Lecture 11:
Risk-Neutral Valuation

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Risk-Neutral Probabilities

We can use an arbitrage argument to set the "right" probability of an upward move (β) as a function of the risk-free rate.

At any point, investors can either (a) hold \$1 stock or (b) invest \$1 at the risk-free rate r.

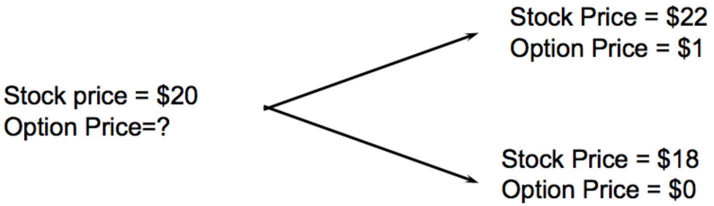
A *risk-neutral* investor would not care which portfolio they owned if they had the same return.

Setting equal the returns from the stock ($\beta\alpha + (1 - \beta)/\alpha$) and the risk-free portfolio ($1 + r$), we can solve for β to determine the *risk-neutral probability*.

But in truth, investors are not risk-neutral. In order to take the riskier investment they must be paid a premium.

Single-Step Option Pricing

Binomial trees price options using the idea of *risk-neutral valuation*.
Suppose a stock price is currently at \$20, and will either be at \$22 or \$18 in three months.
What is the price of a European call option for a strike price of \$21? Clearly, this reduces to determining the probability of the upward price movement.



Risk Neutral Valuation

The risk-neutral investor argument for setting this probability can be applied if we set up two portfolios which are of provably of equal risk and value.

We will construct two riskless portfolios, one involving the stock and the other the risk-free rate.

Using Options to Eliminate Risk

A *riskless* portfolio can be created by buying Δ shares of stock and selling a short position in 1 call option, such that the value of the portfolio is the same whether the stock moves up or down.

If the stock moves to \$22, our portfolio will be worth $\$22\Delta - \$1 \cdot 1$, since we must pay the return of the option we sold.

If the stock moves to \$18, our portfolio will be worth $\$18\Delta - \$0$, since the option we sold is worthless.

A riskless portfolio is constructed by buying $\Delta = 0.25$ shares, since it is the solution of $\$22\Delta - \$1 \cdot 1 = \$18\Delta$.

Valuing the Portfolio

Whether the stock goes up or down, this portfolio is worth $\$4.50$ at the end of the period.

The discounted value of this portfolio today, V, can be computed given the risk-free interest rate r. Thus $V = (4.50)e^{-rt}$.

Since the value of V is equal to owning $\Delta = 0.25$ shares of stock at \$20 per share minus the value f of the option, $f = 20 \cdot 0.25 - V$.

The General Case

In general, if there is an upward price movement, the value at the end of the option is

$$S_0 u\Delta - f_u$$

where $S_0 u$ (f_u) the price of the stock (option) after an upward movement.
If there is a downward price movement, the value at the end of the option is

$$S_0 d\Delta - f_d$$

Setting them equal and solving for Δ yields

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

The present value of the portfolio with a risk-free rate of r is

$$(S_0 u\Delta - f_u)e^{-rT}$$

which can be set up for a cost of $S_0\Delta - f$.
Equating these two and solving for f yields

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

By definition, the value of f must also be

$$f = e^{-rT}\left(\beta f_u + (1-\beta)f_d\right)$$

where β is the probability of an upward movement.
Solving for β we get

$$\beta = \frac{e^{rT} - d}{u - d}$$

Interpreting this Probability

The expected stock price at time T implied by these probabilities is S_0e^{rT}.

This implies that the stock price earns the risk free rate.

The value of an option is its expected payoff in a risk-neutral world discounted at the risk-free rate.

Irrelevance of Stock's Expected Return

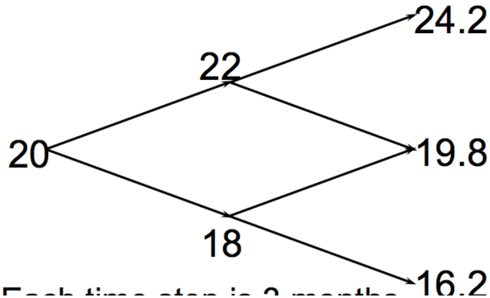
When we value an option in terms of the price of the underlying asset, the probability of up and down movements in the real world is irrelevant, since they can be hedged.

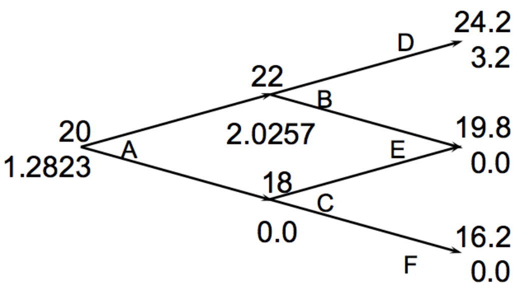
This is an example of a more general result stating that the expected return (drift) on the underlying asset in the *real* world is irrelevant.

The option has to have the risk-neutral valuation, because if not there exists an arbitrage opportunity buying the right portfolio.

Pricing Options with Binomial Trees

The value of the option can be worked backwards from the terminating (basis) condition level by level.





The value of the option on leaf / terminating level is determined because the option price at expiration is completely given by the stock and strike prices.

Finer Gradations

Adding additional levels to the trees allows finer price gradations than just a single up or down.

The price of an option generally converges after about $n = 30$ levels or so.

Note that the number of options needed (Δ) changes at each node/level in the binomial tree. Thus to maintain a riskless portfolio options must be bought and sold continuously, a process known as *delta hedging*.

Generalizing the Model

This binomial tree model can be generalized to include the effects of (1) dividends, by changing the magnitude of the moves in the levels corresponding to dividend periods, (2) changing interest rates, by using the rate appropriate on a given yield curve.

It can also be generalized to allow more than two price movements from each node, say increase, decrease, and unchanged.

Pricing American Options

American options permit execution at any intermediate time point.

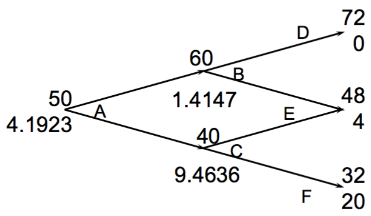
It pays to exercise a non-dividend paying American put early if the underlying stock price is sufficiently low (say 0) due to time-value of money.

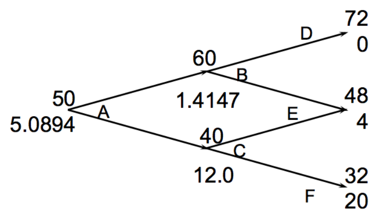
In general, it pays to exercise now whenever the payoff from immediate execution exceeds the value computed for the option at that point.

The options can be priced by using the higher of the two possible valuations at any point in the tree.

American Put Example

Observe the difference between evaluating a put ($S_0 = 50$, strike price $K = 52)$) as European vs. American:





The price at each node is the maximum of $S_K - S_T$ and its European evaluation.

Early Exercise for American Calls

It can be proven that it never pays to execute an American call option early.
Consider a single period for an American call. Start at S_0 and end at $S_0 u$ or $S_0 d$, with payoff f_u and f_d where $0 < f_d < e^{rT} < f_u$.
The no exercise condition $e^{-rT}(pf_u + (1-p)f_d) > S_0 - K$ clearly holds for $K > S_0$. The two other cases are:

- $S_0 d < K \leq S_0$
- $K \leq S_0 d$

Case I: $S_0 d < K \leq S_0$

- $e^{-rT}(pf_u + (1-p)f_d) \geq e^{-rT}pf_u \geq e^{-rT}p(S_0u - K)$.
- Using $p = (e^{rT} - d)/(u - d)$, we therefore need to prove

$$\begin{array}{rccc} & e^{-rT}p(S_0u - K) & > & S_0 - K \\ \Longleftrightarrow & (1 - e^{-rT}p)K & > & (1 - e^{-rT}pu)S_0 \\ \Longleftrightarrow & (u - d - 1 + de^{-rT})K & > & (ue^{-rT} - 1)S_0 d. \end{array}$$

- Proof:

$$\begin{aligned} (u - d - 1 + de^{-rT})K &> (u - d - 1 + de^{-rT})S_0 d && (1) \\ &= (u - d - (u - d)e^{-rT} + ue^{-rT} - 1)S_0 d && (2) \\ &= ((u - d)(1 - e^{-rT}) + ue^{-rT} - 1)S_0 d && (3) \\ &> (ue^{-rT} - 1)S_0 d. && (4) \end{aligned}$$

This completes the proof and shows that an American call would never be exercised early in this case.

Case II: $K \leq S_0 d$

- We have $f_u \geq S_0 u - K$, and $f_d \geq S_0 d - K$.
- We therefore have

$$
\begin{aligned}
e^{-rT}(pf_u + (1-p)f_d) &\geq e^{-rT}(p(S_0 u - K) + (1-p)(S_0 d - K)) && (5) \\
&= e^{-rT}((puS_0 + (1-p)dS_0) - K) && (6) \\
&= e^{-rT}(e^{rT}S_0 - K) && (7) \\
&= S_0 - e^{-rT}K && (8) \\
&> S_0 - K. && (9) \\
& && (10)
\end{aligned}
$$

This shows that American call can never be exercised early in this case either.

Why Monte Carlo Simulation?

Monte Carlo simulation is simpler than dynamic programming to conceive or implement.

When the number of levels gets too high for exhaustive dynamic programming computation (say $n = 1,000,000$), Monte Carlo random walks can still be used to sample the distribution.

Dynamic programming cannot as readily be applied to compute path-dependent distributions (such as Hurst random walks or pricing Asian options) as the state at each node depends on the path used to get there.

How Much is Up (and Down)?

To complete the model, we need to set the magnitude for up and down movements in the binomial tree.

If $S_0 u < S_0 e^{rt}$, the upside for stock ownership is too low, and we are better off investing at the risk-free rate.

If $S_0 d > S_0 e^{rt}$, holding stock guarantees a better return than the risk free rate!

Thus $S_0 u > S_0 e^{rt} > S_0 d$. Otherwise the probability formulae give numbers outside of $[0, 1]$.

This leaves us considerable freedom to set the u, d, and p parameters.