Lecture 12: Depth-First Search

Steven Skiena

Department of Computer Science
State University of New York
Stony Brook, NY 11794–4400

http://www.cs.stonybrook.edu/~skiena

Problem of the Day

Prove that in a breadth-first search on a undirected graph G, every edge in G is either a tree edge or a cross edge, where a cross edge (x, y) is an edge where x is neither is an ancestor or descendent of y.

Connected Components

The *connected components* of an undirected graph are the separate “pieces” of the graph such that there is no connection between the pieces.

Many seemingly complicated problems reduce to finding or counting connected components. For example, testing whether a puzzle such as Rubik’s cube or the 15-puzzle can be solved from any position is really asking whether the graph of legal configurations is connected.

Anything we discover during a BFS must be part of the same connected component. We then repeat the search from any undiscovered vertex (if one exists) to define the next component, until all vertices have been found:

Implementation

```
connected_components(graph *g)
{
        int c;
        int i;

        initialize_search(g);

        c = 0;
        for (i=1; i<=g− >nvertices; i++)
                if (discovered[i] == FALSE) {
                        c = c+1;
                        printf(”Component %d:”,c);
                        bfs(g,i);
                }
}
```

Two-Coloring Graphs

The *vertex coloring* problem seeks to assign a label (or color) to each vertex of a graph such that no edge links any two vertices of the same color.

A graph is *bipartite* if it can be colored without conflicts while using only two colors. Bipartite graphs are important because they arise naturally in many applications.

For example, consider the graph of students and the courses they are registered for. No edges go between student pairs or course pairs, so the graph is bipartite.

Finding a Two-Coloring

We can augment breadth-first search so that whenever we discover a new vertex, we color it the opposite of its parent.

```
twocolor(graph *g)
{
        int i;

        for (i=1; i<=(g->nvertices); i++)
                color[i] = UNCOLORED;

        bipartite = TRUE;

        initialize_search(&g);

        for (i=1; i<=(g->nvertices); i++)
                if (discovered[i] == FALSE) {
                        color[i] = WHITE;
                        bfs(g,i);
                }
}
```

```
process_edge(int x, int y)
{
        if (color[x] == color[y]) {
                bipartite = FALSE;
                printf(”Warning: graph not bipartite, due to (%d,%d)”,x,y);
        }

        color[y] = complement(color[x]);
}


complement(int color)
{
if (color == WHITE) return(BLACK);
if (color == BLACK) return(WHITE);

return(UNCOLORED);
}
```

We can assign the first vertex in any connected component to be whatever color/gender we wish.

Depth-First Search

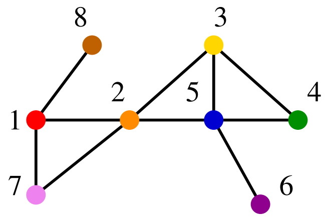
DFS has a neat recursive implementation which eliminates the need to explicitly use a stack.
Discovery and final times are a convenience to maintain.

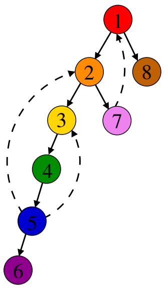
```
dfs(graph *g, int v)
{
        edgenode *p; (* temporary pointer *)
        int y; (* successor vertex *)

        if (finished) return; (* allow for search termination *)

        discovered[v] = TRUE;
        time = time + 1;
        entry_time[v] = time;

        process_vertex_early(v);

        p = g− >edges[v];
        while (p ! = NULL) {
                y = p− >y;
```

```
            if (discovered[y] == FALSE) {
                    parent[y] = v;
                    process_edge(v,y);
                    dfs(g,y);
            }
            else if ((!processed[y]) || (g−>directed))
                    process_edge(v,y);

            if (finished) return;

            p = p−>next;
    }

    process_vertex_late(v);

    time = time + 1;
    exit_time[v] = time;

    processed[v] = TRUE;
}
```

The *Key* Idea with DFS

A depth-first search of a graph organizes the edges of the graph in a precise way.
In a DFS of an undirected graph, we assign a direction to each edge, from the vertex which discover it:

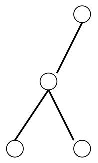




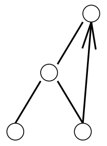
Edge Classification for DFS

Every edge is either:

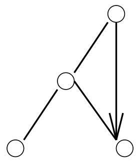
1. A Tree Edge



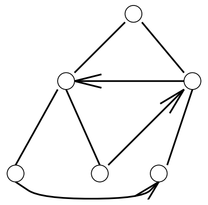
2. A Back Edge
   to an ancestor



3. A Forward Edge
   to a decendant



4. A Cross Edge
   to a different node



On any particular DFS or BFS of a directed or undirected graph, each edge gets classified as one of the above.

Edge Classification Implementation

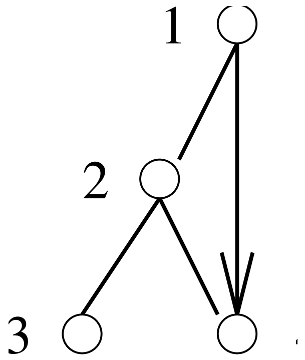
```
int edge_classification(int x, int y)
{
        if (parent[y] == x) return(TREE);
        if (discovered[y] && !processed[y]) return(BACK);
        if (processed[y] && (entry_time[y]¿entry_time[x])) return(FORWARD);
        if (processed[y] && (entry_time[y]¡entry_time[x])) return(CROSS);

        printf(”Warning: unclassified edge (%d,%d)”,x,y);
}
```

DFS: Tree Edges and Back Edges Only

The reason DFS is so important is that it defines a very nice ordering to the edges of the graph.

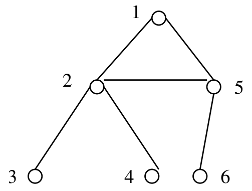
In a DFS of an undirected graph, every edge is either a tree edge or a back edge.

Why? Suppose we have a forward edge. We would have encountered $(4, 1)$ when expanding 4, so this is a back edge.

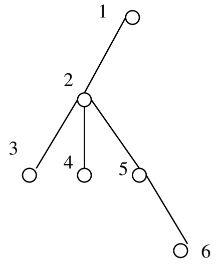


No Cross Edges in DFS

Suppose we have a cross-edge



When expanding 2, we would discover 5, so the tree would look like:



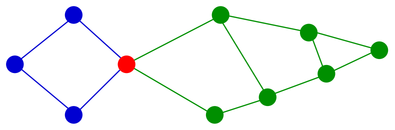
DFS Application: Finding Cycles

Back edges are the key to finding a cycle in an undirected graph.
Any back edge going from x to an ancestor y creates a cycle with the path in the tree from y to x.

```
process_edge(int x, int y)
{
        if (parent[x] ! = y) { (* found back edge! *)
                printf(”Cycle from %d to %d:”,y,x);
                find_path(y,x,parent);
                finished = TRUE;
        }
}
```

Articulation Vertices

Suppose you are a terrorist, seeking to disrupt the telephone network. Which station do you blow up?



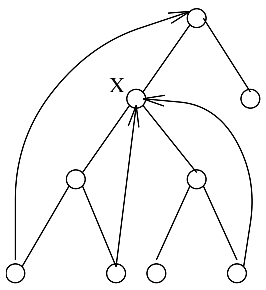
An *articulation vertex* is a vertex of a connected graph whose deletion disconnects the graph.

Clearly connectivity is an important concern in the design of any network.

Articulation vertices can be found in $O(n(m+n))$ – just delete each vertex to do a DFS on the remaining graph to see if it is connected.

A Faster $O(n+m)$ DFS Algorithm

In a DFS tree, a vertex v (other than the root) is an articulation vertex iff v is not a leaf and some subtree of v has no back edge incident until a proper ancestor of v.



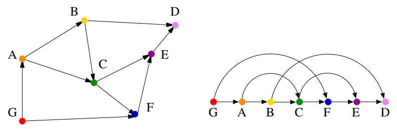
The root is a special case since it has no ancestors.

X is an articulation vertex since the right subtree does not have a back edge to a proper ancestor.

Leaves cannot be articulation vertices

Topological Sorting

A directed, acyclic graph has no directed cycles.

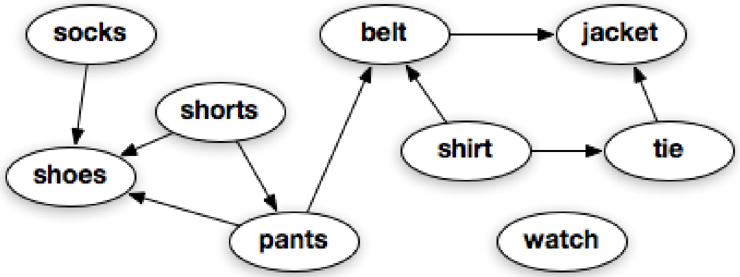


A topological sort of a graph is an ordering on the vertices so that all edges go from left to right.
DAGs (and only DAGs) has at least one topological sort (here G, A, B, C, F, E, D).

Applications of Topological Sorting

Topological sorting is often useful in scheduling jobs in their proper sequence. In general, we can use it to order things given precidence constraints.

Example: Dressing priority schedule



Example: Identifying errors in DNA fragment assembly

Certain fragments are constrained to be to the left or right of other fragments, unless there are errors.

```
A B R A C            A B R A C A D A B R A
A C A D A            ---------------------
A D A B R            A B R A C
D A B R A              R A C A D
R A C A D                A C A D A
                             A D A B R
                               D A B R A
```

Solution – build a DAG representing all the left-right constraints. Any topological sort of this DAG is a consistant ordering. If there are cycles, there must be errors.

Topological Sorting via DFS

A directed graph is a DAG if and only if no back edges are encountered during a depth-first search.

Labeling each of the vertices in the reverse order that they are marked *processed* finds a topological sort of a DAG.

Why? Consider what happens to each directed edge $\{x, y\}$ as we encounter it during the exploration of vertex x:

Case Analysis

- If y is currently *undiscovered*, then we then start a DFS of y before we can continue with x. Thus y is marked *completed* before x is, and x appears before y in the topological order, as it must.
- If y is *discovered* but not *completed*, then $\{x, y\}$ is a back edge, which is forbidden in a DAG.
- If y is *completed*, then it will have been so labeled before x. Therefore, x appears before y in the topological order, as it must.

Topological Sorting Implementation

```
process_vertex_late(int v)
{
        push(&sorted,v);
}


process_edge(int x, int y)
{
        int class;

        class = edge_classification(x,y);

        if (class == BACK)
                printf("Warning: directed cycle found, not a DAG");
}
```

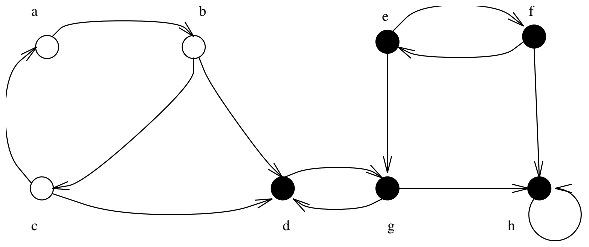
We push each vertex on a stack soon as we have evaluated all outgoing edges. The top vertex on the stack always has no incoming edges from any vertex on the stack, repeatedly popping them off yields a topological ordering.

Backtracking and Depth-First Search

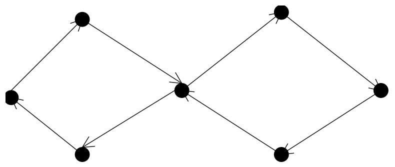
Depth-first search uses essentially the same idea as backtracking.
Both involve exhaustively searching all possibilities by advancing if it is possible, and backing up as soon as there is no unexplored possibility for further advancement. Both are most easily understood as recursive algorithms.

Strongly Connected Components

A directed graph is strongly connected iff there is a directed path between any two vertices.
The strongly connected components of a graph is a partition of the vertices into subsets (maximal) such that each subset is strongly connected.



Observe that no vertex can be in two maximal components, so it is a partition.



There is an elegant, linear time algorithm to find the strongly connected components of a directed graph using DFS which is similar to the algorithm for biconnected components.