# The Fixed Fluctuation Model

In the fixed fluctuation model, all returns are either $\alpha$ or $1/\alpha$.

Such a model is consistent with our random walk model, although we picture the return sequences as being generated by a hostile rather than random adversary.

It can also be thought of *time scaling* model, where we consider each return of $\alpha$ or $1/\alpha$ as one step, regardless of how long it took to take that step.

An $(\alpha, n, k)$-adversary generates length-$n$ binary sequences on $(\alpha, 1/\alpha)$ where exactly $k$ individual returns are profitable.

Thus the optimal offline return is $\alpha^k$.

Let $FMM$ denote the optimal money making algorithm against this adversary and $R_\alpha(n, k)$ be its return. Then:

$$R(n, k) = \max_{b=0}^{1} \quad \min\{(b/\alpha + (1-b))R(n-1, k),$$
$$(\alpha b + (1-b))R(n-1, k-1)\}$$

for boundary conditions $R(n, 0) = 1$ and $R(n, n) = \alpha^n$.

# Results Against Fixed Fluctuation Adversaries

It can be proven that $FFM$ is always better than optimal offline buy and hold.

For the constant rebalancing strategy $(b, 1 - b)$, the optimal rebalancing constant

$$b = \frac{(n/k)(\alpha + 1) - 1}{\alpha - 1}$$

This constant rebalancing strategy is also better than buy and hold − however we assume no transaction costs.

The downside of $FMM$ and other provably money making algorithms is that they cannot invest much money in the earlier periods of the game, which limits total returns.

# Binomial Trees

This kind of recurrence gives rise to a graph of possible paths called a *binomial tree* within the finance literature:
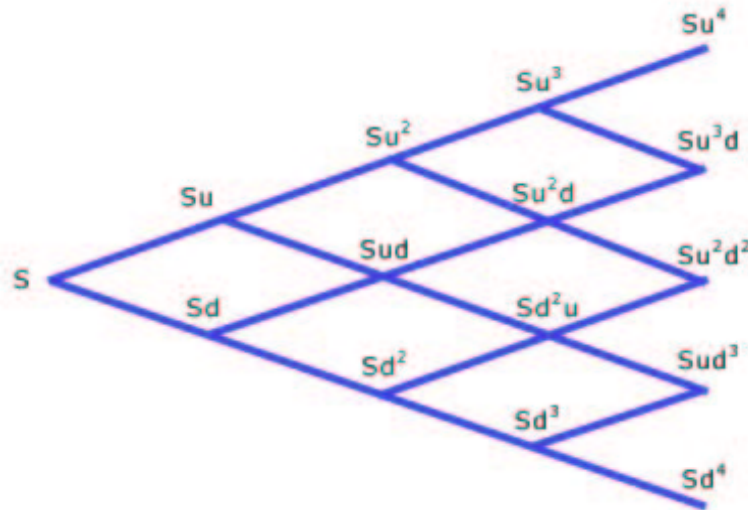


Binomial trees are used compute option prices in a similar manner as we are using them.

We assume that there is given probability of upward ($\beta$) and downward ($1 - \beta$) moves.

We can use an arbitrage argument to set the probability $\beta$ as a function of the risk-free rate.

At any point, investors can either (a) hold stock or (b) invest at the risk-free rate $r$.

A *risk-neutral* investor would not care which portfolio they owned if they had the same return.

Setting equal the returns from the stock ($\beta\alpha + (1 - \beta)/\alpha$) and the risk-free portfolio ($1 + r$), we can solve for $\beta$ to determine the *risk-neutral probability*.

# Risk Neutral Valuation

In truth, investors are not risk-neutral. In order to take the riskier investment they must be paid a premium.

Binomial trees are used to price option using the idea of *risk-neutral* **valuation**.

Suppose a stock price is currently at $20, and will either be at $22 or $18 in three months.

What is the price of a European call option for a strike price of $21? Clearly, this reduces to determining the probability of the upward price movement.

The risk-neutral investor argument for setting this probability can be applied if we set up two portfolios which are of provably of equal risk.

We will construct two riskless portfolios, one involving the stock and the other the risk-free rate.

# Using Options to Eliminate Risk

A *riskless* portfolio can be created by buying $\Delta$ shares of stock and selling a short position in 1 call option, such that the value of the portfolio is the same whether the stock moves up or down.

If the stock moves to $22, our portfolio will be worth $22\Delta - \$1 \cdot 1$, since we must pay the return of the option we sold.

If the stock moves to $18, our portfolio will be worth $18\Delta - \$0$, since the option we sold is worthless.

A riskless portfolio is constructed by buying $\Delta = 0.25$ shares, since it is the solution of $22\Delta - \$1 \cdot 1 = \$18\Delta$.

Whether the stock goes up or down, this portfolio is worth $4.5 at the end of the period.

The discounted value of this portfolio today, $V$, can be computed given the risk-free interest rate $r$.

Since the value of $V$ is equal to owning $\Delta = 0.25$ shares of stock at $20 per share minus the value $f$ of the option, $f = 20 \cdot 0.25 - V$.

# The General Case

In general, if there is an upward price movement, the value at the end of the option is

$$S_0 u \Delta - f_u$$

where $S_0 u$ ($f_u$) the price of the stock (option) after an upward movement.

If there is a downward rice movement, the value at the end of the option is

$$S_0 d \Delta - f_d$$

Setting them equal and solving for $\Delta$ yields

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

The present value of the portfolio with a risk-free rate of $r$ is

$$(S_0 u \Delta - f_u) e^{-rT}$$

which can be set up for a cost of $S_0 \Delta - f$.

Equating these two and solving for $f$ yields

$$f = S_0 \Delta - (S_0 u \Delta - f_u) e^{-rT}$$

By definition, the value of $f$ must also be

$$f = e^{-rT} \left( p f_u + (1 - p) f_d \right)$$

where $p$ is the probability of an upward movement.

Solving for $p$ we get

$$p = \frac{e^{rT} - d}{u - d}$$

# Pricing Options with Binomial Trees

Binomial trees are widely used to price options.

Adding additional levels to the trees allows finer price gradations than just a single up or down.

The value of the option can be worked backwards from the terminating (basis) condition level by level.

The value of the option on terminating level is clear since the stock price determined.

The price of an option generally converges for about $n = 30$ levels

This binomial tree model can be generalized to include the effects of (1) dividends, by changing the magnitude of the moves in the levels corresponding to dividend periods, (2) changing interest rates, by using the rate appropriate on a given yield curve.

It can also be generalized to allow more than two price movements from each node, say increase, decrease, and unchanged.

Once the number of levels gets too high for exhaustive computation, we can use Monte Carlo random walks to price the option.

Note that the number of options needed ($\Delta$) changes at each node/level in the binomial tree. Thus to maintain a riskless portfolio options must be bought and sold continuously, a process known as *delta hedging*.