Reading Group Notes, 9/26/03

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1 A TSP Conjecture of Moshe Dror

Let $S = \{p_1, \ldots, p_n\}$ be a set of n cities, given as points in the Euclidean plane. Let ℓ^* denote the optimal TSP tour length on S, and let ℓ_i^* denote the optimal TSP tour length on the set $S \setminus p_i$.

A conjecture of M. Dror states that, provided $n \geq 3$,

$$\sum_i (\ell^* - \ell_i^*) \leq \ell^*.$$

Restated, our goal is to show that

$$(n-1)\ell^* \leq \sum_i \ell_i^*.$$

Consider the union, U, of optimal tours, T_i^*, on the sets $S \setminus p_i$, for $i = 1, \ldots, n$. Then the degree in U of each node p_i is exactly $2(n-1)$, since each p_i is passed through in exactly $(n-1)$ of the optimal tours. Thus, U represents an Eulerian set of edges (many edges are repeated, possibly), and it is clearly connected. The total length of U is $\sum_i \ell_i^*$. Our goal is to show that the length of U is *at least* $(n-1)\ell^*$. For this, it would suffice to show that we can extract from U a set of $(n-1)$ tours (of all n cities).

Joe Mitchell: *An erroneous "proof":* An Eulerian traversal of U results in an $(n-1)$-repetitive tour, passing through each node p_i $(n-1)$ times. Thus, its length is at least $(n-1)\ell^*$. This concludes the proof.

Alper Yildirim: *Another erroneous "proof":* Another way to finish the proof would be to use Dirac's theorem, which states that any graph with $n \geq 3$ vertices has a Hamilton circuit (a circuit that goes through all the vertices) if the degree of each vertex is at least $n/2$. Since the graph given by the union of n tours has only n vertices and the degree of each vertex is $2(n-1)$, we have that $2(n-1) \geq n/2$ if $n \geq 2$. Thus, one can successively find and remove $(n-1)$ TSP tours. Clearly, each tour is at least as long as the optimal tour. The result follows.

George Hart: In general, a tour which visits each node k times need not be k times as long as the optimal Hamiltonian tour.

Himanshu Gupta: I think the original argument has a flaw: An Eulerian traversal of U results in an $(n-1)$-repetitive tour, passing each node $(n-1)$ times. BUT, that doesn't necessarily mean that it can be represented as a *sequence* (or even a union of) of $(n-1)$ non-repetitive tours, unless you can further characterize the Eulerian traversal of U (as Alper tried). Or is it obvious?

Now, Alper's argument also has a flaw. You can apply Dirac's theorem to remove at most $(3/4)n$ tours after which the degree of each node in reduced U becomes less than $n/2$. In other words, $2(n-1) - 2k > n/2$ for $k < (3/4n - 1)$ only, as each removal of tour reduces the degree of each vertex by 2.

BTW, Dirac's theorem doesn't require connectivity. The following web-page has a beautiful proof for the theorem: http://www.math.uwaterloo.ca/navigation/ideas/articles/honsberger/index.shtml

Alper Yildirim: I agree with Himanshu. I was also not completely convinced that one could produce n-1 TSP tours (i.e., Hamilton circuits) from an Euler cycle. That is why I thought Dirac's theorem would convince me. However, as both George and Himanshu pointed out correctly, that does not work, either, since after removing $3n/4$ tours, the sufficient condition is not satisfied anymore. Furthermore, Dirac's theorem also assumes a connected graph. I am not quite convinced that after each removal of a TSP tour, the resulting graph is still connected. Thanks for the corrections.

Estie Arkin: There is another easy argument that shows that if each node of a graph has degree $\geq n/2$ then the graph is connected. To prove this, suppose by contradiction that the graph is not connected. Look at the smallest connected component; it must have $n/2$ or fewer nodes, so the degrees of nodes in this component are $< n/2$ (as the node degrees are at most the number of nodes in the component minus one).

George Hart: Dirac's theorem applies to graphs, not multigraphs. But the construction is a multigraph—multiple arcs can join a given pair of vertices.

Karl Juhnke: Who says that an Eulerian tour that visits each point in a graph $n-1$ times is at least $n-1$ times as long as the optimal TSP tour?

I'm thinking of a rectangle ABCD with AB=CD=1 and BC=DA=100. Now I have a graph with the edges AB and CD each present five times, and BC and DA each present once. Each node has degree six. There is an Eulerian tour which visits each node three times, namely A-B-A-B-A-B-C-D-C-D-C-D-A, but it is **not** three times as long as the optimal TSP on ABCD.

Furthermore, Alper's way of finishing the proof doesn't work either, because Dirac's theorem only applies when parallel edges aren't allowed, and your construction will have zillions of parallel edges. Plus, even if we could find and remove several Hamilton cycles, the conditions for finding a further cycle would not be true any more, i.e. when the node of each degree is reduced to less than $n/2$ by removal of cycles, we might not be able to find more cycles. Indeed, the graph might not even stay connected!

I still believe the conjecture, but the proof doesn't convince me yet.

Saurabh Sethia: Note that it is easy to see that the length of U is at least $(n-2)\ell^*$. We simply add to each T_i^* a doubled edge linking p_i to its nearest city (at distance δ_i), thereby obtaining a tour on all of S. Thus,

$$\ell^* \leq \ell_i^* + 2\delta_i,$$

so summing over i we get

$$n\ell^* \leq \sum_i \ell_i^* + 2\sum_i \delta_i,$$

which gives, using the fact that $\sum_i \delta_i \leq \ell^*$,

$$(n-2)\ell^* \leq \sum_i \ell_i^*.$$

Rewriting this, we get that the sum of the marginals is at most *twice* the optimal length:

$$\sum_i (\ell^* - \ell_i^*) \leq 2\ell^*$$

Joe Mitchell: So we checked and the Petersen graph example (all edges have length 1, and then make it a complete graph by adding edges of length 2 – this exactly satisfies triangle inequality) seems to be a counterexample for the non-geometric case, as suspected: marginal costs of 2 per node (total of 20), versus $\ell^* = 11$.

Note that we have a simple proof that the sum of marginals is at most $2\ell^*$. Thus, the Petersen graph is pretty close to tight (factor 2).

Valentin Polishchuk: Actually, any hypohamiltonian graph G serves as an example. If it has n nodes, than OPT is $n+1$ (take a HC in $G \setminus 1$, as soon as the HC passes through a node, adjacent to 1, go to 1 and go back to the HC). The $f(i)$'s are all 2 then. It's known that for all but finitely n there exist hypohamiltonian graphs on n nodes, e.g., $P(n, 2)$ are for $n = 5(mod6)$. $P(n, 2)$ are very much alike the graphs from Estie's example – just the nodes in the inner circle are connected in the "every other" way, to form a "star".

Moshe Dror: As for the motivation: I have been doing research on vehicle routing in the past. In particular, propane deliveries. This involves selecting each day a set of points (customers). However it is not clear at all how to price out a cost of visiting a customer. This lead me to issues of cost allocation (cooperative game theory). It is important in this context to know the marginal costs and their relation to the cost of a tour.

Another observation (perhaps useful) for the optimal tour, say T^*, and the marginal values: $m(i)+m(j) \leq 2c(i,j)$, for $(i,j) \in T^*$. That is, any two consecutive marginal values on the optimal tour have to be less or equal 2 times the distance between them.

Dimitris Papamichail I think that the conjecture not only holds for convex polygons, but for the super-class of polygons that, for any node i, the TSP-i is sharing n-2 edges with the TSP and its last edge connects the neighbors of i in the TSP. And I think proving that does not need any euclidean space arguments, but only the triangle inequality.

Joe: Right. I recall that this observation was also made at reading group (perhaps you mentioned, or Petr did?). If the marginals come from tours that are nearly identical (all but the 2 edges) to the full tour, then the problem is easy.

Valentin I think that the following 6 points provide a counterexample to the TSP conjecture in 3D for the L_1 metric:

-1 0 0 ; 1 0 0 ; 0 1 0 ; -1 0 1; 1 0 1 ; 0 1 1 .

In the L_1 metric, OPT is 10, e.g., 1 2 3 6 5 4 1.

The optimal solution on $S \setminus 6$ is 8, e.g., 1 3 2 5 4 1. Thus, $f(6) = 2$ and the sum of the marginals is 12, which is *greater* than 10 (the OPT).

This counterexample is based on a 6-node counterexample for general graphs which Karl and I found some time ago.

Karl: Valentin, your pictures got me started trying to embed the Petersen Graph. I believe that in $\Re^6$ with the L_1 metric, the following points constitute a Petersen Graph where the graph edges have length 2 and the edges in the complement have length 4:

(1, 1, 1, 0, 0, 0) (1, 0, 1, 1, 0, 0) (1, 0, 0, 1, 1, 0) (1, 0, 0, 0, 1, 1) (1, 1, 0, 0, 0, 1) (0, 1, 1, 0, 1, 0) (0, 0, 1, 1, 0, 1) (0, 1, 0, 1, 1, 0) (0, 0, 1, 0, 1, 1) (0, 1, 0, 1, 0, 1)

I mention this because it appears to me to be a counterexample to the TSP conjecture also for the L_2 metric, and L_3, and L_4, etc. The best TSP on all ten nodes is 9 shorts plus 1 long, whereas removing any one node reduces the best TSP to 9 shorts, for a total savings of 10 longs.

Valentin: In Karl's example in L_p the opt is $9 * 2^{1/p} + 4^{1/p}$ (indeed, 9 short and 1 long edge), so $f(i)$'s are $4^{1/p}$ and their sum is $10 * 41/p > 9 * 2^{1/p} + 4^{1/p} = opt$, so it is indeed a counterexample for L_p metric for any p!

Reading Group, 11/21/03 Karl pointed out another example based on the prism, which shows that the TSP conjecture fails to hold for the L_2 metric in 3D: Consider a unit equilateral triangular (right) prism of height 0.75. (The triangle's sides are each of unit length.) The length of an optimal tour is 5.5 (four unit edges of the triangles, plus two vertical edges), while the sum of the marginals is 1, as is easily checked. (We chose the height to be as small as possible so that the optimal tour has the claimed form.)