Lecture 2:
Asymptotic Notation

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Problem of the Day

The *knapsack problem* is as follows: given a set of integers $S = \{s_1, s_2, \ldots, s_n\}$, and a given target number T, find a subset of S which adds up exactly to T. For example, within $S = \{1, 2, 5, 9, 10\}$ there is a subset which adds up to $T = 22$ but not $T = 23$.

Find counterexamples to each of the following algorithms for the knapsack problem. That is, give an S and T such that the subset is selected using the algorithm does not leave the knapsack completely full, even though such a solution exists.

Solution

- Put the elements of S in the knapsack in left to right order if they fit, i.e. the first-fit algorithm?
- Put the elements of S in the knapsack from smallest to largest, i.e. the best-fit algorithm?
- Put the elements of S in the knapsack from largest to smallest?

The RAM Model of Computation

Algorithms are an important and durable part of computer science because they can be studied in a machine/language independent way.
This is because we use the RAM model of computation for all our analysis.

- Each "simple" operation (+, -, =, if, call) takes 1 step.
- Loops and subroutine calls are *not* simple operations. They depend upon the size of the data and the contents of a subroutine. "Sort" is not a single step operation.

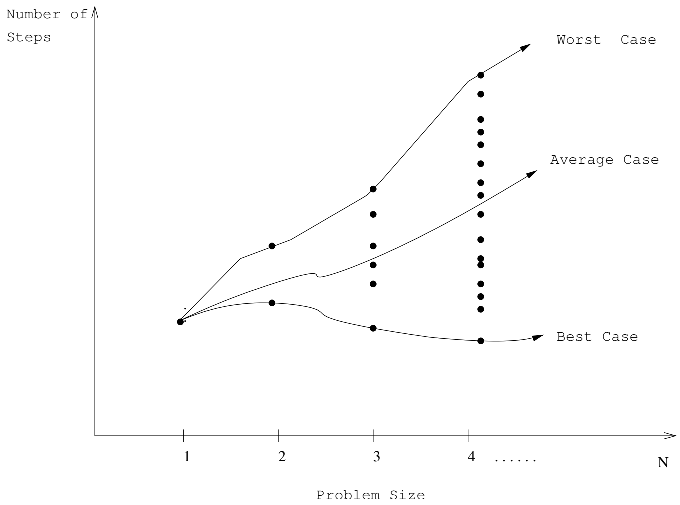
- Each memory access takes exactly 1 step.

We measure the run time of an algorithm by counting the number of steps.

This model is useful and accurate in the same sense as the flat-earth model (which *is* useful)!

Worst-Case Complexity

The *worst case complexity* of an algorithm is the function defined by the maximum number of steps taken on any instance of size n.



Best-Case and Average-Case Complexity

The *best case complexity* of an algorithm is the function defined by the minimum number of steps taken on any instance of size n.

The *average-case complexity* of the algorithm is the function defined by an average number of steps taken on any instance of size n.

Each of these complexities defines a numerical function: time vs. size!

Our Position on Complexity Analysis

What would the reasoning be on buying a lottery ticket on the basis of best, worst, and average-case complexity?

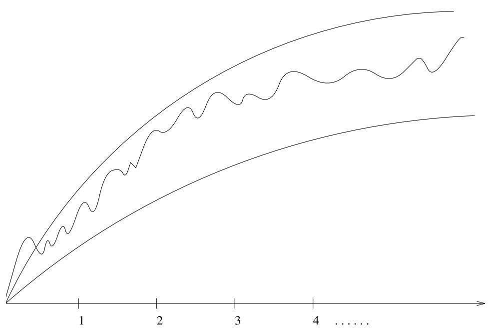
Generally speaking, we will use the worst-case complexity as our preferred measure of algorithm efficiency.

Worst-case analysis is generally easy to do, and "usually" reflects the average case. Assume I am asking for worst-case analysis unless otherwise specified!

Randomized algorithms are of growing importance, and require an average-case type analysis to show off their merits.

Exact Analysis is Hard!

Best, worst, and average case are difficult to deal with because the *precise* function details are very complicated:



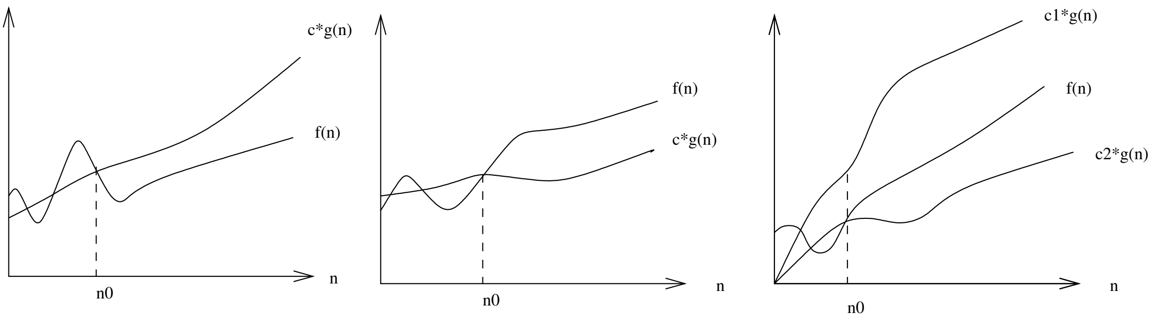
It easier to talk about *upper and lower bounds* of the function. Asymptotic notation (O, Θ, Ω) are as well as we can practically deal with complexity functions.

Names of Bounding Functions

- $g(n) = O(f(n))$ means $C \times f(n)$ is an *upper bound* on $g(n)$.
- $g(n) = \Omega(f(n))$ means $C \times f(n)$ is a *lower bound* on $g(n)$.
- $g(n) = \Theta(f(n))$ means $C_1 \times f(n)$ is an upper bound on $g(n)$ and $C_2 \times f(n)$ is a lower bound on $g(n)$.

C, C_1, and C_2 are all constants independent of n.

O, Ω, and Θ



(a) (b) (c)

The definitions imply a constant n_0 *beyond which* they are satisfied. We do not care about small values of n.

Formal Definitions

- $f(n) = O(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or below $c \cdot g(n)$.
- $f(n) = \Omega(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or above $c \cdot g(n)$.
- $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1, and c_2 such that to the right of n_0, the value of $f(n)$ always lies between $c_1 \cdot g(n)$ and $c_2 \cdot g(n)$ inclusive.

Big Oh Examples

$$3n^2 - 100n + 6 \ = \ O(n^2) \ because \ 3n^2 > 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 \ = \ O(n^3) \ because \ .01n^3 > 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 \ \neq \ O(n) \ because \ c \cdot n < 3n^2 \ \ when \ \ n > c$$

Think of the equality as meaning *in the set of functions*.

Big Omega Examples

$$3n^2 - 100n + 6 \;=\; \Omega(n^2) \; because \; 2.99n^2 < 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 \;\neq\; \Omega(n^3) \; because \; 3n^2 - 100n + 6 < n^3$$
$$3n^2 - 100n + 6 \;=\; \Omega(n) \; because \; 10^{10^{10}} n < 3n^2 - 100n + 6$$

Big Theta Examples

$$3n^2 - 100n + 6 = \Theta(n^2)\ because\ O\ \ and\ \ \Omega$$
$$3n^2 - 100n + 6 \neq \Theta(n^3)\ because\ O\ \ only$$
$$3n^2 - 100n + 6 \neq \Theta(n)\ because\ \Omega\ \ only$$

Big Oh Addition/Subtraction

Suppose $f(n) = O(n^2)$ and $g(n) = O(n^2)$.

- What do we know about $g'(n) = f(n) + g(n)$? Adding the bounding constants shows $g'(n) = O(n^2)$.
- What do we know about $g''(n) = f(n) - |g(n)|$? Since the bounding constants don't necessary cancel, $g''(n) = O(n^2)$

We know nothing about the lower bounds on g' and g'' because we know nothing about lower bounds on f and g.