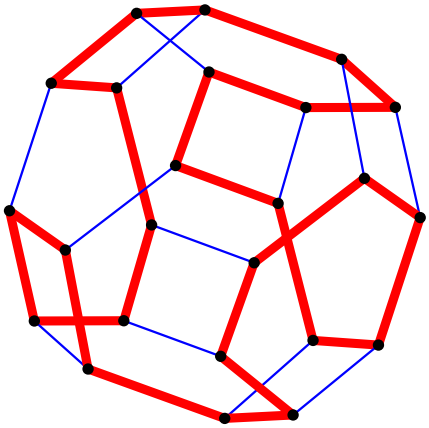
Computational Discrete Mathematics with Combinatorica



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What is *Combinatorica*?

Combinatorica is an extension to *Mathematica* that provides over 450 functions to do discrete mathematics on the computer.

It is distributed as a standard add-on package with every copy of *Mathematica*.

It can be loaded by executing

```
<<DiscreteMath`Combinatorica`
```

A *Combinatorica* user can count, enumerate, sample, visualize, and manipulate discrete structures such as permutations, combinations, integer and set partitions, Young tableaux, partially ordered sets, trees, and graphs.

Combinatorica in Action

Enumerating, Counting, and Sampling.

```
In[2]:= Partitions[6, 3]
Out[2]= {{3, 3}, {3, 2, 1}, {3, 1, 1, 1}, {2, 2, 2},
         {2, 2, 1, 1}, {2, 1, 1, 1, 1},
         {1, 1, 1, 1, 1, 1}}
```

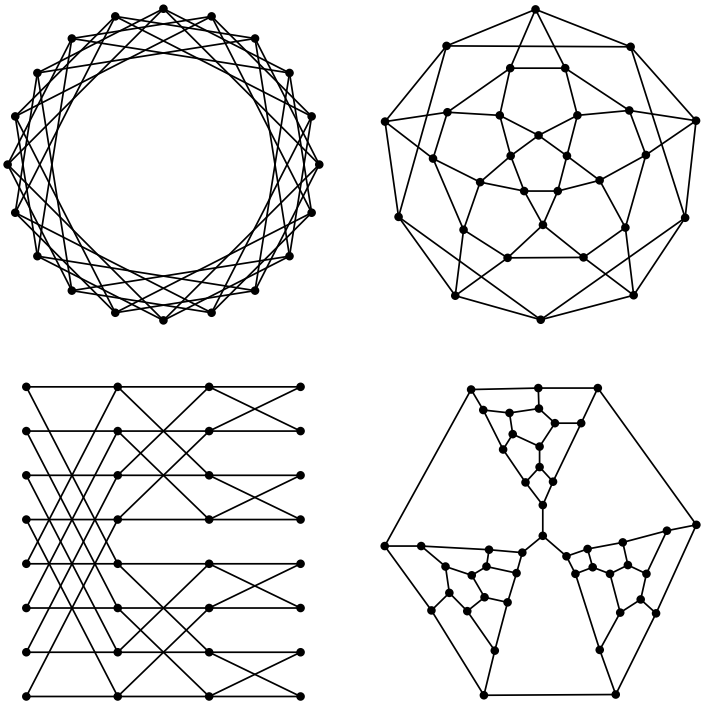
```
In[3]:= Table[NumberOfGraphs[n], {n, 10, 20}] // ColumnForm
Out[3]= 12005168
        1018997864
        165091172592
        50502031367952
        29054155657235488
        31426485969804308768
        64001015704527557894928
        24593586415353293268371977 6
        17875777251456117005478781908 48
        24637809253125004524383007491432768
        64549012279579984185616463849074274944 0
```

```
In[4]:= RandomTableau[{4, 3, 1, 1}]//TableForm
Out[4]//TableForm= 1   2   5   9
                   3   4   8
                   6
                   7
```

Combinatorica in Action

Constructing Graphs.

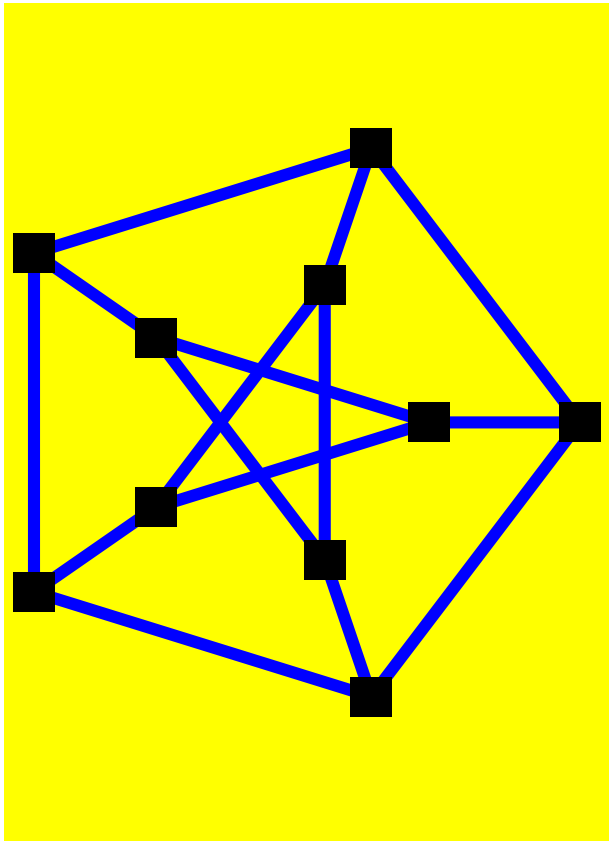
```
In[5]:= ShowGraphArray[{{CirculantGraph[20, {3, 5}],
        LineGraph[DodecahedralGraph]},
        {ButterflyGraph[3], TutteGraph}}];
```



Combinatorica in Action

Drawing Graphs.

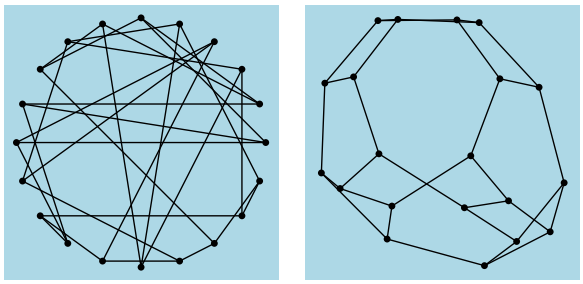
```
In[6]:= ShowGraph[PetersenGraph,
            VertexStyle -> Box[Large],
            EdgeStyle -> Thick, EdgeColor -> Blue,
            Background -> Yellow];
```



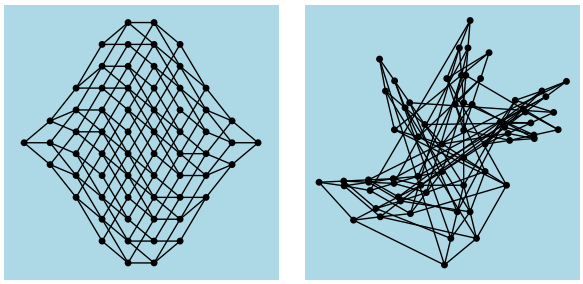
Combinatorica in Action

Drawing Graphs.

```
In[7]:= ShowGraphArray[{g = RegularGraph[3, 20],
            SpringEmbedding[g, 100]},
            Background -> LightBlue];
```



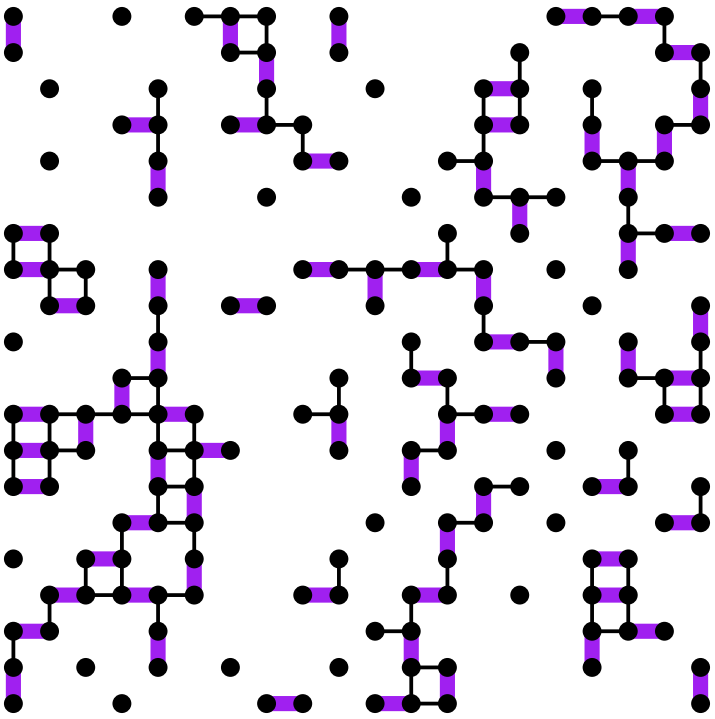
```
In[8]:= ShowGraphArray[g = GridGraph[4, 4, 4];
            {RankedEmbedding[g, {1}], RadialEmbedding[g]},
            Background -> LightBlue];
```



Combinatorica in Action

Graph Algorithms.

```
In[9]:= g = InduceSubgraph[GridGraph[20, 20],
          RandomSubset[400]];
        ShowGraph[Highlight[g, {xm = BipartiteMatching[g]}
                  HighlightedEdgeColors -> {Purple}]];
```



```
In[10]:= {Length[xm], Length[MaximalMatching[g]]}
Out[10]:= {77, 74}
```

Combinatorica in Action

Graph Algorithms.

```
In[11]:= VertexColoring[MycielskiGraph[5]]
Out[11]= {1, 2, 2, 3, 1, 3, 1, 2, 1, 4, 2, 1, 2, 2,
          3, 1, 3, 1, 2, 1, 5, 2, 4}
```

```
In[12]:= Expand[wheelpoly[z_] =
                ChromaticPolynomial[Wheel[7], z]]
                     2        3        4       5       6    7
Out[12]=62 z  - 191 z  + 240 z  - 160 z  + 60 z  - 12 z  + z
```

```
In[13]:= Table[wheelpoly[z], {z, 1, 10}]
Out[13]= {0, 0, 6, 264, 3660, 24600, 109410, 373296,
          1058904, 2621520}
```

```
In[14]:= HamiltonianCycle[Hypercube[4]]
Out[14]= {1, 2, 3, 4, 8, 5, 6, 7, 11, 10, 9, 12, 16, 15,
          14, 13, 1}
```

Combinatorica: A Brief History

Combinatorica was initially released in 1990 with 230 functions and about 2500 lines of code.

It is not a high-performance algorithms library (e.g., LEDA), but a mathematical research tool and a prototyping environment.

Combinatorica received a 1991 EDUCOM award for distinguished mathematics software.

In 2003, we released a "new *Combinatorica*." The new package is essentially a complete rewrite of *Combinatorica*. Over 80% of the functions have been rewritten and the package has grown to more than twice its original length.

Combinatorica: A Brief History

Some of the highlights of the new package (relative to the old) are:

- Dramatic improvement in the running time.
- Significant improvement in the functionality and flexibility of most existing functions.
- Coverage of additional topics in discrete mathematics and algorithms.
- Many new graph constructors.
- Significantly improved graph drawing that allows for color, multiple edge and vertex styles, self-loops and multiple parallel edges, graph captions, zoom-in and zoom-out, and many other new features.

Mathematica Under the Hood: Usually a Boon

Mathematica provides sophisticated machinery for symbolic manipulation, random number generation, function timing, and portable graphics.

```
In[15]:= poly1 = DihedralGroupIndex[3, x]
                3
         x[1]     x[1] x[2]    x[3]
Out[15]= ----- + --------- + ----
          6          2         3
```

```
In[16]:= poly2 = Expand[poly1 /.  Table[x[i] ->
                      (a^i + b^i + c^i), {i, 3}]]

          3    2        2    3    2              2
Out[16]= a  + a  b + a b  + b  + a  c + a b c + b  c +

           2      2    3
         a c  + b c  + c
```

Mathematica Under the Hood: Usually a Boon

Mathematica provides a flexible, high-level language. *Combinatorica*, with more than 450 functions has about 6500 lines of code at an average of less than 15 lines per function.

```
Partitions[n_Integer, maxpart_Integer] :=
        Block[{$RecursionLimit = Infinity},
              Join[Map[(Prepend[#,maxpart])&,
                        Partitions[n-maxpart,maxpart]],
                   Partitions[n,maxpart-1]
              ]
        ]
```

Mathematica Under the Hood: Sometimes a Curse

Mathematica is an interpreted language and so programs in *Mathematica* are much slower than programs in languages such as C.

Mathematica seems to follow the "wolf-RAM" model of computation. If an algorithm takes $O(f(n))$ running time in the RAM model of computation, there is no guarantee that a faithful implementation with run in $O(f(n))$ time.

Combinatorica for Research and Education

Combinatorica seems to have an eclectic set of users: lawyers, anthropologists, economists, biologists, physicists, mathematicians, high school students, etc.

Research Applications of *Combinatorica* include:

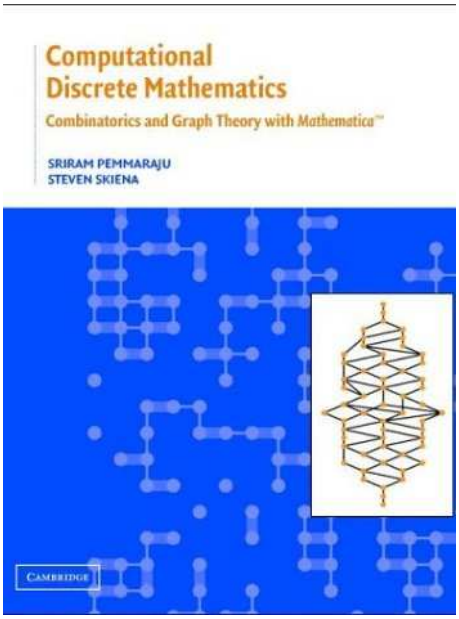
- Unfolding Convex Polytopes (Namiki and Fakuda).
- Implementing Graph Grammars (Valiente).
- Generalized Lights Out (Cowen, et. al).
- Topologies of Renal Glomerular Microvascular Networks (Wahl, et. al).

Combinatorica Resources

Demo notebooks, graph databases, tutorials, etc. are available at

`www.combinatorica.com`

The definitive guide: S. Pemmaraju and S. Skiena, *Computational Discrete Mathematics: Combinatorics and Graph Theory in Mathematica*, Cambridge University Press, 2003.



The Title Page Picture

This defines a binary relation between pairs of 4-permutations that differ by an adjacent swap.

```
In[17]:=atr = MemberQ[
        Table[p = #1; {p[[i]], p[[i + 1]]} =
              {p[[i + 1]], p[[i]]}; p,
              {i, Length[#1] - 1}], #2] &;
```

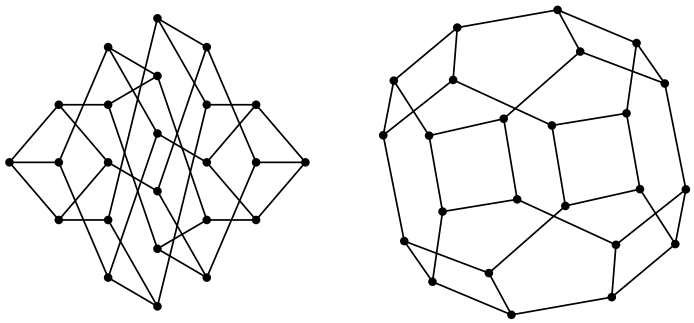
Here we construct a representation of the binary relation as an undirected graph.

```
In[18]:= g = MakeGraph[Permutations[4], atr,
                       Type -> Undirected];
```

The Title Page Picture

Here we display the graph in two different ways.

```
In[19]:= ShowGraphArray[{RankedEmbedding[g, {1}],
                         SpringEmbedding[g, 100]}];
```



The latter is a *truncated octahedron*, a 14-faced polyhedron that can be obtained by lopping off each of the 6 corners of a regular octahedron.