Spring 2006

Hole-Free Polygons and Guarding

Reading Group: March 10, 2006 *Scribe: Girishkumar Sabhnani and Joe Mitchell*

1 Overview

In the October 28, 2005 meeting, Joe Mitchell defined a new class of simple polygons called *hole-free polygons* with the motivation that this new class of polygons may have enough special structure that there is a polynomial-time algorithm to compute the minimum number of guards, or at least a constant factor approximation. In general simple polygons, the *Art Gallery/Guarding* problem is *NP-Hard*. We also study the relationship between hole-free polygons and "streets."

2 Hole-Free Polygons

Definition 1 *A* hole-free polygon *is a simple polygon P in which a hole cannot be created by illuminating a finite set of points in P. More precisely, the union, $\cup_i VP(p_i)$ of the visibility polygons for any finite set $S = \{p_1, \ldots, p_k\} \subset P$ of points in P does not have a hole (necessarily convex) fully interior to P.*

Note that a hole in $\cup_i VP(p_i)$ must be convex since it is created by the intersection of half-planes. Also a hole-free polygon must have at least six vertices as at least three reflex vertices are needed to form the three sides of the convex hole. Figure 1 shows a hole-free polygon with the minimum number of vertices (6).

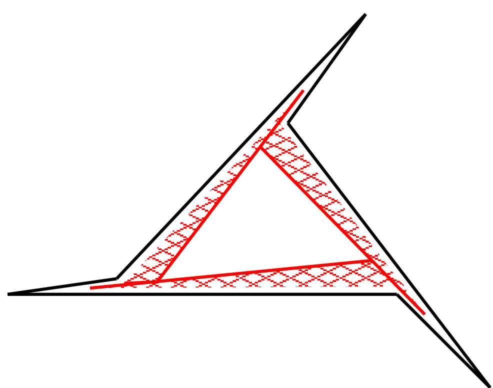

Figure 1: Godfried's favorite polygon serves as example of a hole-free polygon with the fewest possible vertices (6).

2.1 Detection of Hole-Free Polygons

We have an $O(n^4)$ algorithm to detect if a polygon is hole-free.

Theorem 2 *Given a simple polygon P with n vertices, one can detect if P is hole-free in time $O(n^4)$.*

Proof To be provided. $\square$

2.2 Star-Shaped vs. Hole-Free Polygons

Theorem 3 *If P is star-shaped, then it must be hole-free.*

Proof To be provided. □

Note that a hole-free polygon need not be star-shaped; see Figure 3 (Right).

3 Relationship with Streets

Streets are a special class of simple polygons with structure that makes them easier to handle in some problems.

Definition 4 *A* street *is a simple polygon whose boundary can be partitioned into two chains such that illluminating either one of the chains illuminates the entire polygon.*

There are two special types of streets – *walkable streets* and *straight-walkable streets*.

Definition 5 *There is a* walk *in a simple polygon P, if there exist two points, s and t, on the boundary of P such that two points can be moved from s to t, each on one of the boundary chains, such that the line segment connecting the two points is always fully contained in the polygon. See Figure 2.*

Definition 6 *A* walkable street *is a street in which there exists a walk.*

Definition 7 *A* straight-walkable street *is a street in which there exists a walk for which the two points move monotonically from s to t 9one cw, one ccw), never backtracking.*

Figure 2 shows examples of a walkable street, a straight-walkable street, and a street that is not walkable.

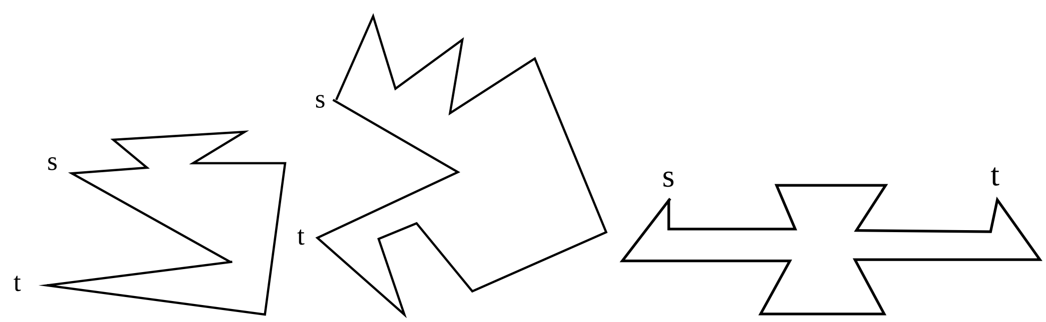

Figure 2: Left: Walkable Street, Center: Straight-Walkable Stree, Right: A Street that is not walkable

Contradicting the section name, hole-free polygons are not well related with the class of streets, as we have examples showing hole-free polygons that are streets and that are not, and vice-versa. Refer to Figures 3 4.

4 Hardness Results

It is known (Megiddo and Tamir) that it is NP-hard to cover a set of lines in the plane with a minimum number of points such that at least one point lies on every line. This fact gives an easy proof of NP-hardness for computing a minimum-cardinality set of guards in a simple polygon, using a construction known as a *spike box*, as shown in Figure 5. It is a bounding box surrounding all the intersection points of the arrangement, with skinny triangles ("spikes") protruding from the rectangle where the lines extend outside the rectangle. Thus, if an efficient algorithm exists to guard the spike

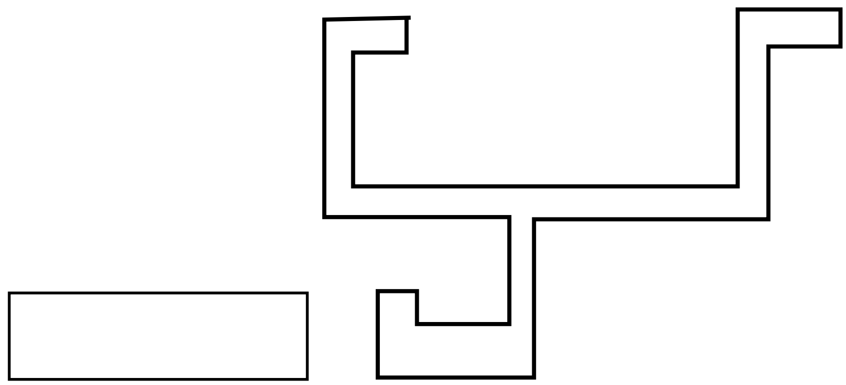

Figure 3: Left:Hole-free polygon that is a street, Right: Hole-free polygon that is not a street

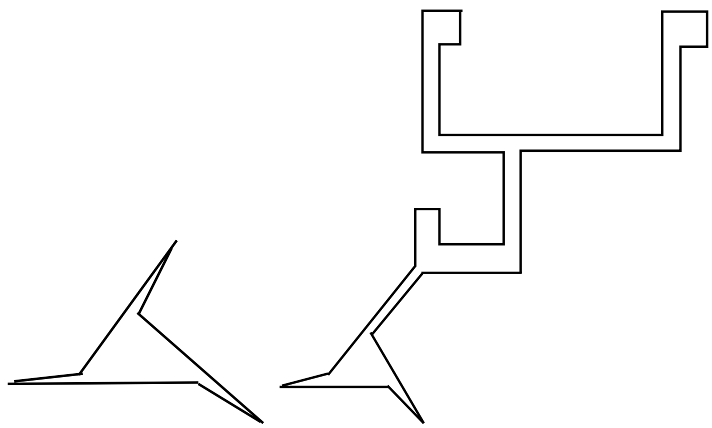

Figure 4: Left:A street that is not hole-free, Right: A polygon that is neither hole-free nor a stree.

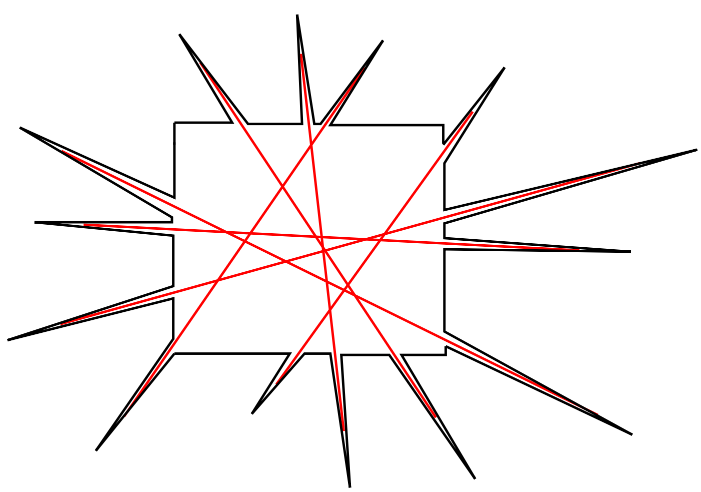

Figure 5: Spike box showing hardness for guarding simple polygons.

box we have solved the earlier problem because if all the spikes are seen by the guards, those points will also cover all the lines.

Thus, guarding the simple polygons in general is *NP-Hard*.

One can see that a spike box itself is a street, and hence even guarding streets is NP-hard. (This is also shown, by a different proof, in Nilsson's thesis.)

We observe that a reduced spike box construction shows that guarding a walkable street is also NP-hard, as shown in Figure 6.

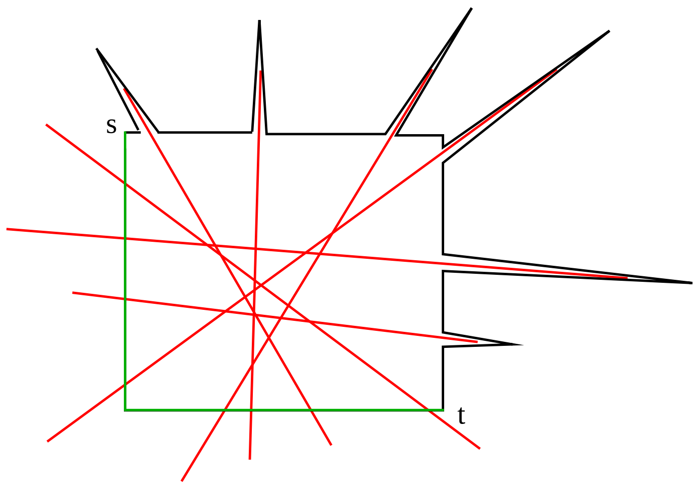

Figure 6: A reduced spike box showing that a walkable street is NP-hard to guard.

It is an open problem to determine the complexity of guarding straight-walkable streets.

5 Summary

The definition of the new class of hole-free polygons seems interesting, and not much is known about guarding such polygons. It would be interesting to find an algorithm that computes an optimal guard set for hole-free polygons, or even a constant-factor approximation of the optimal guard set.