The Kelly Criterion

How To Manage Your Money
When You Have an Edge

The First Model

- You play a sequence of games
- If you win a game, you win W dollars for each dollar bet
- If you lose, you lose your bet
- For each game,
  - Probability of winning is p
  - Probability of losing is $q = 1 - p$
- You start out with a bankroll of B dollars

The First Model, con't

- You bet some percentage, f, of your bankroll on the first game --- You bet fB
- After the first game you have B_1 depending on whether you win or lose
- You then bet the same percentage f of your new bankroll on the second game --- You bet fB_1
- And so on
- The problem is what should f be to maximize your "expected" gain
  - That value of f is called the Kelly Criterion

Kelly Criterion

- Developed by John Kelly, a physicist at Bell Labs
  - 1956 paper “A New Interpretation of Information Rate” published in the Bell System Technical Journal
    - Original title “Information Theory and Gambling”
  - Used Information Theory to show how a gambler with inside information should bet
- Ed Thorpe used system to compute optimum bets for blackjack and later as manager of a hedge fund on Wall Street
  - 1962 book “Beat the Dealer: A Winning Strategy for the Game of Twenty One”

Not So Easy

- Suppose you play a sequence of games of flipping a perfect coin
  - Probability is ½ for heads and ½ for tails
  - For heads, you win 2 dollars for each dollar bet
    - You end up with a total of 3 dollars for each dollar bet
  - For tails, you lose your bet
- What fraction of your bankroll should you bet
  - The odds are in your favor, but
    - If you bet all your money on each game, you will eventually lose a game and be bankrupt
    - If you bet too little, you will not make as much money as you could

Bet Everything

- Suppose that your bankroll is 1 dollar and, to maximize the expected (mean) return, you bet everything ($f = 1$)
- After 10 rounds, there is one chance in 1024 that you will have 59,049 dollars and 1023 chances in 1024 that you will have 0 dollars
  - Your expected (arithmetic mean) wealth is 57.67 dollars
  - Your median wealth is 0 dollars
- Would you bet this way to maximize the arithmetic mean of your wealth?

Winning W or Losing Your Bet

- You play a sequence of games
- In each game, with probability p, you win W for each dollar bet
- With probability $q = 1 – p$, you lose your bet
- Your initial bankroll is B
- What fraction, f, of your current bankroll should you bet on each game?

Win W or Lose your Bet, con't

- In the first game, you bet fB
  - Assume you win. Your new bankroll is $B_1 = B + WfB = (1 + fW)\,B$
  - In the second game, you bet fB_1: $fB_1 = f(1 + fW)\,B$
  - Assume you win again. Your new bankroll is $B_2 = (1 + fW)\,B_1 = (1 + fW)^2\,B$
  - If you lose the third game, your bankroll is $B_3 = (1 - f)\,B_2 = (1 + fW)^2 * (1 - f)\,B$

Win W or Lose your Bet, con't

- Suppose after *n* games, you have won w games and lost *l* games
  - Your total bankroll is
    $B_n = (1 + fW)^w * (1 - f)^l B$
  - The gain in your bankroll is
    $Gain_n = (1 + fW)^w * (1 - f)^l$
- Note that the bankroll is growing (or shrinking) exponentially

Win W or Lose your Bet, con’t

- The possible values of your bankroll (and your gain) are described by probability distributions
- We want to find the value of *f* that maximizes, in some sense, your “expected” bankroll (or equivalently your “expected” gain)
- There are two ways we can think about finding this value of *f*
  - They both yield the same value of *f* and, in fact, are mathematically equivalent
- We want to find the value of *f* that maximizes
  - The geometric mean of the gain
  - The arithmetic mean of the log of the gain

Finding the Value of *f* that Maximizes the Geometric Mean of the Gain

- The geometric mean, G, is the limit as n approaches infinity of the n^{th} root of the gain
  $G = lim_{n\text{->}oo} ((1 + fW)^{w/n} * (1 - f)^{l/n})$

  which is

  $G = (1 + fW)^p * (1 - f)^q$
- For this value of G, the value of your bankroll after n games is
  $B_n = G^n * B$

The Intuition Behind the Geometric Mean

- If you play *n* games with a probability *p* of winning each game and a probability *q* of losing each game, the expected number of wins is *pn* and the expected number of loses is *qn*
- The value of your bankroll after *n* games

  $B_n = G^n * B$

  is the value that would occur if you won exactly *pn* games and lost exactly *qn* games

Finding the Value of *f* that Maximizes the Geometric Mean of the Gain, con't

- To find the value of *f* that maximizes *G*, we take the derivative of

  $G = (1 + fW)^p * (1 - f)^q$

  with respect to f, set the derivative equal to 0, and solve for *f*

  $(1 + fW)^p *(-q (1 - f)^{q-1}) + Wp(1 + fW)^{p-1} * (1 - f)^q = 0$

- Solving for f gives

  $f = (pW - q) / W$

  $= p - q / W$

- This is the Kelly Criterion for this problem

Finding the Value of *f* that Maximizes the Arithmetic Mean of the Log of the Gain

- Recall that the gain after *w* wins and *l* losses is
  $Gain_n = (1 + fW)^w * (1 - f)^l$
- The log of that gain is

  $\log(Gain_n) = w * \log(1 + fW) + l * \log(1 - f)$

- The arithmetic mean of that log is the

  $\lim_{n \to \infty} (w/n * \log(1 + fW) + l/n * \log(1 - f))$

  which is

$p * \log(1 + fW) + q * \log(1 - f)$

Finding the Value of *f* that Maximizes the Arithmetic Mean of the Log of the Gain, con't

- To find the value of *f* that maximizes this arithmetic mean we take the derivative with respect to *f*, set that derivative equal to 0 and solve for *f*

  $$pW / (1 + fW) - q / (1 - f) = 0$$

- Solving for *f* gives

  $$f = (pW - q) / W$$
  $$= p - q / W$$

- Again, this is the Kelly Criterion for this problem

Equivalent Interpretations of Kelly Criterion

- The Kelly Criterion maximizes
  – Geometric mean of wealth
  – Arithmetic mean of the log of wealth

Relating Geometric and Arithmetic Means

- Theorem

  The log of the geometric mean of a random variable equals the arithmetic mean of the log of that variable

Intuition About the Kelly Criterion for this Model

- The Kelly criterion

  $f = (pW - q) / W$

  is sometimes written as

  $f = \mathit{edge} / \mathit{odds}$

  - *Odds* is how much you will win if you win
    - At racetrack, *odds* is the tote-board odds
  - *Edge* is how much you expect to win
    - At racetrack, *p* is your inside knowledge of which horse will win

Examples

- For the original example *($W = 2$, $p = ½$)*
  f = *.5 - .5 / 2 = .25*

  G = 1.0607
  - After 10 rounds (assuming *B = 1*)
    - Expected (mean) final wealth = 3.25
    - Median final wealth = 1.80
- By comparison, recall that if we bet all the money (f = 1)
  - After 10 rounds (assuming B = 1)
    - Expected (mean) final wealth = 57.67
    - Median final wealth = 0

More Examples

- If $pW - q = 0$, then $f = 0$
  - You have no advantage and shouldn’t bet anything
  - In particular, if $p = 1/2$ and $W = 1$, then again $f = 0$

Winning W or Lose L (More Like Investing)

- If you win, you win *W* If you lose, you lose *L.*
  L is less than 1
- Now the geometric mean, G, is
  $G = (1 + fW)^p * (1 - fL)^q$
- Using the same math, the value of *f* that maximizes *G* is
  $f = (pW - qL)/WL$
  $= p/L - q/W$
- This is the Kelly Criterion for this problem

An Example

- Assume *p = ½, W = 1, L = 0.5*. Then

  f = .5

  G = 1.0607
- As an example, assume *B = 100*. You play two games
  - Game 1 you bet *50* and lose (*25*) . *B* is now 75
  - Game 2 you bet ½ of new *B* or *37.50*. You win. *B* is *112.50*
- By contrast, if you had bet your entire bankroll on each game,
  - After Game 1, *B* would be 50
  - After Game 2, *B* would be *100*

Shannon's Example

- Claude Shannon (of Information Theory fame) proposed this approach to profiting from random variations in stock prices based on the preceding example
- Look at the example as a stock and the "game" as the value of the stock at the end of each day
  - If you "win," since $W = 1$, the stock doubles in value
  - If you "lose." since $L = ½$, the stock halves in value

Shannon’s Example, con”t

- In the example, the stock halved in value the first day and then doubled in value the second day, ending where it started
  - If you had just held on to the stock, you would have broken even
- Nevertheless Shannon made money on it
  - The value of the stock was never higher than its initial value, and yet Shannon made money on it
    - His bankroll after two days was 112.50
  - Even if the stock just oscillated around its initial value forever, Shannon would be making $(1.0607)^n$ gain in *n* days

An Interesting Situation

- If L is small enough, f can be equal to 1 (or even larger)
  - You should bet all your money

An Example of That Situation

- Assume $p = ½$, $W = 1$, $L = 1/3$. Then

  $f = 1$

  $G = 1.1547$

- As an example, assume $B = 100$. You play two games
  - Game 1 you bet *50* and lose 16.67 B is now *66.66*
  - Game 2 you bet *66.66* and win. B is now *133.33*

A Still More Interesting Situation

- If *L* were *0* (you couldn’t possibly lose) *f* would be infinity
  - You would borrow as much money as you could (beyond your bankroll) to bet all you possibly could.

N Possible Outcomes (Even More Like Investing)

- There are n possible outcomes, x_i ,each with probability p_i
  - You buy a stock and there are n possible final values, some positive and some negative
- In this case

$$G = \prod (1 + f x_i)^{p_i}$$

N Outcomes, con't

- The arithmetic mean of the log of the gain is

$$R = \log G = \sum p_i * \log(1 + fx_i)$$

- The math would now get complicated, but if $fx_i << 1$ we can approximate the log by the first two terms of its power expansion

$$\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \ldots$$

to obtain

$$R = \log G \approx f \sum p_i x_i - f^2 \sum p_i x_i^2 / 2$$

N Outcomes, con't

- Taking the derivative, setting that derivative equal to 0, and solving for *f* gives

$$f = (\sum p_i x_i)/(\sum p_i x_i{}^2)$$

which is very close to the *mean/variance*

- The variance is

$$\sum p_i x_i{}^2 - (\sum p_i x_i)^2$$

Properties of the Kelly Criterion

- Maximizes
  - The geometric mean of wealth
  - The arithmetic mean of the log of wealth
- In the long term (an infinite sequence), with probability 1
  - Maximizes the final value of the wealth (compared with any other strategy)
  - Maximizes the median of the wealth
    - Half the distribution of the “final” wealth is above the median and half below it
  - Minimizes the expected time required to reach a specified goal for the wealth

Fluctuations using the Kelly Criterion

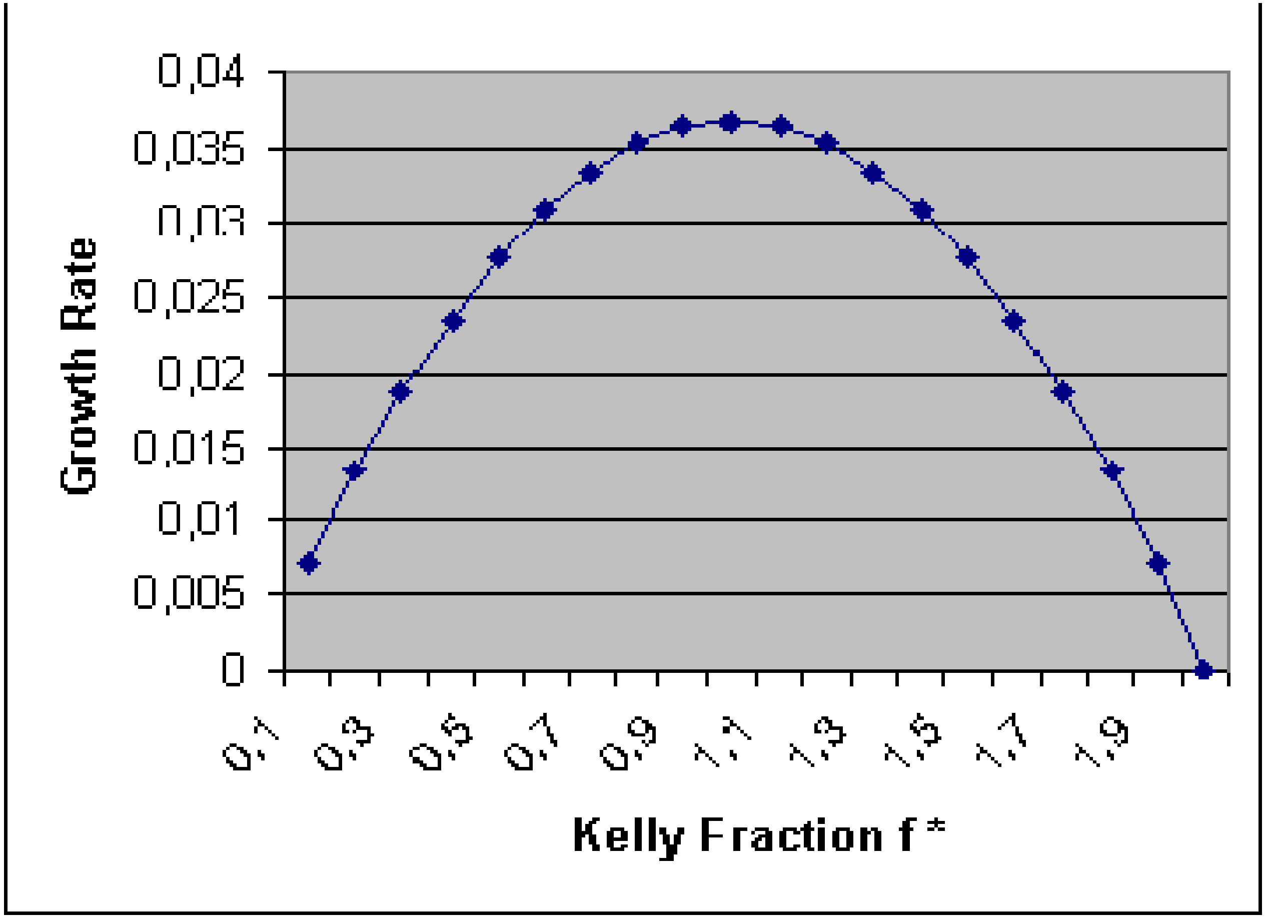
- The value of *f* corresponding to the Kelly Criterion leads to a large amount of volatility in the bankroll
  - For example, the probability of the bankroll dropping to *1/n* of its initial value at some time in an infinite sequence is *1/n*
    - Thus there is a 50% chance the bankroll will drop to ½ of its value at some time in an infinite sequence
  - As another example, there is a 1/3 chance the bankroll will half before it doubles

An Example of the Fluctuation

Varying the Kelly Criterion

- Many people propose using a value of *f* equal to ½ (half Kelly) or some other fraction of the Kelly Criterion to obtain less volatility with somewhat slower growth
  - Half Kelly produces about 75% of the growth rate of full Kelly
  - Another reason to use half Kelly is that people often overestimate their edge.

Growth Rate for Different Kelly Fractions



Some People Like to Take the Risk of Betting More than Kelly

- Consider again the example where $p = ½, W = 1, L = 0.5, B = 100$

  Kelly value is $f = .5 \quad G = 1.060$

  Half Kelly value is $f = .25 \quad G = 1.0458$

- Suppose we are going to play only 4 games and are willing to take more of a risk

  We try $f = .75 \quad G = 1.0458$

  and $f = 1.0 \quad G = 1.0$

All the Possibilities in Four Games

		Final Bankroll			
		½ Kelly	**Kelly**	**1½ Kelly**	**2 Kelly**
		f = .25	**f = .5**	**f = .75**	**f = 1.0**
4 wins 0 losses	(.06)	244	506	937	1600
3 wins 1 loss	(.25)	171	235	335	400
2 wins 2 losses	(.38)	120	127	120	100
1 win 3 losses	(.25)	84	63	43	25
0 wins 4 losses	(.06)	59	32	15	6
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Arithmetic Mean		128	160	194	281
Geometric Mean		105	106	105	100

I Copied This From the Web

If I maximize the expected square-root of wealth and you maximize expected log of wealth, then after 10 years you will be richer 90% of the time. But so what, because I will be much richer the remaining 10% of the time. After 20 years, you will be richer 99% of the time, but I will be fantastically richer the remaining 1% of the time.

The Controversy

- The math in this presentation is not controversial
- What is controversial is whether you should use the Kelly Criterion when you gamble (or invest)
  - You are going to make only a relatively short sequence of bets compared to the infinite sequence used in the math
    - The properties of infinite sequences might not be an appropriate guide for a finite sequence of bets
    - You might not be comfortable with the volatility
  - Do you really want to maximize the arithmetic mean of the log of your wealth (or the geometric mean of your wealth)?
    - You might be willing to take more or less risk

Some References

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- http://www-stat.wharton.upenn.edu/~steele/Courses/434F2005/Context/Kelly%20Resources/Samuelson1979.pdf
  - Famous paper that critiques the Kelly Criterion in words of one syllable
- http://en.wikipedia.org/wiki/Kelly_criterion
- http://www.castrader.com/kelly_formula/index.html
  - Contains pointers to many other references