Lecture 12:
The Black-Scholes Model

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The Black-Scholes-Merton Model

Analyzing the Binomial tree model with infinitely time small steps gives the Black-Scholes option pricing model, which says the value of a stock option is determined by six factors:

- S, the current price of the underlying stock
- y, the dividend yield of the underlying stock
- K, the strike price specified in the option contract
- r, the risk-free interest rate over the life of the option contract
- T, the time remaining until the option contract expires
- σ, the price volatility of the underlying stock.

The Pricing Formula

The price of a call option on a single share of common stock is: $C = Se^{yT}N(d_1) - Ke^{rT}N(d_2)$
The price of a put option on a single share of common stock is: $P = Ke^{rT}N(d_2) - Se^{yT}N(d_1)$
and

$$d_1 = \frac{\ln(S/K) + (r - y + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Formulae Details

Three common functions are used to price call and put option prices:

- e^{-rt}, or $\exp(-rt)$, is the *natural exponent* of the value of rt (in common terms, it is a discount factor)
- $\ln(S/K)$ is the natural log of the "moneyness" term, S/K. $e = 2.71828$ is the base of the natural log
- $N(d_1)$ and $N(d_2)$ denotes the standard cumulative normal probability for the values of d_1 and d_2. It is the probability that a random draw from a normal dist. will be $< d$.

Pricing Example

Suppose you are given the following inputs:

- $S = \$50$ (current stock price)
- $y = 2\%$ (dividend yield)
- $K = \$45$ (strike price)
- $T = 3$ months (or 0.25 years)
- $s = 25\%$ (stock volatility)
- $r = 6\%$ (risk-free interest rate)

Computing d_1 and d_2

$$d_1 = \frac{\ln(S/K) + (r - y + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(50/45) + (0.06 - 0.02 + 0.25^2/2)0.25}{0.25\sqrt{0.25}}$$

$$= \frac{0.10536 + 0.07125 \times 0.25}{0.125}$$

$$= 0.98538$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.98538 - 0.25\sqrt{0.25} = 0.86038$$

To compute $N(d_1)$ and $N(d_2)$, we can either look it up in a normal distribution table, or call a library function like NORMSDIST(x) in Excel.
We can use the fact that $N(-d_1) = 1 - N(d_1)$ in case the library does not accept negative arguments.

Computing the Call and Put Price

Call Price:

$$
\begin{aligned}
C &= Se^{yT}N(d1) - Ke^{rT}N(d2) && (1)\\
&= \$50 \times e^{-(0.02)(0.25)} \times 0.83778 \; - \; \$45xe^{-(0.06)(0.25)} \times 0.80521 && (2)\\
&= \$50 \times 0.99501 \times 0.83778 \; - \; \$45 \times 0.98511 \times 0.80521 = \$5.985. && (3)
\end{aligned}
$$

Put Price:

$$
\begin{aligned}
P &= Ke^{rT}N(d2) - Se^{yT}N(d1) && (4)\\
&= \$45 \times e^{-(0.06)(0.25)} \times 0.19479 \; - \; \$50 \times e^{-(0.02)(0.25)} \times 0.16222 && (5)\\
&= \$45 \times 0.98511 \times 0.19479 \; - \; \$50 \times 0.99501 \times 0.16222 = \$0.565. && (6)
\end{aligned}
$$

Daily to Annual Volatility

The volatility σ is the standard deviation of the continuously compounded rate of return in on year.

The standard deviation of the return in time Δt is $\sigma\sqrt{\delta t}$.

Assuming there are 252 trading (instead of 365 real) days in a year provides a way to convert observed daily standard deviations to annual volatility.

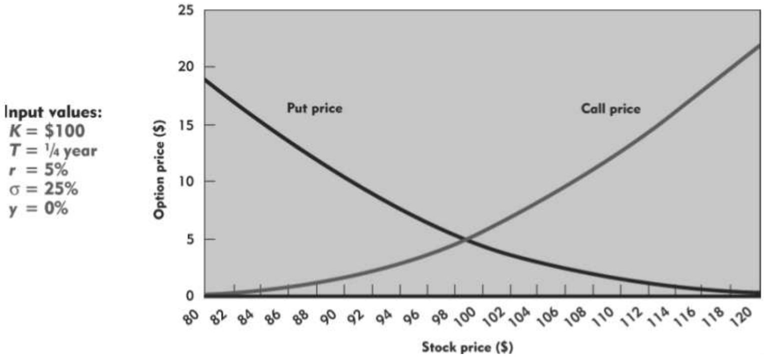
Thus a 25% annual volatility maps to a 1.57% daily volatility

Significance of the Black-Scholes Formula

Having a closed form means that options can be priced on a calculator instead of a computer (or extremely rapidly on a computer).

It also means that the influence of individual factors on price can be studied analytically instead of experimentally.

Impact of Current Price

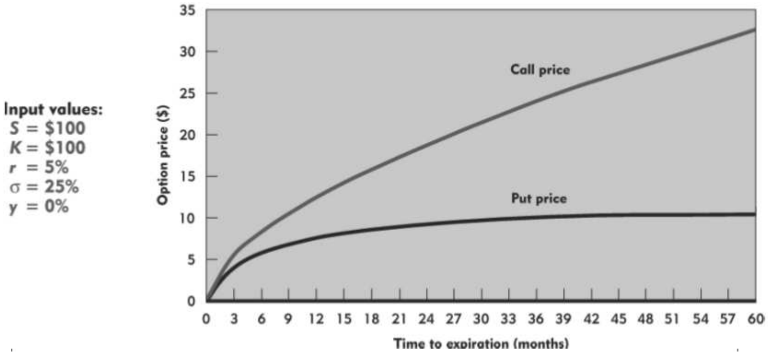


Call and put prices are approximately equal when $S = K$.

As S becomes very large, c tend to $S - Ke^{-rT}$ and p tends to zero.

As S becomes very small, p tend to $Ke^{-rT} - S$ and c tends to zero.

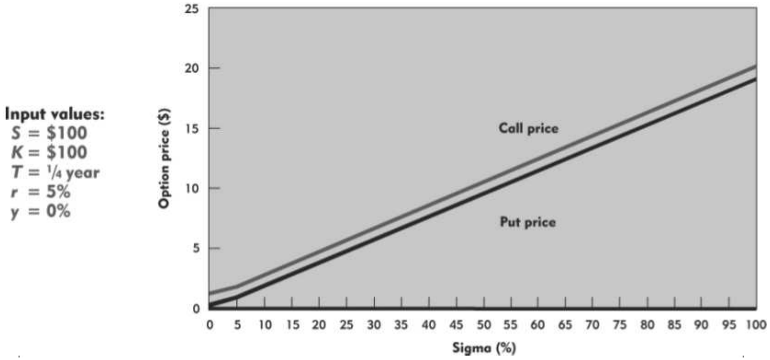
Impact of Expiration Date



The put price is less than that of the call because equal percentage up/down moves are *not* equal dollar moves.

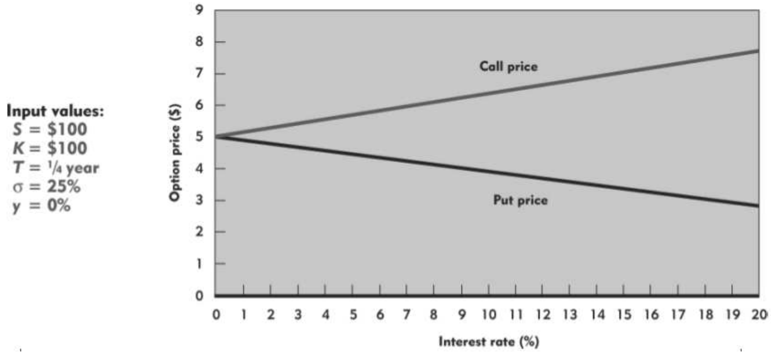
European put prices do not always increase with expiration date.

Impact of Volatility



Why does the call option have value at $\sigma = 0$ but not the put?
Of the six pricing factors, only volatility is not directly observable.
Again, calls have more dollar upside than puts.

Impact of Interest Rates



The present-value *cost* of exercising the call (K) decreases at higher rates.
The present value received by the put (K) decreases at higher rates.

Assumptions of the Model

In the short time period Δt, the return on a stock of price S is normally distributed:

$$\frac{\Delta S}{S} \approx \phi(\mu \Delta t, \sigma^2 \Delta t)$$

where μ is the expected return and σ is the volatility.
It follows that the actual price S_t are lognormally distributed.
We assume (1) trading is continous, (2) short-selling is allowed, and (3) there are no transaction costs.
The drift μ, volatility σ, and risk-free rate r are all constant for the period (some of which can be relaxed)

Risk Neutral Valuation

We assume there are no riskless arbitrage opportunities.
BS is based on the same principles of risk-neutral valuation underlying binomial trees.
The option price and stock price depend on the same underlying source of uncertainty.
We can form a portfolio consisting of stock and option to eliminate this source of uncertainty.
The portfolio is instantaneously riskless and must instantaneously earn the risk free rate (i.e. risk-neutral valuation)

Where Does the Black-Scholes Formula Come From?

It is derived using stochastic calculus and partial differential equation methods beyond the scope of the course.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

More intuitively, it is the continuous-time limit of the binomial tree method with particular values for upward and downward sets.

By analogy, the binomial theorem is a closed form for stock (not option) prices under discrete (not continuous) additive (not multiplicative) moves of ± 1 (not functions of σ).

Drift and Volatility

To complete the model, we need to set the magnitude for up and down movements in the binary tree. Suppose we choose u, d, and p as follows:

$$u = 1 + \sigma\sqrt{\delta t}$$

$$d = 1 - \sigma\sqrt{\delta t}$$

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}$$

Realizing Drift and Volatility

The expected asset price change in one time step with these parameters is

$$puS + (1 - p)dS = (1 + \mu\delta t)S$$

The variance of the change in asset prices is $S^2\sigma^2\delta t - S^2\mu^2\delta t^2$, so the standard deviation of returns is $\sigma\sqrt{\delta t}$

Thus these parameters create a process with drift μ and volatility σ. Black-Scholes uses risk-neutral valuation, so $\mu = r$.

Other u and d values are also popular in binominal trees, which realize the desired volatility with $u \times d = 1$.

Differential Equations

Many financial quantities are naturally expressed as differential equations, which define functions in terms of rates of change.

Define $M(t)$ as the amount in a bank account earning compound interest at rate r as a function of time t.

The change in wealth $M(t+dt) - M(t) \approx dM = rM(t)dt$ as $dt \longrightarrow 0$.

The equation $M(t) = M(0)e^{rt}$ solves this equation because

$$D[M(0)e^{rt}] = M(0)e^{rt}r$$

so indeed $dM/dt = rM(t)$ for $M(0) = 1$.

The Differential Equation

Any security whose price is dependent on the stock price satisfies the differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

The particular security being valued is determined by the boundary conditions of the equation.
In a forward contract the boundary condition is $f = S - K$ when $t = T$, and the solution is

$$f = S - Ke^{-r(T-t)}$$

Limitations of the Black-Scholes Model

- The log-normal return distribution it assumes is often violated.
- The continous model does not allow for jumps in the underlying stock prices.
- Volatility of the stock is considered constant during the option's lifetime.

More sophisticated models can be readily evaluated as binomial trees, with analytic results more difficult to obtain.