Lecture 15: Backtracking

Steven Skiena

Department of Computer Science
State University of New York
Stony Brook, NY 11794–4400

http://www.cs.stonybrook.edu/~skiena

Topic: Problem of the Day

Problem of the Day

Let $G = (V, E)$ be a directed, weighted graph such taht all weights are positive. Let v and w be two vertices in G, and $k \leq |V|$ be an integer. Design an algorithm to find the shortest path from v to w that contains exactly k edges. Note that the path need not be simple.

Questions?

Topic: Backtracking

Sudoku

							1	2
				3	5			
			6				7	
7						3		
			4			8		
1								
			1	2				
	8						4	
	5					6		

6	7	3	8	9	4	5	1	2
9	1	2	7	3	5	4	8	6
8	4	5	6	1	2	9	7	3
7	9	8	2	6	1	3	5	4
5	2	6	4	7	3	8	9	1
1	3	4	5	8	9	2	6	7
4	6	9	1	2	8	7	3	5
2	8	7	3	5	6	1	4	9
3	5	1	9	4	7	6	2	8

Solving Sudoku

Solving Sudoku puzzles involves a form of exhaustive search of possible configurations.

However, exploiting constraints to rule out certain possibilities for certain positions enables us to *prune* the search to the point people can solve Sudoku by hand.

Backtracking is the key to implementing exhaustive search programs correctly and efficiently.

Backtracking

Backtracking is a systematic method to iterate through all possible configurations of a search space. It is a general algorithm which must be customized for each application.

We model our solution as a vector $a = (a_1, a_2, ..., a_n)$, where each element a_i is selected from a finite ordered set S_i.

Such a vector might represent an arrangement where a_i contains the ith element of the permutation. Or the vector might represent a given subset S, where a_i is true if and only if the ith element of the universe is in S.

The Idea of Backtracking

At each step in the backtracking algorithm, we start from a given partial solution, say, $a = (a_1, a_2, ..., a_k)$, and try to extend it by adding another element at the end.

After extending it, we test whether what we have so far is a complete solution.

If not, the critical issue is whether the current partial solution a is potentially extendible to a solution.

- If so, recur and continue.
- If not, delete the last element from a and try another possibility for that position if one exists.

Questions?

Topic: Backtracking Implementation

Recursive Backtracking

Backtrack(a, k)
if a is a solution, print(a)
else {
    $k = k + 1$
    compute S_k
    while $S_k \neq \emptyset$ do
        a_k = an element in S_k
        S_k = $S_k - a_k$
        Backtrack(a, k)
}

Backtracking and DFS

Backtracking is really just depth-first search on an implicit graph of configurations.

- Backtracking can easily be used to iterate through all subsets or permutations of a set.
- Backtracking ensures correctness by enumerating all possibilities.
- For backtracking to be efficient, we must prune dead or redundent branches of the search space whenever possible.

Backtracking Implementation

```
void backtrack(int a[], int k, data input) {
    int c[MAXCANDIDATES];       /* candidates for next position */
    int nc;                     /* next position candidate count */
    int i;                      /* counter */

    if (is_a_solution(a, k, input)) {
        process_solution(a, k,input);
    } else {
        k = k + 1;
        construct_candidates(a, k, input, c, &nc);
        for (i = 0; i < nc; i++) {
            a[k] = c[i];
            make_move(a, k, input);
            backtrack(a, k, input);
            unmake_move(a, k, input);

            if (finished) {
                return;         /* terminate early */
            }
        }
    }
}
```

`is_a_solution(a,k,input)`

This Boolean function tests whether the first k elements of vector a are a complete solution for the given problem.

The last argument, `input`, allows us to pass general information into the routine to evaluate whether a is a solution.

`construct_candidates(a,k,input,c,nc)`

This routine fills an array c with the complete set of possible candidates for the kth position of a, given the contents of the first $k-1$ positions.
The number of candidates returned in this array is denoted by `nc`.

`process_solution(a,k)`

This routine prints, counts, or somehow processes a complete solution once it is constructed.

Backtracking ensures correctness by enumerating all possibilities. It ensures efficiency by never visiting a state more than once.

Because a new candidates array c is allocated with each recursive procedure call, the subsets of not-yet-considered extension candidates at each position will not interfere with each other.

Questions?

Topic: Constructing Subsets by Backtracking

Constructing all Subsets

To construct all 2^n subsets, set up an array/vector of n cells, where the value of a_i is either true or false, signifying whether the ith item is or is not in the subset.

To use the notation of the general backtrack algorithm, $S_k = (true, false)$, and v is a solution whenever $k \geq n$.

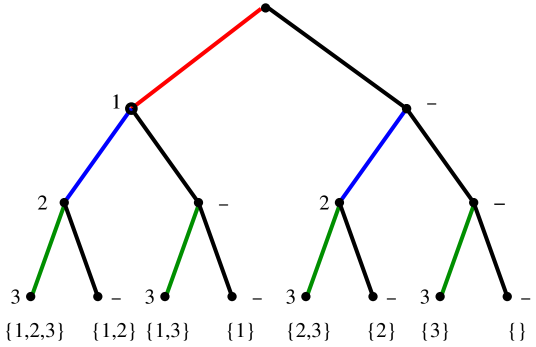
Subset Generation Tree / Order

What order will this generate the subsets of $\{1, 2, 3\}$?

$$(1) \rightarrow (1, 2) \rightarrow (1, 2, 3) \rightarrow (1, 2, -) \rightarrow (1, -) \rightarrow (1, -, 3) \rightarrow$$

$$(1, -, -) \rightarrow (1, -) \rightarrow (1) \rightarrow (-) \rightarrow (-, 2) \rightarrow (-, 2, 3) \rightarrow$$

$$(-, 2, -) \rightarrow (-, -) \rightarrow (-, -, 3) \rightarrow (-, -, -) \rightarrow (-, -) \rightarrow (-) \rightarrow ()$$



Using Backtrack to Construct Subsets

We can construct all subsets of n items by iterating through all 2^n length-n vectors of *true* or *false*, letting the ith element denote whether item i is (or is not) in the subset.

Thus the candidate set $S_k = (true, false)$ for all positions, and a is a solution when $k \geq n$.

```
int is_a_solution(int a[], int k, int n) {
    return (k == n);
}

void construct_candidates(int a[], int k, int n, int c[], int *nc) {
    c[0] = true;
    c[1] = false;
    *nc = 2;
}
```

Process the Subsets

Here we print the elements in each subset, but you can do whatever you want – like test whether it is a vertex cover solution...

```
void process_solution(int a[], int k, int input) {
    int i;    /* counter */

    printf("{");
    for (i = 1; i <= k; i++) {
        if (a[i] == true) {
            printf(" %d", i);
        }
    }

    printf(" }\n");
}
```

Main Routine: Subsets

Finally, we must instantiate the call to `backtrack` with the right arguments.

```
void generate_subsets(int n) {
    int a[NMAX];                         /* solution vector */

    backtrack(a, 0, n);
}
```

Questions?

Topic: Constructng Permutations by Backtracking

Constructing all Permutations

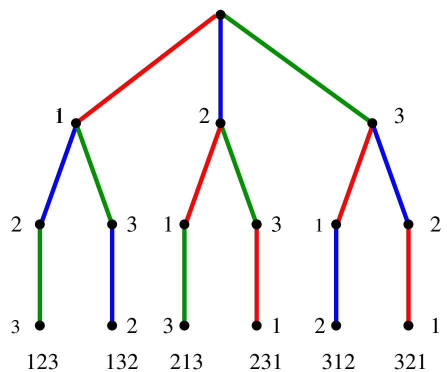
How many permutations are there of an n-element set?

To construct all $n!$ permutations, set up an array/vector of n cells, where the value of a_i is an integer from 1 to n which has not appeared thus far in the vector, corresponding to the ith element of the permutation.

To use the notation of the general backtrack algorithm, $S_k = (1, \ldots, n) - v$, and v is a solution whenever $k \geq n$.

Permutation Generation Tree / Order

$$\begin{array}{rcl}
(1) & \rightarrow & (1,2) \rightarrow (1,2,3) \rightarrow (1,2) \rightarrow (1) \rightarrow (1,3) \rightarrow \\
(1,3,2) & \rightarrow & (1,3) \rightarrow (1) \rightarrow () \rightarrow (2) \rightarrow (2,1) \rightarrow \\
(2,1,3) & \rightarrow & (2,1) \rightarrow (2) \rightarrow (2,3) \rightarrow (2,3,1) \rightarrow (2,3) \rightarrow () \\
(2) & \rightarrow & () \rightarrow (3) \rightarrow (3,1)(3,1,2) \rightarrow (3,1) \rightarrow (3) \rightarrow \\
(3,2) & \rightarrow & (3,2,1) \rightarrow (3,2) \rightarrow (3) \rightarrow ()
\end{array}$$



Constructing All Permutations

To avoid repeating permutation elements, $S_k = \{1, \ldots, n\} - a$, and a is a solution whenever $k = n$:

```
void construct_candidates(int a[], int k, int n, int c[], int *nc) {
    int i;                  /* counter */
    bool in_perm[NMAX];     /* what is now in the permutation? */

    for (i = 1; i < NMAX; i++) {
        in_perm[i] = false;
    }

    for (i = 1; i < k; i++) {
        in_perm[a[i]] = true;
    }

    *nc = 0;
    for (i = 1; i <= n; i++) {
        if (!in_perm[i]) {
            c[ *nc ] = i;
            *nc = *nc + 1;
        }
    }
}
```

Auxilliary Routines

Completing the job of generating permutations requires specifying `process_solution` and `is_a_solution`, as well as setting the appropriate arguments to `backtrack`. All are essentially the same as for subsets:

```
void process_solution(int a[], int k, int input) {
    int i;     /* counter */

    for (i = 1; i <= k; i++) {
        printf(" %d", a[i]);
    }
    printf("\n");
}


int is_a_solution(int a[], int k, int n) {
    return (k == n);
}
```

Main Program: Permutations

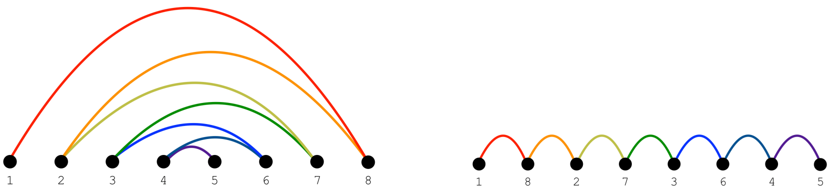
```
void generate_permutations(int n) {
    int a[NMAX];                            /* solution vector */

    backtrack(a, 0, n);
}
```

Questions?

Topic: Backtracking Contest

The Backtracking Contest: Bandwidth

The *bandwidth problem* takes as input a graph G, with n vertices and m edges (ie. pairs of vertices). The goal is to find a permutation of the vertices on the line which minimizes the maximum length of any edge.



The bandwidth problem has a variety of applications, including circuit layout, linear algebra, and optimizing memory usage in hypertext documents.

Computing the Bandwidth of a Graph

The bandwidth problem is NP-complete, meaning you will not be able to find an algorithm with polynomial worst-case running time.

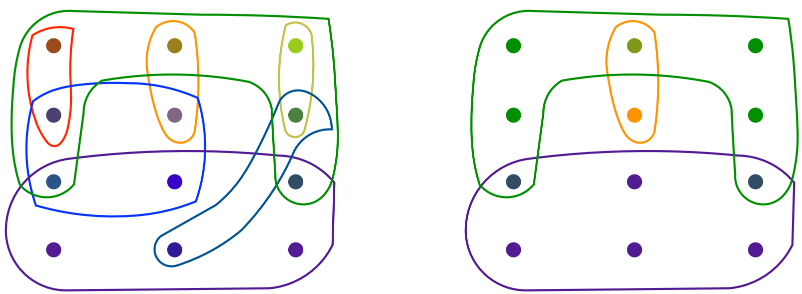
It remains NP-complete even for restricted classes of trees.

A backtracking program which iterates through all the $n!$ possible permutations and computes the length of the longest edge for each one gives an easy $O(n! \cdot m)$ algorithm.

But the goal of this assignment is to find as practically good an algorithm as possible, so try to avoid constructing all permutations.

The Backtracking Contest: Set Cover

The *set cover* problem takes as input a collection of subsets $S = \{S_1, \ldots, S_m\}$ of the universal set $U = \{1, \ldots, n\}$. The goal is to find the smallest subset of the subsets T such that $\cup_{i=1}^{|T|} T_i = U$.



Computing the Minimum Set Cover

Set cover arises when you try to efficiently acquire or represent items that have been packaged in a fixed set of lots. You want to get all the items, buying as few lots as possible.

Finding *a* cover is easy, because you can always buy one of each lot. But a small set cover will do the same job for less money.

A backtracking program which iterates through all the 2^m possible subsets and tests whether it represents a cover gives an easy $O(2^m \cdot nm)$ algorithm.

But the goal of this assignment is to find as practically good an algorithm as possible, so try to avoid constructing all subsets.

Rules of the Game

1. Everyone does this assignment separately: you are not allowed to work with your partner on this program.
2. If you do not completely understand what the problem is, you have no chance of producing a working program. *Ask for a clarification or explanation!!!!!*
3. There will data files of many different sizes. Test on the smaller files first. Do not be afraid to create your own test data to help debug your program.
4. The data files are available via the course WWW page.
5. You will be graded on how fast your program is, not on style. No credit will be given for incorrect programs.

6. You can run your program on whatever computer you have access to, although it should also run on a machine we have access to.
7. You are to turn in the source files of your program, along with a brief description of any interesting optimizations, sample runs, and the time it takes on sample data files. Report the largest test file your program could handle in one minute or less of wall clock time.
8. The top five self-reported times / largest sizes will be collected and tested by me to determine the winner.

Producing Efficient Programs

- Don't optimize prematurely: Don't worry about recursion vs. iteration until you have worked out the best way to prune the tree. That is where the money is.
- Choose your data structures for a reason: What operations will you be doing? Is the case of insertion/deletion more crucial than fast retrieval?

  When in doubt, keep it simple, stupid (KISS).
- Let the profiler determine where to do final tuning: Your program is probably spending time where you don't expect.

Questions?

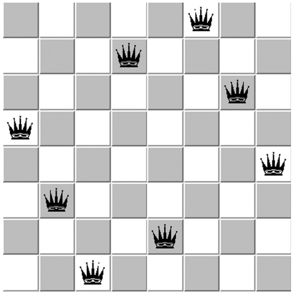
Topic: Problem of the Day

Problem of the Day

A *derangement* is a permutation p of $\{1, \ldots, n\}$ such that no item is in its proper position, i.e. $p_i \neq i$ for all $1 \leq i \leq n$. Write an efficient backtracking program with pruning that constructs all the derangements of n items.

Questions?

The Eight-Queens Problem



The eight queens problem is a classical puzzle of positioning eight queens on an 8×8 chessboard such that no two queens threaten each other.

Eight Queens: Representation

What is concise, efficient representation for an n-queens solution, and how big must it be?

Since no two queens can occupy the same column, we know that the n columns of a complete solution must form a permutation of n. By avoiding repetitive elements, we reduce our search space to just $8! =$ 40,320 – clearly short work for any reasonably fast machine.

The critical routine is the candidate constructor. We repeatedly check whether the kth square on the given row is threatened by any previously positioned queen. If so, we move on, but if not we include it as a possible candidate:

Candidate Constructor: Eight Queens

```
void construct_candidates(int a[], int k, int n, int c[], int *ncandidates) {
    int i, j;                   /* counters */
    bool legal_move;            /* might the move be legal? */

    *ncandidates = 0;
    for (i = 1; i <= n; i++) {
        legal_move = true;
        for (j = 1; j < k; j++) {
            if (abs((k)-j) == abs(i-a[j])) {     /* diagonal threat */
                legal_move = false;
            }
                        if (i == a[j]) {         /* column threat */
                legal_move = false;
            }
        }
        if (legal_move) {
            c[*ncandidates] = i;
            *ncandidates = *ncandidates + 1;
        }
    }
}
```

Auxiliary Routines

The remaining routines are simple, particularly since we are only interested in counting the solutions, not displaying them:

```
void process_solution(int a[], int k, int input) {
    solution_count ++;
}

int is_a_solution(int a[], int k, int n) {
    return (k == n);
}
```

Finding the Queens: Main Program

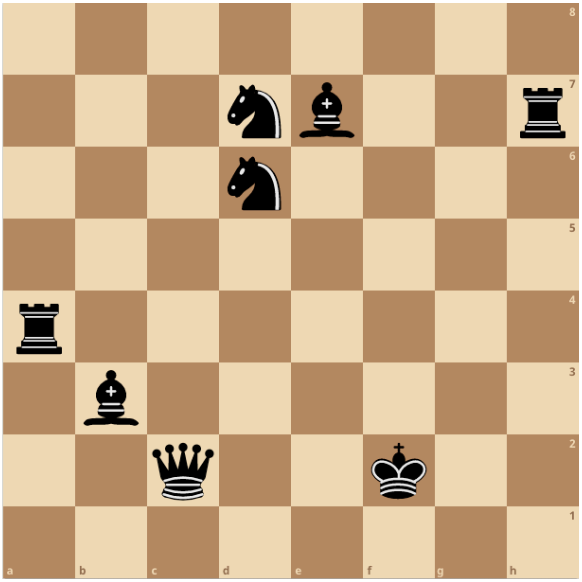
```
void nqueens(int n) {
    int a[NMAX];          /* solution vector */

    solution_count = 0;
    backtrack(a, 0, n);
    printf("n=%d  solution_count=%d\n", n, solution_count);
}
```

This program can find the 365,596 solutions for $n = 14$ in minutes.

Topic: Covering the Chess Board

Can Eight Pieces Cover a Chess Board?

Consider the 8 main pieces in chess (king, queen, two rooks, two bishops, two knights). Can they be positioned on a chessboard so every square is threatened?



Combinatorial Search

Only 63 square are threatened in this configuration. Since 1849, no one had been able to find an arrangement with bishops on different colors to cover all squares.

We can resolve this question by searching through all possible board configurations *if* we spend enough time.

We will use it as an example of how to attack a combinatorial search problem.

With clever use of backtracking and pruning techniques, surprisingly large problems can be solved by exhaustive search.

How Many Chess Configurations Must be Tested?

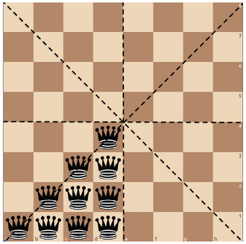
Picking a square for each piece gives us the bound:

$$64!/(64-8)! = 178,462,987,637,760 \approx 10^{15}$$

Anything much larger than 10^8 is unreasonable to search on a modest computer in a modest amount of time.

Exploiting Symmetry

However, we can exploit symmetry to save work. With reflections along horizontal, vertical, and diagonal axis, the queen can go in only 10 non-equivallent positions.
Even better, we can restrict the white bishop to 16 spots and the queen to 16, while seeing all distinct configurations.



$$16 \times 16 \times 32 \times 64 \times 2080 \times 2080 = 2,268,279,603,200 \approx 10^{12}$$

Covering the Chess Board

In covering the chess board, we prune whenever we find there is a square which we *cannot* cover given the initial configuration!

Specifically, each piece can threaten a certain maximum number of squares (queen 27, king 8, rook 14, etc.) We *prune* whenever the number of unthreated squares exceeds the sum of the maximum remaining coverage.

As implemented by a graduate student project, this backtrack search eliminates 95% of the search space, when the pieces are ordered by decreasing mobility.

With precomputing the list of possible moves, this program could search 1,000 positions per second.

End Game

But this is still too slow!

$$10^{12}/10^3 = 10^9 \text{ seconds } > 1000 \text{ days}$$

Although we might further speed the program by an order of magnitude, we need to prune more nodes!
By using a more clever algorithm, we eventually were able to prove no solution existed, in less than one day's worth of computing.
You too can fight the combinatorial explosion!

Questions?