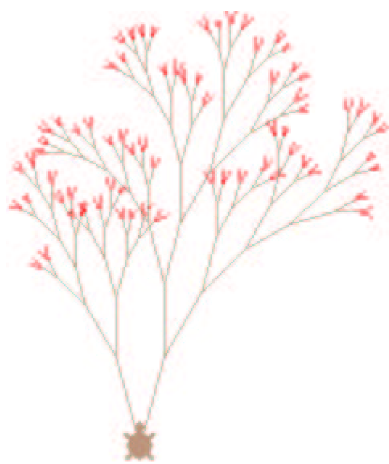
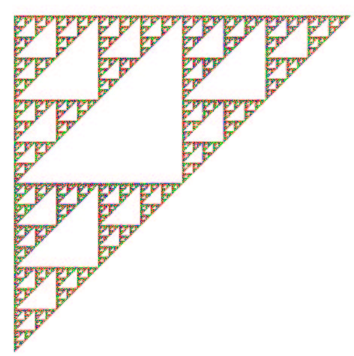
Fractals

Fractals are unusual, imperfectly defined, mathematical objects that observe *self-similarity*, that the parts are somehow self-similar to the whole.

This *self-similarity* process implies that fractals are *scale-invariant*, you cannot distinguish a small part from the larger structure, e.g. a tree branching process.





Fractals are interesting because (1) many phenomena in nature are self-similar and scale invariant, and (2) traditional mathematics/geometry does not capture the properties of these shapes.

Mandelbrot defines a *fractal set* as one in which the *fractal dimension* is strictly greater than the topological dimension.

The Hurst exponent in R/S analysis is somewhat akin to a fractal dimension.

Fractal Models in Finance: Motivation

As has been discussed previously, real financial returns are not accurately modeled by a normal distribution, because there is too much mass at the tails.

In other words, outlying events occur much more frequently in financial returns than they should if normally distributed.

We would expect that the standard deviation of returns should grow at a rate $t^{0.5}$, where t is the amount of time, if they were normally distributed.

In fact, for historical periods on the Dow Jones up to about $t \approx 1000$ days, the standard deviation grows faster (0.53), and then drops dramatically (0.25).

Thus risk does increase with holding period up to a point, but then favors long-term investors.

We seek distributions which model such phenomenon better.

The Fractal Market Hypothesis

It is fairly easy to observe the difference between the real distributions of returns and the normal distribution.

It is fairly easy to construct distributions which fit observed data better than normal.

However, what tends to be lacking is an explanation of how the distribution arises.

The Fractal Market Hypothesis (Peters, 1994) states:

- Markets are stable when they contain investors with large numbers of different time horizons, thus ensuring ample liquidity.

- Technical analysis factors are more important than fundamental factors in short time horizons. Fundamental factors become more important in long time horizons.

- If the validity of fundamental information changes, long-term investors either stop trading or trade on technical factors. However, the market becomes less stable without the long-term horizon investors.

- Prices reflect a combination of short-term and long-term valuations, where short-term valuations are more volatile.

- If a security has no tie to the economic cycle (e.g. currency), there is no long-term trend, so trading, liquidity, and short-term information dominate

R/S Analysis: Introduction

A central tool of fractal data modeling is *rescaled range* or *R/S analysis*.

We have seen how unbiased coin flipping leads to the the expected the difference between the number of heads and tails in an n tosses of an unbiased coin is $\approx c\sqrt{n}$.

This is characteristic of random walk models and Brownian motion, that $R \approx cT^h$, where R is the distance traveled, T is time, c is a constant and $h = 0.5$ is the exponent.

How could we discover this exponent h experimentally, for say, a run of coin flipping? We could build up time series $R_i = |(\sum_{j=1}^{i} c_i)|$, where $c_i = 1$ if the ith flip heads and $c_i = -1$ if the ith flip is tails.

By taking the log of both sides of the equation,

$$\log R = \log c + h \log T$$

Thus by doing a linear regression on the log scaled distance and time series we will discover c and, more importantly, h.

Hurst's Analysis

In any process modeled by an unbiased random walk, $h = 0.5$.

A hydrologist studying floods, H. E. Hurst, did similar analysis on 847 years worth of data on overflows of the Nile River, and found an exponent $h = 0.91$.

This implies that the accumulated deviation from expectation was growing much faster than expected from an unbiased random walk of *independent* observations.

Another way to think about this is that extreme events (e.g. long runs of heads/tails, or heavy floods) were more common than

It implied that the individual observations where not really independent – although they were not accurately modeled by a simple autoregressive process.

Hurst did a similar analysis of many different processes, including rainfall, temperature, and sunspot data, and consistently got $h \approx 0.7$. This quantity h is now called the *Hurst exponent*.

Interpretations of Hurst Exponents

A Hurst exponent of $h = 0.5$ implies an *independent* process. It does require that it be Gaussian, just independent. R/S analysis is *non-parametric*, meaning there is no assumption / requirement of the shape of the underlying distribution.

Hurst exponents of $0.5 < h \leq 1$ imply a *persistent* time series characterized by *long memory effects*.

Hurst exponents of $0 \leq h < 0.5$ imply an *anti-persistent* time series, which covers less distance than a random process. Such behavior is observed in *mean-reverting processes*, although that assumes that the process has a stable mean.

R/S Time Series Analysis

Given a time sequence of observations x_t, define the series

$$X(t,\tau) = \sum_{u=1}^{t} (x_t - \bar{x}_\tau)$$

where

$$\bar{x}_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} x_t$$

Thus $X(t,\tau)$ measures difference sum of the observations time 1 to t compared to the average of the first τ observations.

The *self-adjusted range* $R(\tau)$ is defined

$$R(\tau) = \max_{t=1}^{\tau} X(t,\tau) - \min_{t=1}^{\tau} X(t,\tau)$$

The standard deviation as a function τ is

$$S(\tau) = \sqrt{\left(\frac{1}{\tau} \sum_{t=1}^{\tau} (x_t - \bar{x}_\tau)^2\right)}$$

Finally, the *self-rescaled, self adjusted range* is

$$R/S(\tau) = R(\tau)/S(\tau)$$

Basically, it compares the largest amount of change happens over the initial τ terms to what would be expected given their variance.

The asymptotic behavior of $R/S(\tau)$ is provably

$$R/S(\tau) = ((\pi/2)\tau)^{1/2}$$

for any independent random process with finite variance.

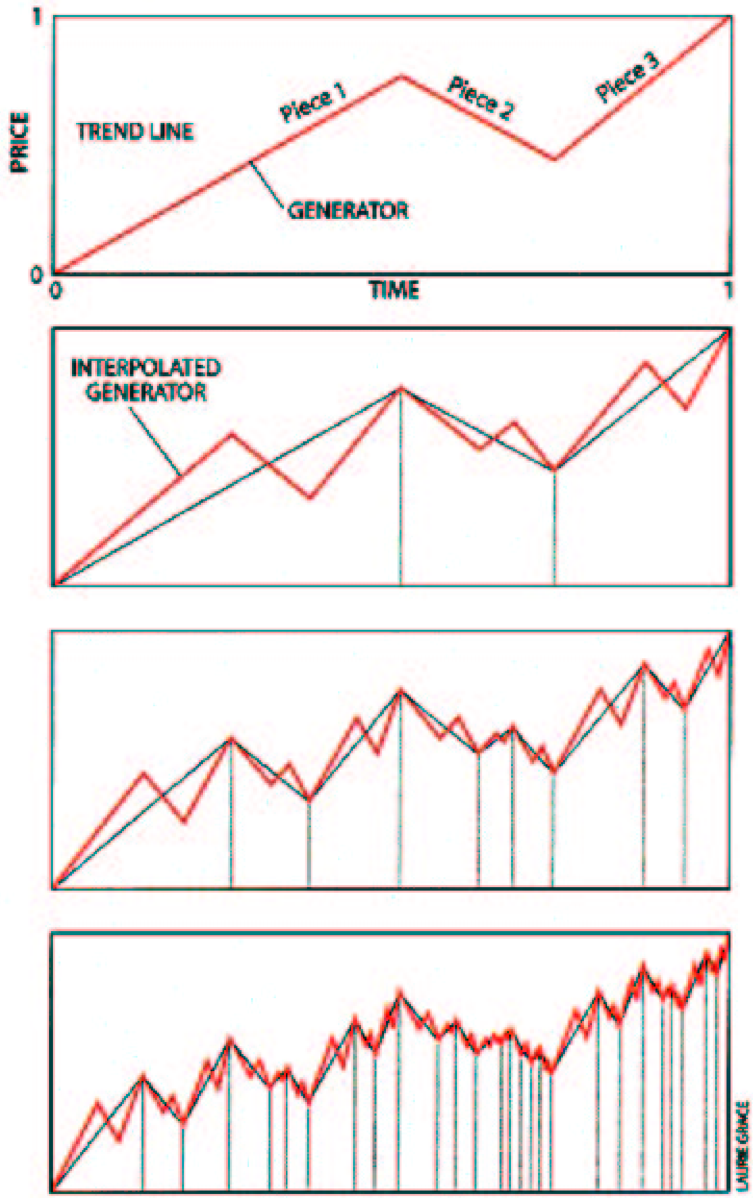
In plotting $\log(R/S(\tau))$ against $\log(\tau)$, we expect to get a line whose slope determines the Hurst exponent.

In practice for financial data, this line fits a straight line up to some τ, and then breaks down.

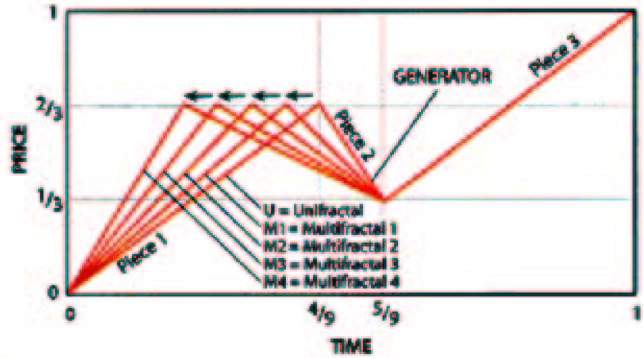
This τ gives some idea of a cycle time, over which there is dependence upon the past.

Generating Multi-fractal Time-Series

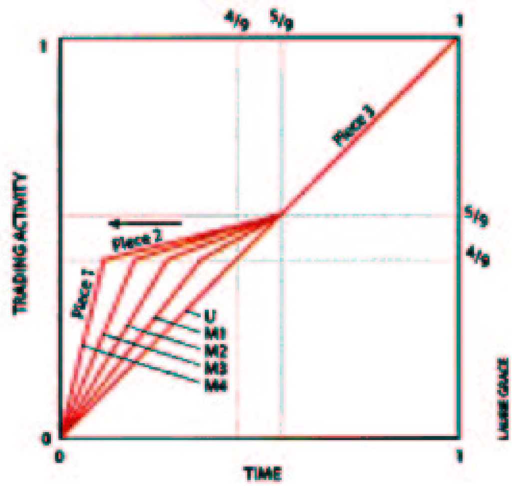
Mandelbrot proposed a multi-fractal generating process to generate time-series which have the persistence effects resembling financial time series:



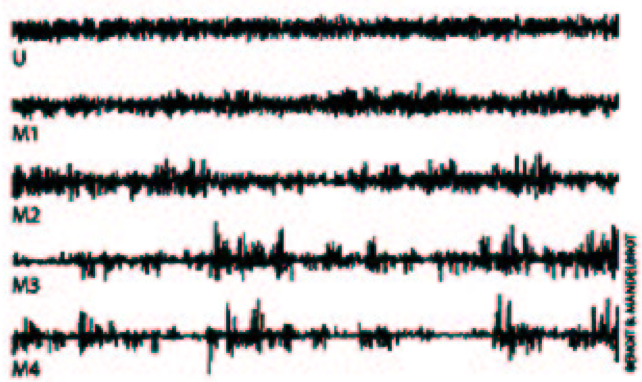
2 MOVING A PIECE of the fractal generator to the left . . .



3 . . . causes the same amount of market activity in a shorter time interval for the first piece of the generator and the same amount in a longer interval for the second piece . . .



4 . . . Movement of the generator to the left causes market activity to become increasingly volatile.



Straddle Strategies for Hurst Random Walks

Another process which yields burstier walks than coin-flipping states that we will repeat the previous step (up or down) with probability h and reverse with probability $1 - h$.

If $h = 0.5$ this is just coin flipping, but other exponents/probabilities yield interesting walks.

We found that $h = 0.57$ matched price behavior better than $h = 0.5$.

Consider the following betting strategy. Buy a stock now. Sell it when it gets up to price u, or drops down to price l.

For $h = 0.5$, is there a strategy (i.e. u, l pair) which will return an expected profit (e.g. 10, 100)? The answer is no – why?

What about $h > 0.5$? The answer is yes – why?

Is this a winning idea? Only if (1) prices really are modeled by Hurst walks, (2) transaction costs are insignificant, (3) there is not an upward trend where buy and hold becomes more profitable. . .