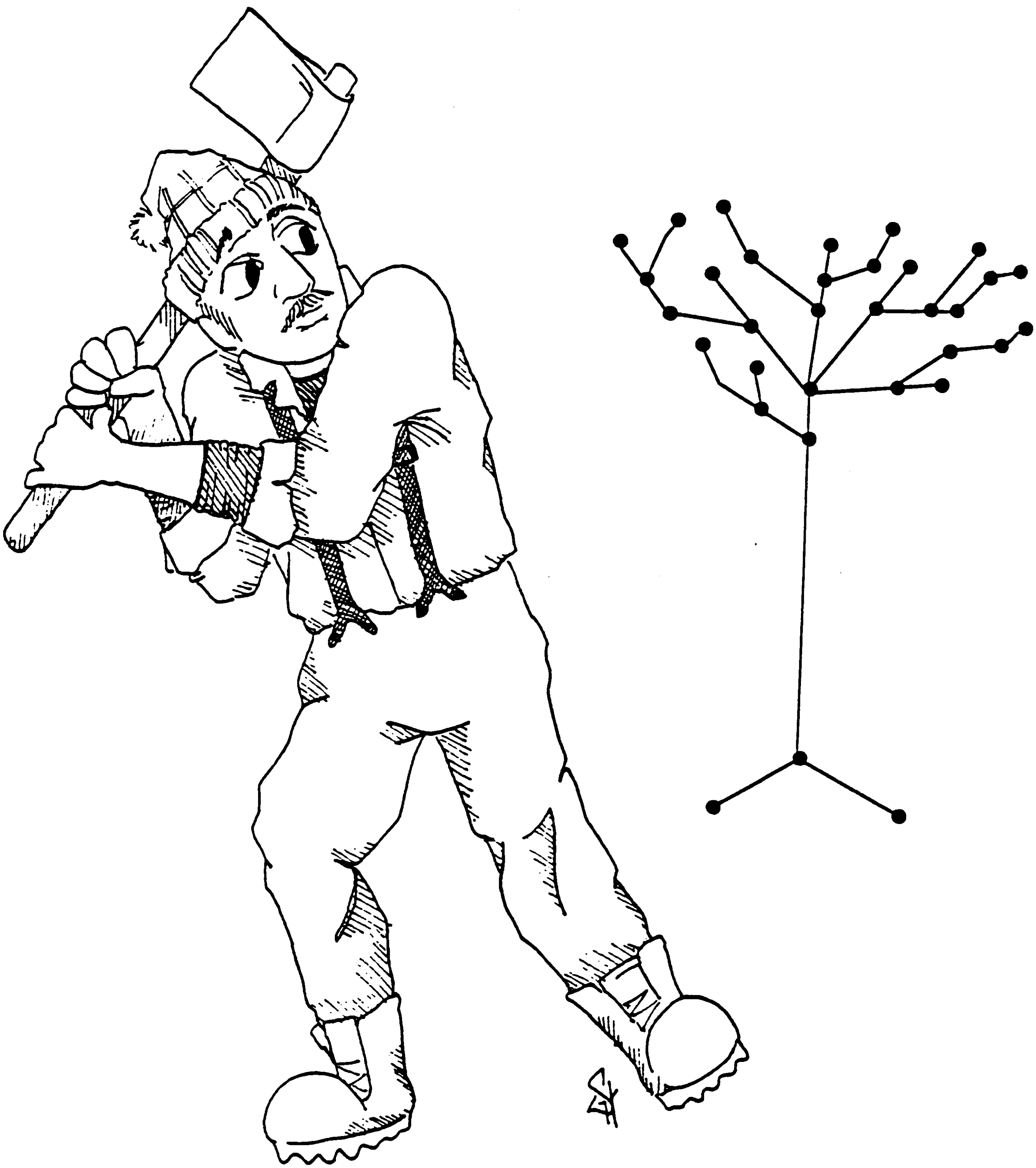
(6.4) A combination of how many finger and hyperplane probes are necessary to determine P?

We conjecture (based on no evidence) that the lower bound remains the same and that the $3n$ probes may be achieved by whatever combination of finger and hyperplane probes is desired.

(6.5) How many probes are required for determination given access to both finger and x–ray probes. Clearly $3n$ is a lower bound but does access to the x–ray probe help?

CHAPTER 7.

CUT–SET PROBES



The problems we have discussed thus far in this thesis have all involved probing purely geometric objects, usually convex polytopes. But the notion of probing leads to interesting problems even if we stretch the definition of geometric object. Whitehead claimed that graph theory represented "the slums of topology". This chapter is devoted to an interesting problem which might be considered to be "the slums of graph theory"!

Consider a graph $G=(V,E)$ whose n vertices are points in general position in the plane (that is, no three points are collinear) and whose edges are all straight line segments between pairs of vertices. We assume that the positions of the vertices are known, but nothing about the edges is specified. A ***cut-set probe*** returns the number of edges cut by a specified line. We show that all such graphs are completely reconstructible with $C(n,2)=\binom{n}{2}$ cut-set probes, prove that $C(n,2)$ probes are necessary, generalize the problem to arbitrary cut-sets, and propose several open problems concerning cut-set probes.

We note that it is not obvious that cut-set sizes should permit reconstruction. Complete knowledge of other graph parameters, such as degree sequences [9] do not have this property. A problem similar to ours involves the complexity of determining properties of graphs given queries of the form "is edge $\{i,j\}$ in the graph?". Karp has conjectured that monotone graph properties such as connectivity are ***evasive***, meaning that in the worst case all edges must be queried to determine whether the graph has this property. Lower bound results [51,78] show such properties are within a multiplicative constant of being evasive. Our results show that reconstructing graphs from cut-set probes is evasive in a similar sense. The problem of reconstructing graphs from cut-set sizes was posed by Dean [13].

Section 7.1 provides tight upper and lower bounds for determination of graphs with cut-set probes. Section 7.2 considers more specialized graph problems, the open ones of which are emphasised in section 7.3.

7.1. Reconstructing Graphs from Cut–set Sizes

If G is a straight line graph as specified, we note that only $C(n,2)$ probes provide potentially useful information, since there are only $C(n,2)$ partitions into two subsets which can be defined by lines. The set of all such probes are sufficient for reconstruction:

Theorem 7.1: $\binom{n}{2}$ cut–set probes are sufficient for edge reconstruction.

Proof: We show that the membership of edge $\{i,j\}$ in G can be tested from the four probes which uniquely partition the remaining $n-2$ vertices into those to the left and the right of the line defined by i and j. Let P_1, P_2, P_3 and P_4 be the probes defined in Figure 7.1.

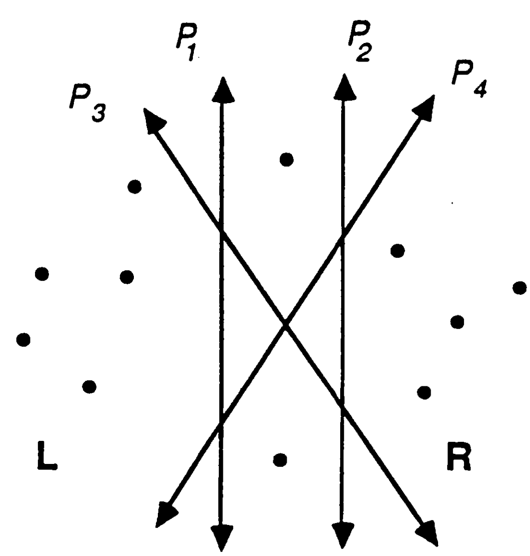


Figure 7.1: The probes necessary to determine whether $\{i,j\} \in E$.

Let L be the set of vertices to the left of the line through i and j and R be the vertices to the right of this line. It follows that each vertex in L is to the left of each of the four probes, and each vertex in R is to their right. Let d_{il} be the number of edges $\{i,v\}\in E, v\in L$, and similarly define the quantities d_{ir}, d_{jl}, and d_{jr}. Finally, let m be the number of edges $\{l,r\}\in E, l\in L, r\in R$ and e be the number of edges $\{i,j\}\in E$. Thus e is a 0/1 variable unless G is a multigraph.

The cut–set sizes returned by the four probes are functions of these quantities:

$$P_1 = m + d_{il} + d_{jl}$$

$$P_2 = m + d_{ir} + d_{jr}$$

$$P_3 = m + d_{il} + d_{jr} + e$$

$$P_4 = m + d_{jl} + d_{ir} + e$$

Thus $P_3 + P_4 - P_1 - P_2 = 2e$, and the membership of any edge in G is determined. □

Note that there is no requirement that the straight line graph be a planar embedding. To show that $C(n,2)$ probes are necessary for the determination of a graph, we consider a special class of graphs. We define a *balanced* graph on n vertices $B(V,E)$ to be one with vertices $V=\{v_1..v_n\}$ and edges $E=\{\{v_i,v_j\} \mid j=(i+1+2k) \bmod n,\ 0 \le k < n/2\}$. Figure 7.2 shows balanced graphs on six and eight vertices. Specificly, we are interested in balanced graphs where the vertices represent a convex n–gon, n even, with the vertices labeled in angular order. The cut–set sizes of such graphs share an interesting property:

Lemma 7.2: The cut–set size of any partition $A \cup B = V$ of a convex embedding of a balanced graph, $|V| = n$ for even n, equals $(|A| \cdot |B| + c)/2$, $0 \le c \le 1$.

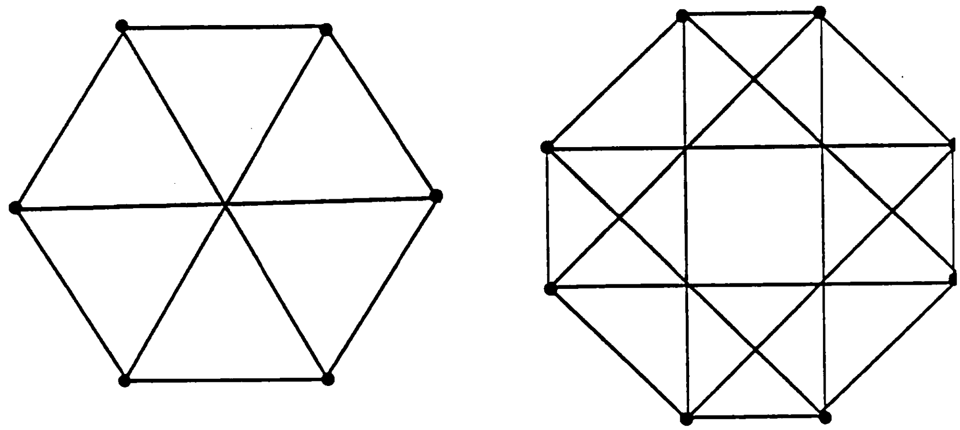


Figure 7.2: Balanced graphs on six and eight vertices.

Proof: There are two cases, depending upon the parity of the sets. If both sets contain an odd number of vertices, $\lceil |A|/2 \rceil \lceil |B|/2 \rceil + \lfloor |A|/2 \rfloor \lfloor |B|/2 \rfloor$ edges are cut, so $c=1$. If not, then B is a set of even size. Each vertex in A is connected to $|B|/2$ of B, so $c=0$. □

Thus each cut-set consists of approximately one half of all the possible edges between elements of A and B. Since clearly we can simulate probing the complement of G instead of G if it is advantageous, this means that each cut provides the minimum possible amount of information concerning which edges are in the graph.

Theorem 7.3: For an even number of vertices, $C(n,2)$ cut-set probes are necessary to determine a graph.

Proof: Consider a convex embedding of a balanced graph on n vertices, where n is even. Any cut-set probe l intersects exactly two edges on the convex n-gon, which together are incident upon either three or four vertices. In the case of four vertices, let $\{a,b\}$ and $\{c,d\}$ be the edges of the n-gon. By the definition of a balanced graph, $\{a,c\}$ and $\{b,d\}$ must also be edges of B for the appropriate choice of labelings of the vertices. However, a graph

$G'=(V',E')$, $V'=V$ and $E'=E-\{\{a,c\},\{b,d\}\}\cup\{\{a,d\},\{b,c\}\}$, as shown in Figure 7.3a, has identical cut–set sizes as G along every probing line except l.

In the case of three vertices, let $\{a,b\}$ and $\{b,c\}$ be the edges of the n-gon. Such is the case when l partitions V into $\{b\}$ and $V-\{b\}$. Replacing both of them with the edge $\{a,c\}$, as in Figure 7.3b, yields a graph which has identical cut–set sizes for all probes but l. Thus any collection of $C(n,2)-1$ probes is insufficient to verify, let alone determine G. □

Corollary: Graphs whose vertices are not in general position cannot be reconstructed.

The corollary follows since there do not exist $C(n,2)$ distinct probes if the vertices are not in general position. We note that the theorem does not resolve the problem for graphs with an odd number of vertices, since it is easy to show that all graphs on three vertices can be verified in only two probes. However, three probes are still necessary for determination.

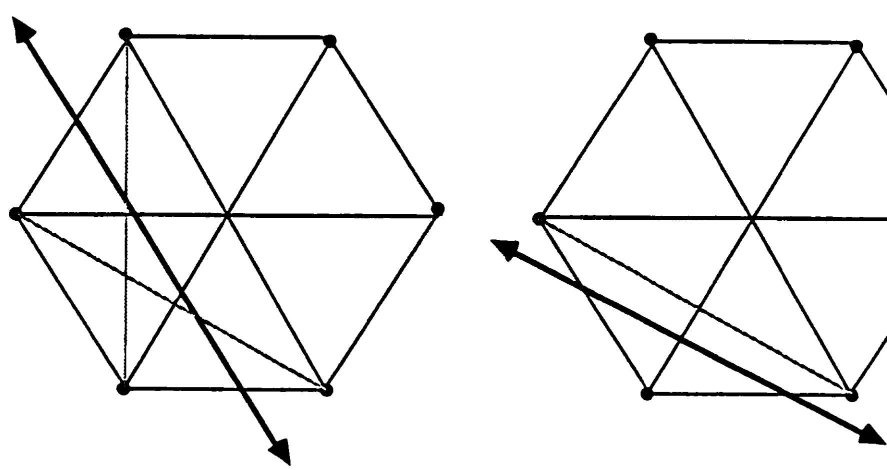


Figure 7.3: Graphs which differ in only one cut–set size.

Sparse or near complete graphs can be verified more efficiently than balanced graphs. It is easily seen that a graph with n vertices and m edges can be verified in $4\min\{m, C(n,2)-m\}+n-1$ cut-set probes. As shown in Theorem 7.1, the presence (or absence) of an edge in G can be tested with four probes. Thus all the edges in G (or the complement of G) can be verified in $4m$ probes. Sweeping the vertices of G by a line which is not parallel to a direction of the point set defines $n-1$ probes such that every edge of G is intersected by at least one probe. Thus we can prove that there are only m edges in the graph, the m verified in the first step. Therefore, it is clear that planar graphs can be verified in $O(n)$ probes.

Every cut-set probe returns a number between 0 and $n^2/4$, which is the largest size of a cut-set in a graph. This maximum occurs for partitions $A \cup B = V$ of complete graphs where $|A| = |B| = n/2$. We note that there are $2^{\Theta(n^2 \log n)}$ possible sequences of probe results, $C(n,2)$ values between 0 and $n^2/4$. There are only $2^{O(n \log n)}$ combinatorially distinct point sets in the plane [32], each of which can support exactly $2^{C(n,2)}$ distinct labeled graphs. Thus there are at most $2^{O(n^2)}$ realizable sequences, so most sequences cannot represent probing outcomes for a graph.

An example of an unrealizable cut-set sequence is $(k,k,k, \cdots ,k)$, except for $k=0$ or $k=2$, for $n>2$. The empty graph realizes the case of $k=0$ and a ring around a convex set of vertices realizes $k=2$. To show that no other k is realizable, we note that in any configuration of n points in general position, there are at least three edges of the convex hull. These may or may not be edges of the graph. Consider one of these edges, connecting vertices a and b, and let l be a line that separates $\{a,b\}$ from $V-\{a,b\}$. For a configuration to realize the specifed cut-set sequence, both of a and b must be of degree k and k edges must cross l. There are two cases, depending upon whether there is an edge between a and b. If so, $k-1$ edges incident upon each vertex must cross l, so $k = 2(k-1)$ and $k = 2$. If not, k edges incident upon both a and b cross l, so $k = 2k$ and $k = 0$.

A graph on n vertices has $2^{n-1}-1$ distinct cut–sets, vastly more than the $C(n,2)$ straight line partitions of points in a plane. Generalizing our notion of a cut–set probe to determine the size of arbitrary cut–sets of a graph permits the possibility of a better strategy than in Theorem 7.1. Using the notion of cut–set sequences, we prove our strategy is optimal within at least a logarithmic factor.

Theorem 7.4: At least $\dfrac{n(n-1)}{4(\log_2(n)-1)}$ generalized cut–set probes are necessary to determine a graph.

Proof: For k cut–set probes to determine a graph on n vertices, there must be at least as many distinct probing outcomes as there are graphs. Since there are $2^{C(n,2)}$ labeled graphs on n vertices and at most $(n^2/4+1)^k$ outcomes of k cut–set probes:

$$\left(\frac{n^2}{4}+1\right)^k \geq 2^{\binom{n}{2}}$$

Taking the binary logarithm of both sides and simplifying gives a lower bound on k. Replacing $(n^2+1)/4$ by $n^2/4$ gives the slightly weaker but cleaner result. □

7.2. Determining Graph Properties

Any problem that can be asked for general graphs can be asked for more restricted subgraphs. Specifically, is it easier to determine a planar graph? How about a dense or sparse graph? These questions are wide open, but we present an improved determination result for trees:

Theorem 7.5: $O(n \log n)$ generalized cut–set probes are sufficient to determine a tree T.

Proof: First, the degree sequence of T is determined in n probes by considering the partition where each vertex is a singleton.

Note that a tree must always contain at least two vertices of degree one. Let i be such a vertex. By using the relationships of Theorem 7.1 and values of d_i and d_j, we can solve for d_{il} and d_{ir}. Using these we can perform a binary search to locate the edge, simulate the deletion of this edge, and repeat for the rest of the tree. □

This strategy will work for regular cut–set probes if a fast strategy for determining the degree sequence of G can be found.

An interesting, although unrelated, graph probing problem comes directly from the notion of ***electrical tomography*** [31]. In electrical tomography, a number of pairs of probe points are made and the electrical resistance between the two points is measured. The measured resistance of each will be a function of the entire organ, not just the line between the two points. This can be modeled by assuming a graph between the points, such that each arc of the graph contains a fixed resistor. Further, the network obeys the classical laws for parallel and series resistance. The problem, given such a network, is how many probes are necessary to determine the resistors associated with each edge.

It is not obvious that network can be determined for all graphs. Gilbert and Shepp [31] claim that the resistances can be determined if the graph is complete. As a counter–example to the general result, they present three networks of five vertices and eight edges which yield identical measurements. However, two of these networks contain open (infinite resistance) edges. Whether networks of finite resistance are uniquely reconstructible is an interesting open question.

7.3. Conclusions and Open Problems

We have shown that $C(n,2)$ cut–set probes are sufficient to determine a straight line graph whose vertices are in general position. Further, this bound is tight for verification and determination when n is even. These results bring up several open questions:

(7.1) Are fewer probes necessary for special types of graphs, such as planar graph embeddings? Lower bounds for restricted graphs such as trees and planar graphs can be based on enumeration results [40] for the class of graph. However, Cayley's formula leads to only an $\Omega(n)$ lower bound for labeled trees.

(7.2) Are fewer probes needed to determine the number of edges in a graph? How about the degree sequence of G? For the special case of convex vertex sets, n probes suffice, since the degree of any vertex can be determined in one probe.

(7.3) Is there a general condition for testing whether a set of cut–set sizes can correspond to the $C(n,2)$ probes of a graph? This may be a very difficult problem since it relates to point configurations which are combinatorially different.

(7.4) Can $C(n,2)$ be shown to be necessary for arbitrary cut–sets? If not, consider the three dimensional problem, with the vertices in general position in E^3 and a probe measuring the number of edges cut by a plane. Clearly, $C(n,2)$ is sufficient since all probes can be directed normal to a common plane, reducing the problem to two dimensions. However, there are

$\approx n^3$ cut–sets defined by the point set in E^3, possibly allowing a better strategy.

We observe that the cut–set sequence of a graph is an alternate representation for the graph. It would be interesting to see if there are any algorithmic implications for such a representation.

The results of this chapter were largely non–geometric, and suggest interesting problems follow from probing purely mathematical objects. For example, consider the problem of polynomial interpolation. It assumes an oracle which evaluates a polynomial P at a given point and asks how many calls to the oracle will be necessary to determine P of n coefficients, at most t of which are non–zero. Clearly n are necessary when $n = t$, but the situation changes when P is degenerate. Ben–Or and Tiwari [5] prove that $2t$ evaluations are necessary and sufficient in the non–adaptive case, where the ith evaluation point are is not a function of the previous $i-1$ evaluations, but the points must be given in batch. Coppersmith has observed that $t+1$ evaluations are sufficient in the adaptive case [48], and clearly t are necessary.

CHAPTER 8.

CONCLUSIONS

This thesis has explored a variety of different models in geometric probing. For each of these, we have demonstrated that convex polygons can be reconstructed in a linear number of probes. Table 8.1 summerizes our results for determining and verifying convex polygons under various probing models. Along the way, we solved a host of related problems and uncovered even more.

To assess the significance and potential of this work, it is worth noting the wide range of research areas which probing touches upon. We hope that this thesis inspires fruitful work in one or more of the following directions:

Tomography and Remote Sensing

Perhaps the most surprising result in this thesis is that a linear number of x–ray probes are sufficient to reconstruct a convex polygon. This is made possible by the interactive nature of our strategy, the fact that we used the results of previous probes in planning the

	Determination		Verification	
Probe	Lower Bound	Upper Bound	Lower Bound	Upper Bound
finger probe	$3n$	$3n$	$2n$	$2n$
hyperplane probe	$3n+1$	$3n+1$	$2n$	$2n$
x–ray probe	$2n$	$5n+19$	n	$3n/2$
half–plane probe	$2n$	$8n+6$	$n/2$	$n+1$

Figure 8.1: Summary of main results.

next. The potential exists for real tomographic systems to work in this way, given improvements in the speed of signal processors, which would greatly decrease the amount of radiation patients are exposed to. Also, our success with using new sensing models such as half-plane probes could stimulate work on physical devices which behave according to this model.

Algorithmic Implications

A probing strategy defines a representation for the object being probed. For some of our models, particularly cut-set probes, it would be interesting to see whether this representation has any useful properties for geometric or graph algorithms. The paradigm of probing is natural for optimization problems and deserves more attention.

Combinatorial Geometry

Rota has said that "combinatorics needs more theory and less theorems". While this is no doubt true, it is the flood of interesting problems which makes combinatorics attractive. The notion of k-projections shows that interesting combinatorial problems arise from analyzing probing strategies.

Geometric Probing

The most important aspect of this thesis is the codification of all major results to date in geometric probing, and identifying the most interesting open problems which remain. I have enjoyed working in this area, and hope that this thesis inspires someone to continue the work. Specifically, the best remaining problems concern developing a model-based strategy for line probes, reconstructing polytopes in E^3 from cross-sectional area and half-space probes, and generalizing the probing in rounds results to more than two probes per round.

Finally, we hope that the ideas of this thesis can be applied outside the realm of computer science and mathematics, to problems of the real world. For example, what discovery could have a bigger impact on the world than determining whether congressional probes ever determine anything.

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16. Abstracts

Many sensing problems can be formulated as problems in geometric probing; that is, given geometric models of a sensor and an object, what information can be determined about the object and how efficiently can it be found.

This thesis discusses problems for finger, hyperplane, x-ray, half-space, and cut-set probing models, as well as aggregates for several of these probes.

We present all major results to date in probing as well as collect significant open problems

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