CSE 613: Parallel Programming

Lecture 7
(Analyzing Divide-and-Conquer Algorithms)

Rezaul A. Chowdhury

Department of Computer Science
SUNY Stony Brook
Spring 2019

A Useful Recurrence

Consider the following recurrence:

$$T(n) = \begin{cases} \Theta(1), & if\ n \leq 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise; \end{cases}$$

where, $a \geq 1$ and $b > 1$.

Arises frequently in the analyses of *divide-and-conquer* algorithms.

Recall the following from the analyses of QSort (quicksort) in lecture 1.

Serial: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Parallel (with serial partition): $T(n) = T\left(\frac{n}{2}\right) + \Theta(n)$

Parallel (with parallel partition): $T(n) = T\left(\frac{n}{2}\right) + \Theta(\log n)$

How the Recurrence Unfolds

$$T(n) = \begin{cases} \Theta(1), & if\ n \leq 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise. \end{cases}$$

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$$\boldsymbol{T(n)}$$

$$\downarrow$$

$$\boldsymbol{f(n)} + \boldsymbol{aT\left(\frac{n}{b}\right)}$$

How the Recurrence Unfolds

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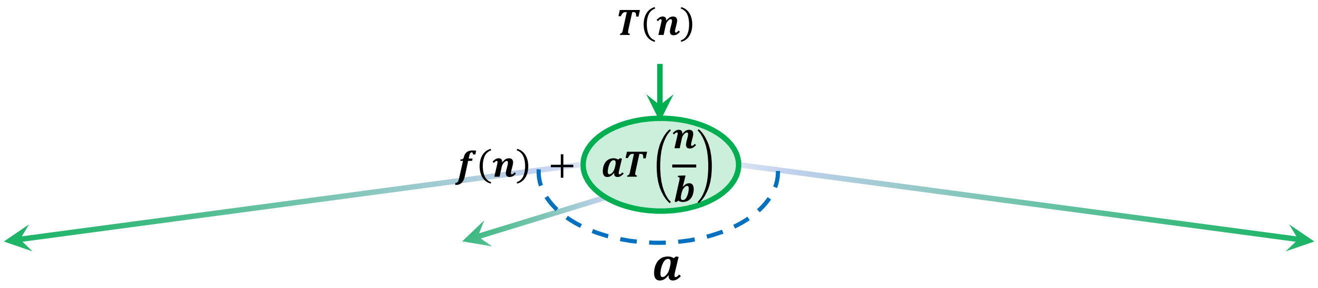

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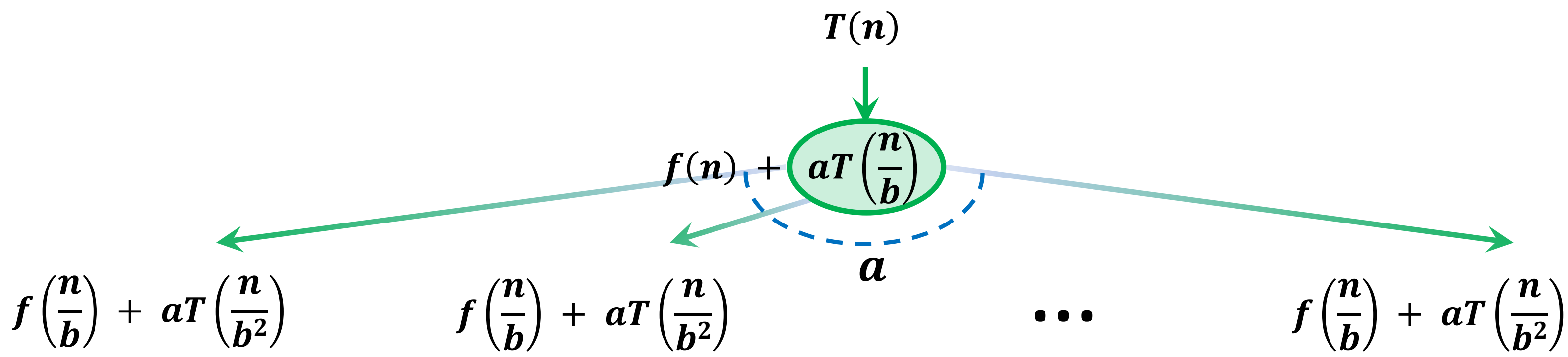

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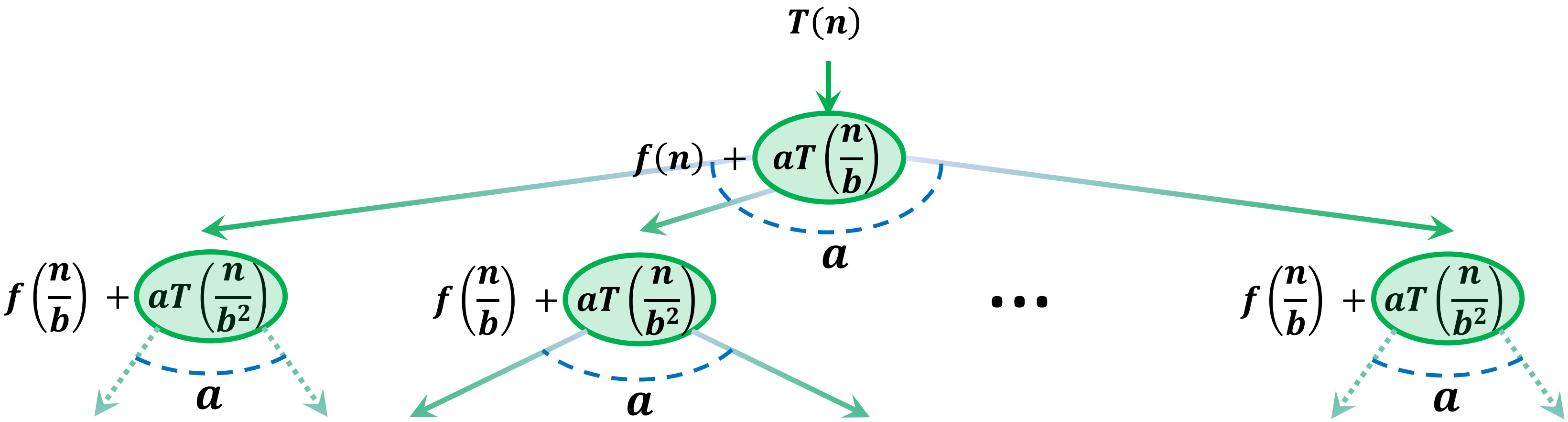

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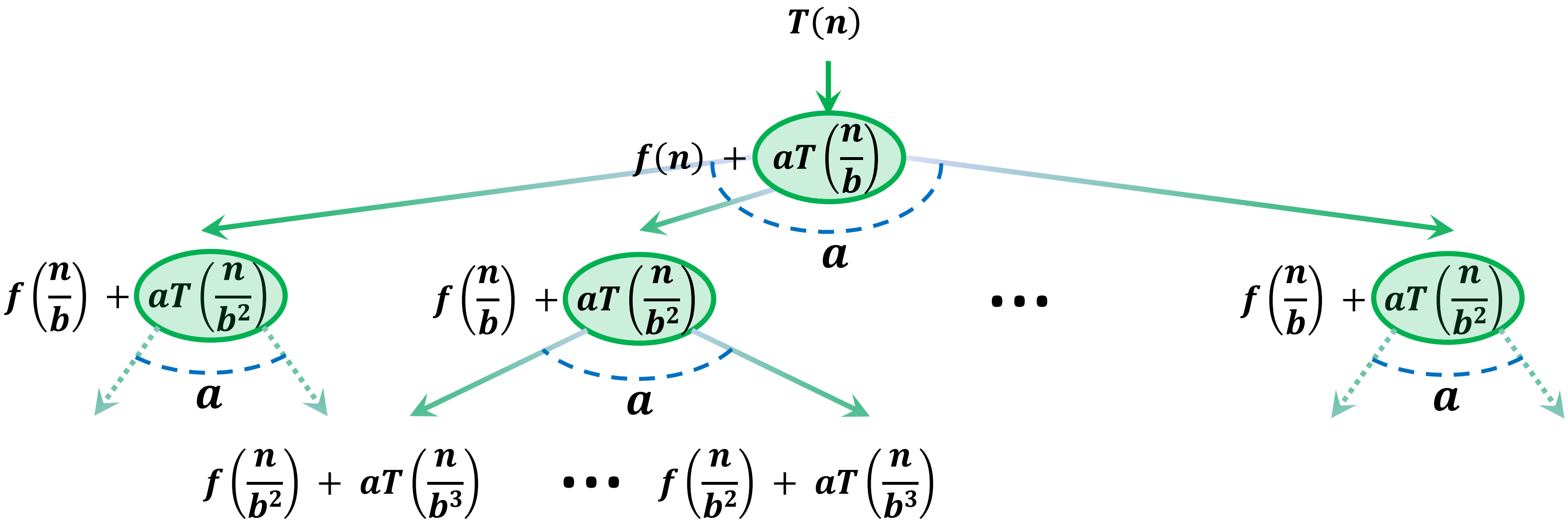
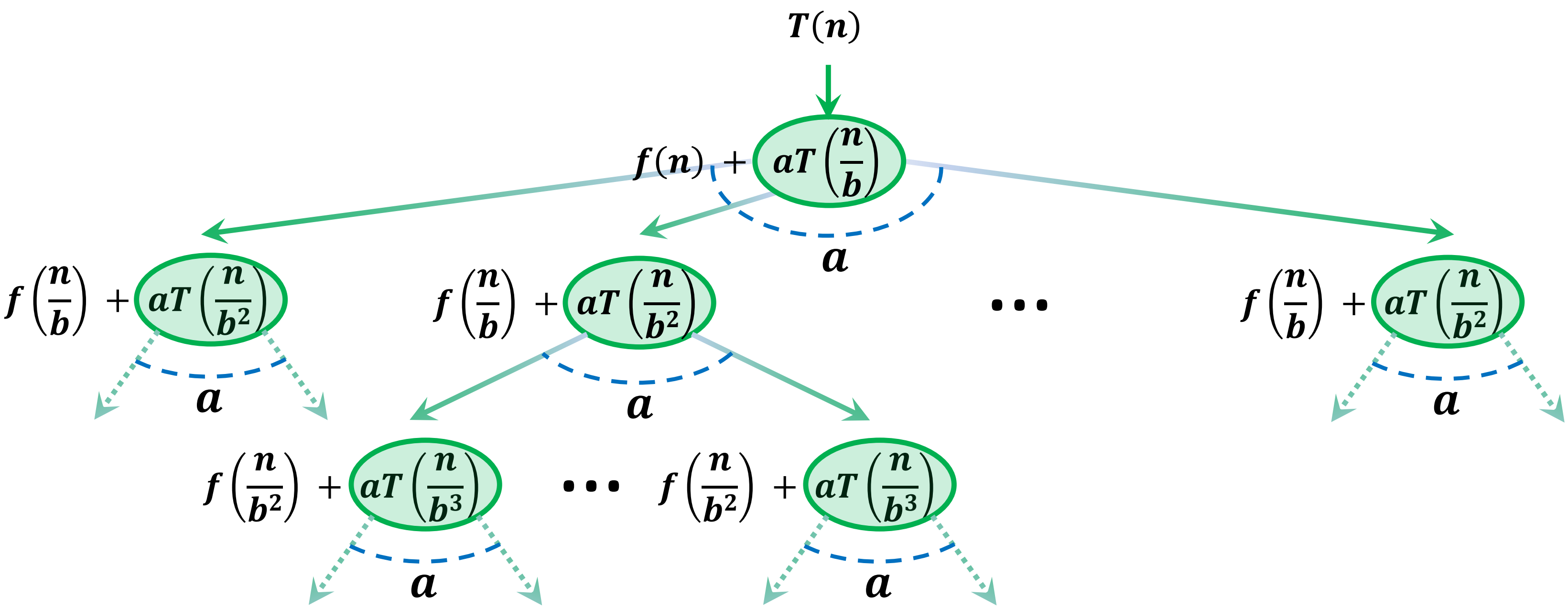


How the Recurrence Unfolds

$$T(n) = \begin{cases} \Theta(1), & if\ n \le 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise. \end{cases}$$



How the Recurrence Unfolds

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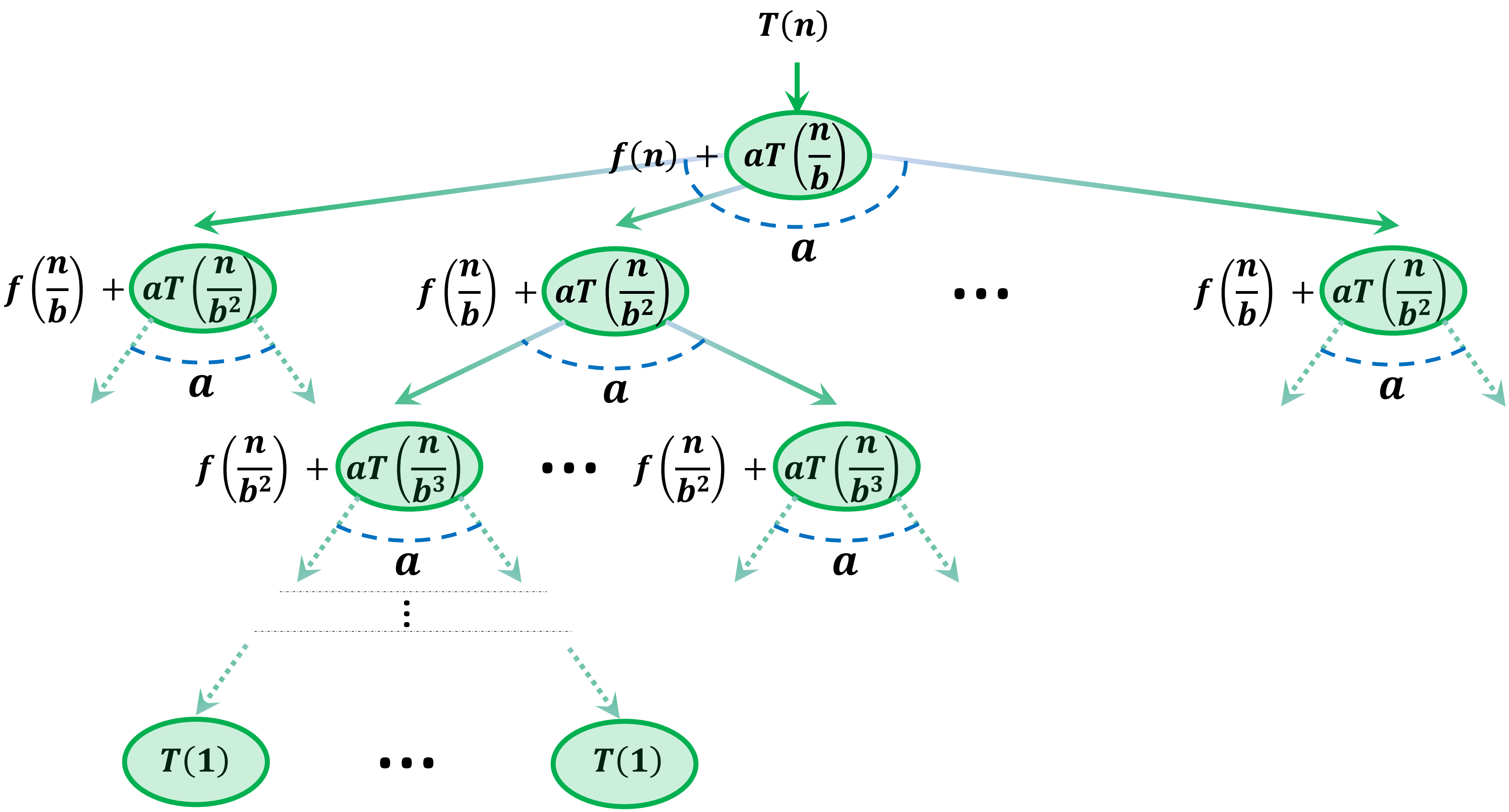
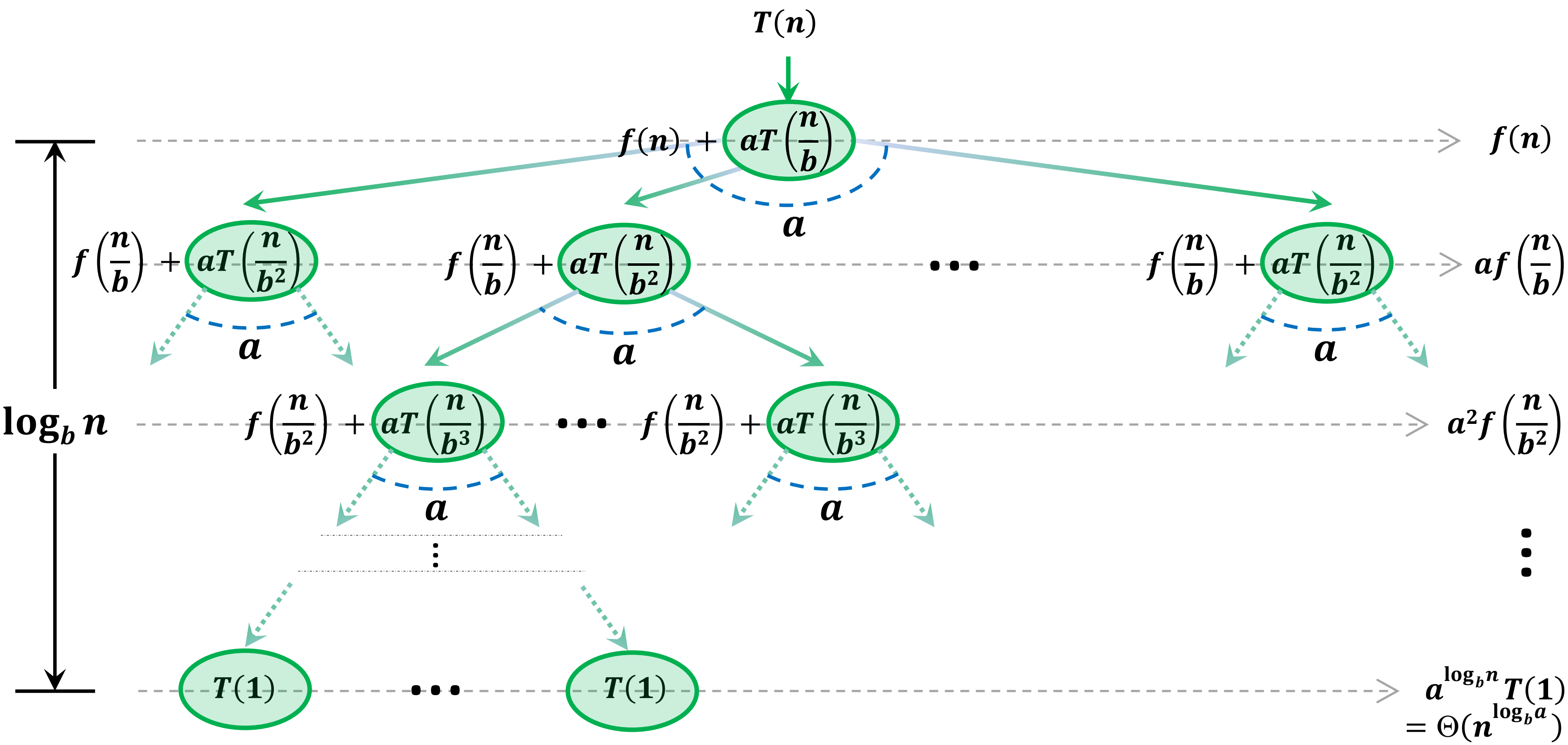


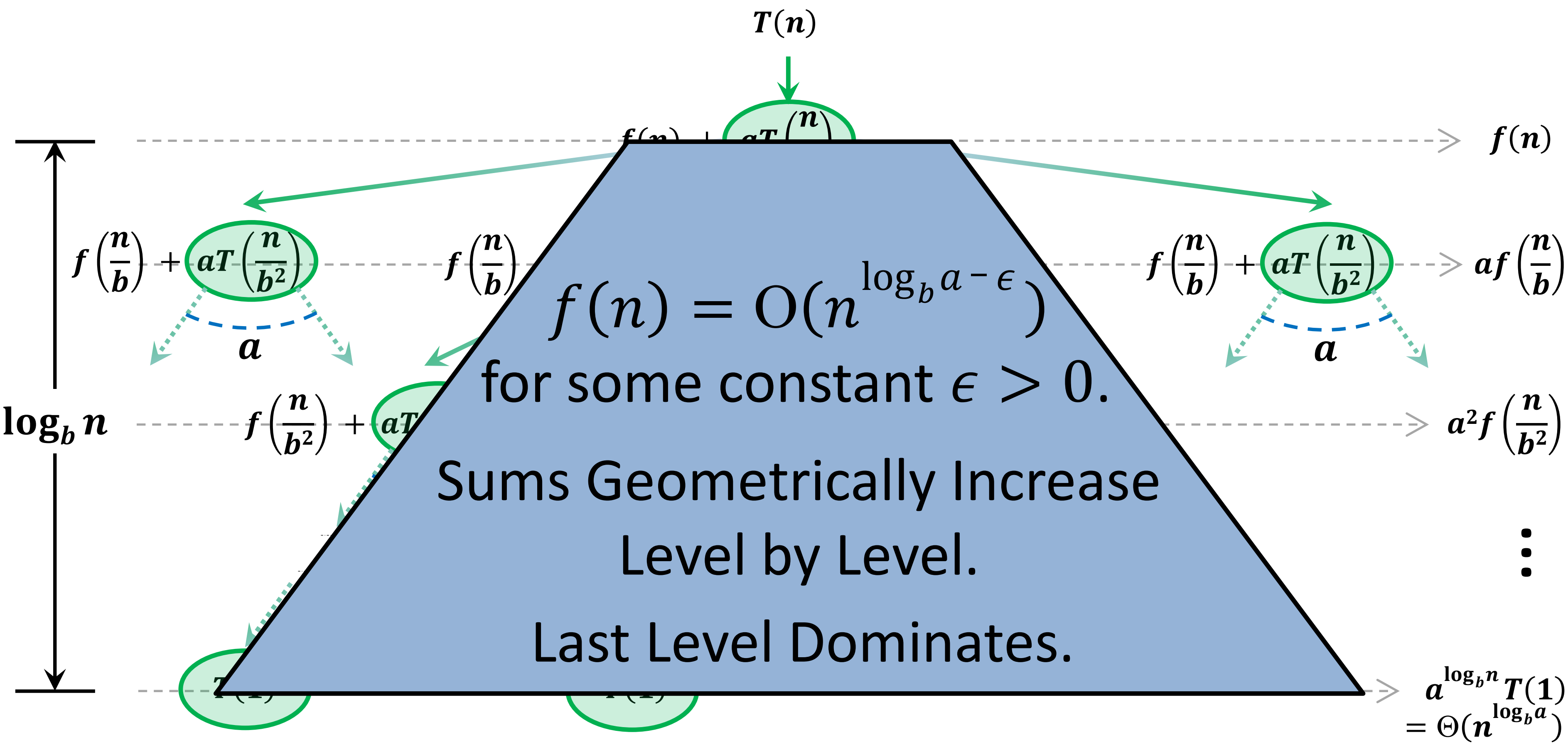
How the Recurrence Unfolds

$$T(n) = \begin{cases} \Theta(1), & \text{if } n \le 1, \\ aT\left(\frac{n}{b}\right) + f(n), & \text{otherwise.} \end{cases}$$



How the Recurrence Unfolds: Case 1

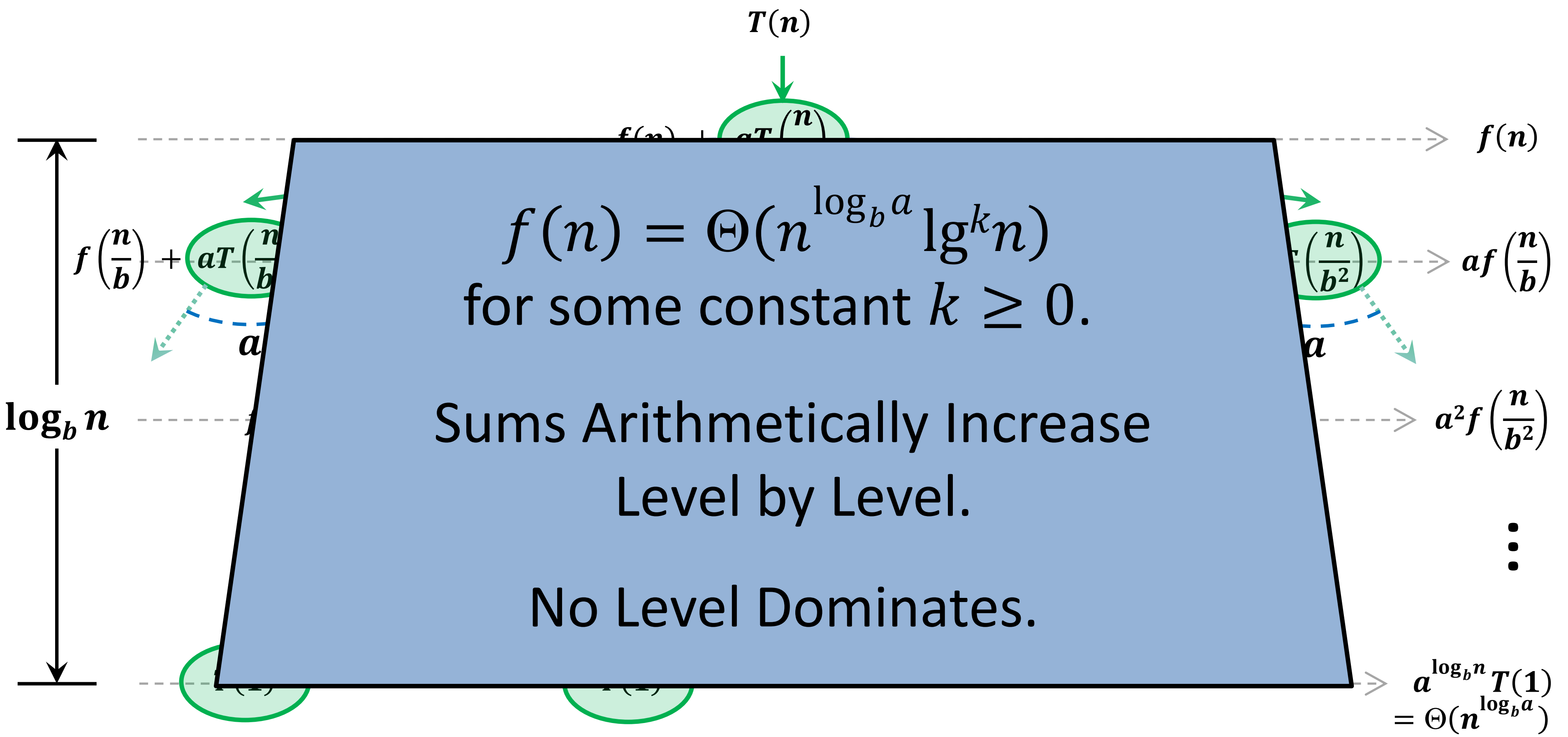
$$T(n) = \begin{cases} \Theta(1), & if\ n \le 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise. \end{cases}$$



$$T(n) = \Theta(n^{\log_b a})$$

How the Recurrence Unfolds: Case 2

$$T(n) = \begin{cases} \Theta(1), & if\ n \leq 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise. \end{cases}$$



$$f(n) = \Theta(n^{\log_b a} \lg^k n)$$

for some constant $k \geq 0$.

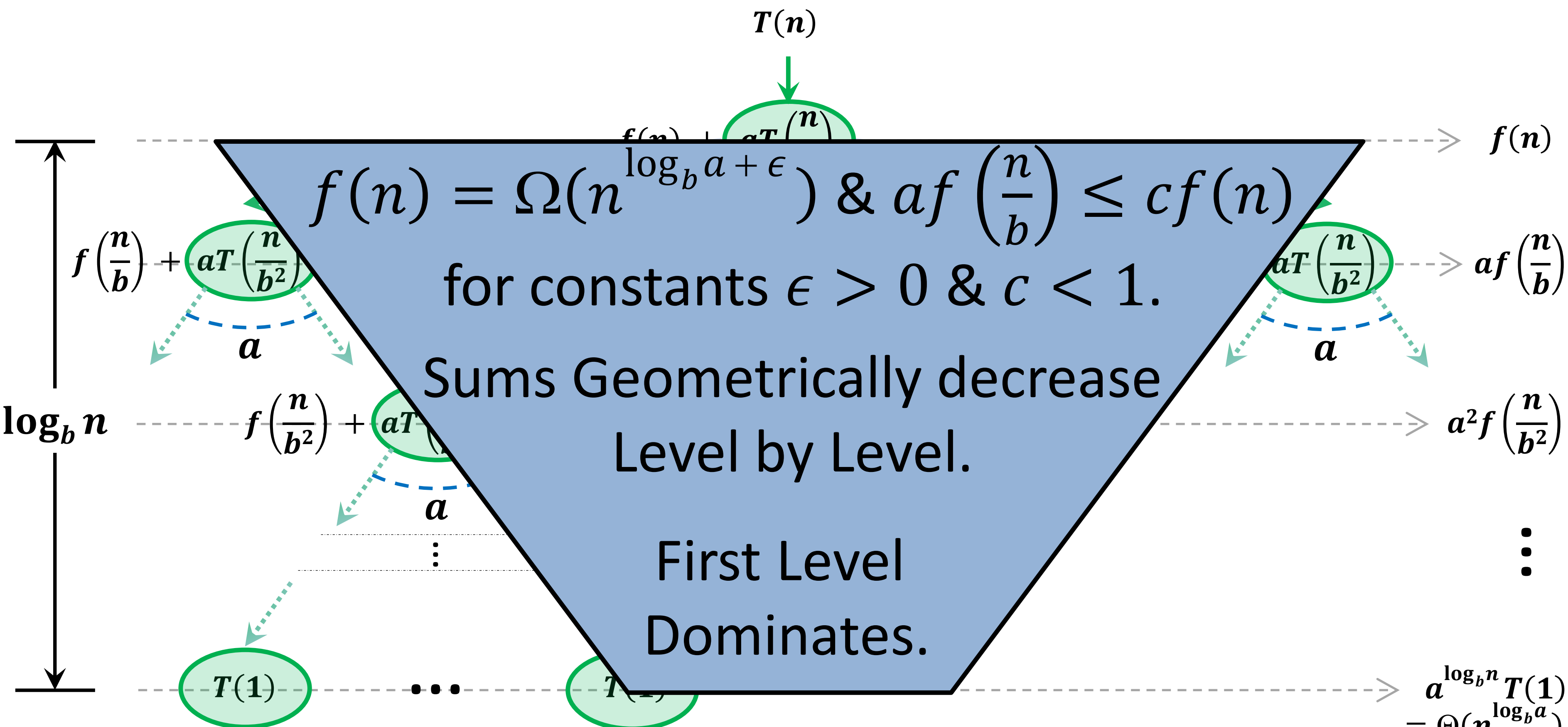
Sums Arithmetically Increase Level by Level.

No Level Dominates.

$$T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$$

How the Recurrence Unfolds: Case 3

$$T(n) = \begin{cases} \Theta(1), & if\ n \le 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise. \end{cases}$$



$f(n) = \Omega(n^{\log_b a + \epsilon})$ & $af\left(\frac{n}{b}\right) \le cf(n)$

for constants $\epsilon > 0$ & $c < 1$.

Sums Geometrically decrease Level by Level.

First Level Dominates.

$$T(n) = \Theta(f(n))$$

The Master Theorem

$$T(n) = \begin{cases} \Theta(1), & if\ n \leq 1, \\ aT\left(\frac{n}{b}\right) + f(n), & otherwise\ (a \geq 1, b > 1). \end{cases}$$

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$

$$T(n) = \Theta(n^{\log_b a})$$

Case 2: $f(n) = \Theta\left(n^{\log_b a} \lg^k n\right)$ for some constant $k \geq 0$.

$$T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$$

Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ and $af\left(\frac{n}{b}\right) \leq cf(n)$

for constants $\epsilon > 0$ and $c < 1$.

$$T(n) = \Theta\left(f(n)\right)$$

Back to QSort Complexities

Now let's try the QSort (quicksort) recurrences from lecture 1.

Serial: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Master Theorem Case 2: $T(n) = \Theta(n \log n)$

Parallel (with serial partition): $T(n) = T\left(\frac{n}{2}\right) + \Theta(n)$

Master Theorem Case 3: $T(n) = \Theta(n)$

Parallel (with parallel partition): $T(n) = T\left(\frac{n}{2}\right) + \Theta(\log n)$

Master Theorem Case 2: $T(n) = \Theta(\log^2 n)$

More Example Applications of Master Theorem

Karatsuba's Algorithm: $T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n)$

Master Theorem Case 1: $T(n) = \Theta\left(n^{\log_2 3}\right)$

Strassen's Matrix Multiplication: $T(n) = 7T\left(\frac{n}{2}\right) + \Theta\left(n^2\right)$

Master Theorem Case 1: $T(n) = \Theta\left(n^{\log_2 7}\right)$

Fast Fourier Transform: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Master Theorem Case 2: $T(n) = \Theta(n \log n)$

Recurrences not Solvable using the Master Theorem

Example 1: $T(n) = \sqrt{n}\, T\left(\frac{n}{2}\right) + n$

$a = \sqrt{n}$ is not a constant

Example 2: $T(n) = 2T\left(\frac{n}{\log n}\right) + n^2$

$b = \log n$ is not a constant

Example 3: $T(n) = \frac{1}{2}T\left(\frac{n}{2}\right) + n^2$

$a = \frac{1}{2}$ is not ≥ 1

Example 4: $T(n) = 2T\left(\frac{4n}{3}\right) + n$

$b = \frac{3}{4}$ is not > 1.

Recurrences not Solvable using the Master Theorem

Example 5: $T(n) = 3T\left(\frac{n}{2}\right) - n$

$f(n) = -n$ is not positive

Example 6: $T(n) = 2\,T\left(\frac{n}{2}\right) + n^2 \sin n$

violates regularity condition of case 3

Example 7: $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

$f(n) = \mathrm{O}\left(n^{\log_b a}\right)$, but $\neq \mathrm{O}\left(n^{\log_b a - \epsilon}\right)$ for any constant $\epsilon > 0$

Example 8: $T(n) = T\left(\frac{n}{2}\right) + 2T\left(\frac{n}{4}\right) + n$

a and b are not fixed

Multithreaded
Matrix Multiplication

Parallel Iterative MM

Iter-MM (Z, X, Y) { X, Y, Z are $n \times n$ matrices, where n is a positive integer }

1. *for* $i \leftarrow 1$ *to* n *do*
2. *for* $j \leftarrow 1$ *to* n *do*
3. $Z[\,i\,][\,j\,] \leftarrow 0$
4. *for* $k \leftarrow 1$ *to* n *do*
5. $Z[\,i\,][\,j\,] \leftarrow Z[\,i\,][\,j\,] + X[\,i\,][\,k\,] \cdot Y[\,k\,][\,j\,]$

Par-Iter-MM (Z, X, Y) { X, Y, Z are $n \times n$ matrices, where n is a positive integer }

1. *parallel for* $i \leftarrow 1$ *to* n *do*
2. *parallel for* $j \leftarrow 1$ *to* n *do*
3. $Z[\,i\,][\,j\,] \leftarrow 0$
4. *for* $k \leftarrow 1$ *to* n *do*
5. $Z[\,i\,][\,j\,] \leftarrow Z[\,i\,][\,j\,] + X[\,i\,][\,k\,] \cdot Y[\,k\,][\,j\,]$

Parallel Iterative MM

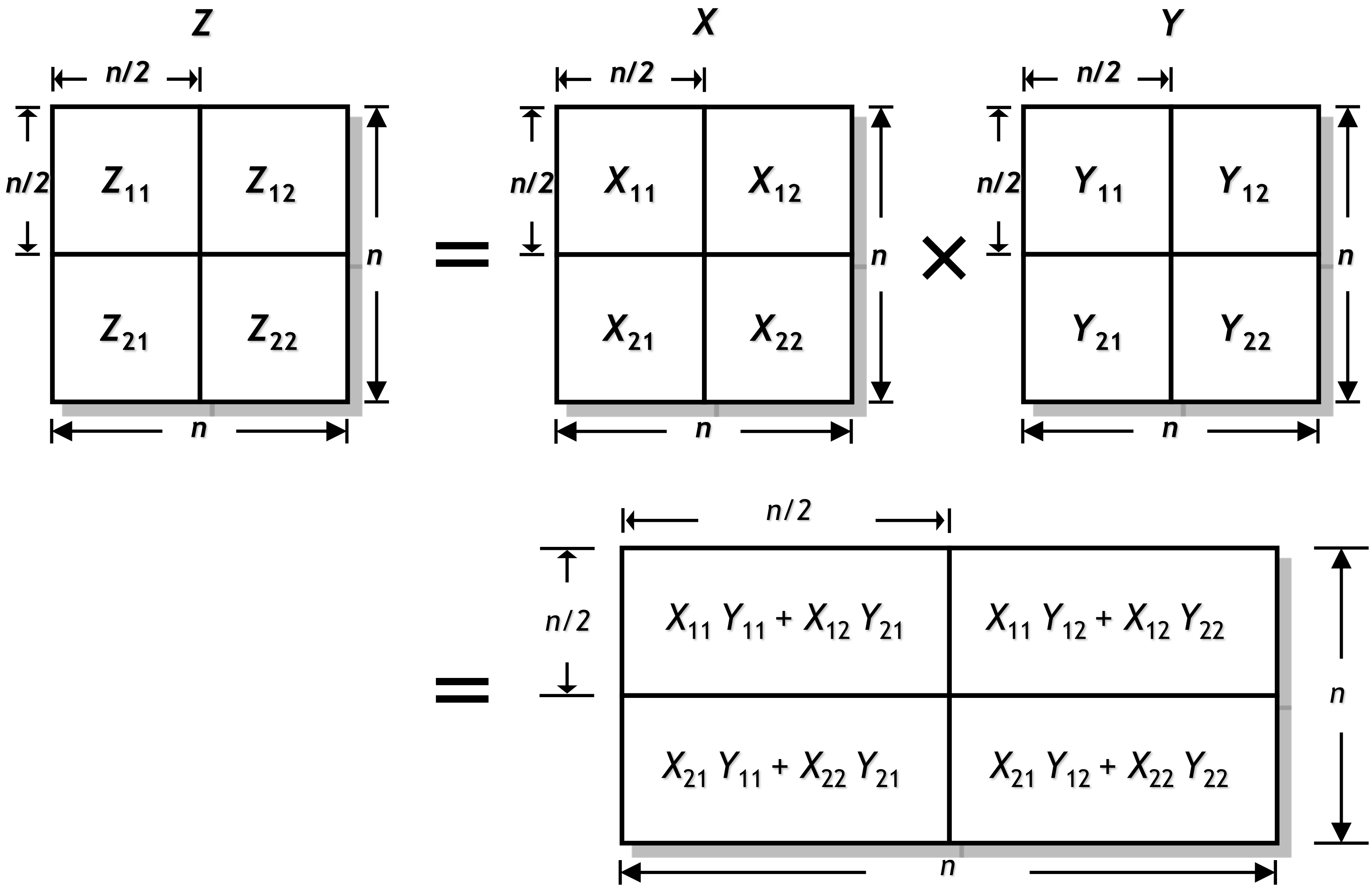
```
Par-Iter-MM ( Z, X, Y )        { X, Y, Z are n × n matrices,
                                 where n is a positive integer }
1. parallel for i ← 1 to n do
2.     parallel for j ← 1 to n do
3.         Z[ i ][ j ] ← 0
4.         for k ← 1 to n do
5.             Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

Work: $T_1(n) = \Theta(n^3)$

Span: $T_\infty(n) = \Theta(\log n + \log n + n) = \Theta(n)$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(n^2)$

Parallel Recursive MM



Parallel Recursive MM

Par-Rec-MM (Z, X, Y) { X, Y, Z *are* $n \times n$ *matrices, where* $n = 2^k$ *for integer* $k \geq 0$ }

if $n = 1$ *then*

    $Z \leftarrow Z + X \cdot Y$

else

    spawn *Par-Rec-MM* (Z_{11}, X_{11}, Y_{11})

    spawn *Par-Rec-MM* (Z_{12}, X_{11}, Y_{12})

    spawn *Par-Rec-MM* (Z_{21}, X_{21}, Y_{11})

    Par-Rec-MM (Z_{22}, X_{21}, Y_{12})

    sync

    spawn *Par-Rec-MM* (Z_{11}, X_{12}, Y_{21})

    spawn *Par-Rec-MM* (Z_{12}, X_{12}, Y_{22})

    spawn *Par-Rec-MM* (Z_{21}, X_{22}, Y_{21})

    Par-Rec-MM (Z_{22}, X_{22}, Y_{22})

    sync

endif

Parallel Recursive MM

```
Par-Rec-MM ( Z, X, Y )    { X, Y, Z are n × n matrices,
                            where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else
    spawn Par-Rec-MM ( Z11, X11, Y11 )
    spawn Par-Rec-MM ( Z12, X11, Y12 )
    spawn Par-Rec-MM ( Z21, X21, Y11 )
          Par-Rec-MM ( Z22, X21, Y12 )
    sync
    spawn Par-Rec-MM ( Z11, X12, Y21 )
    spawn Par-Rec-MM ( Z12, X12, Y22 )
    spawn Par-Rec-MM ( Z21, X22, Y21 )
          Par-Rec-MM ( Z22, X22, Y22 )
    sync
endif
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

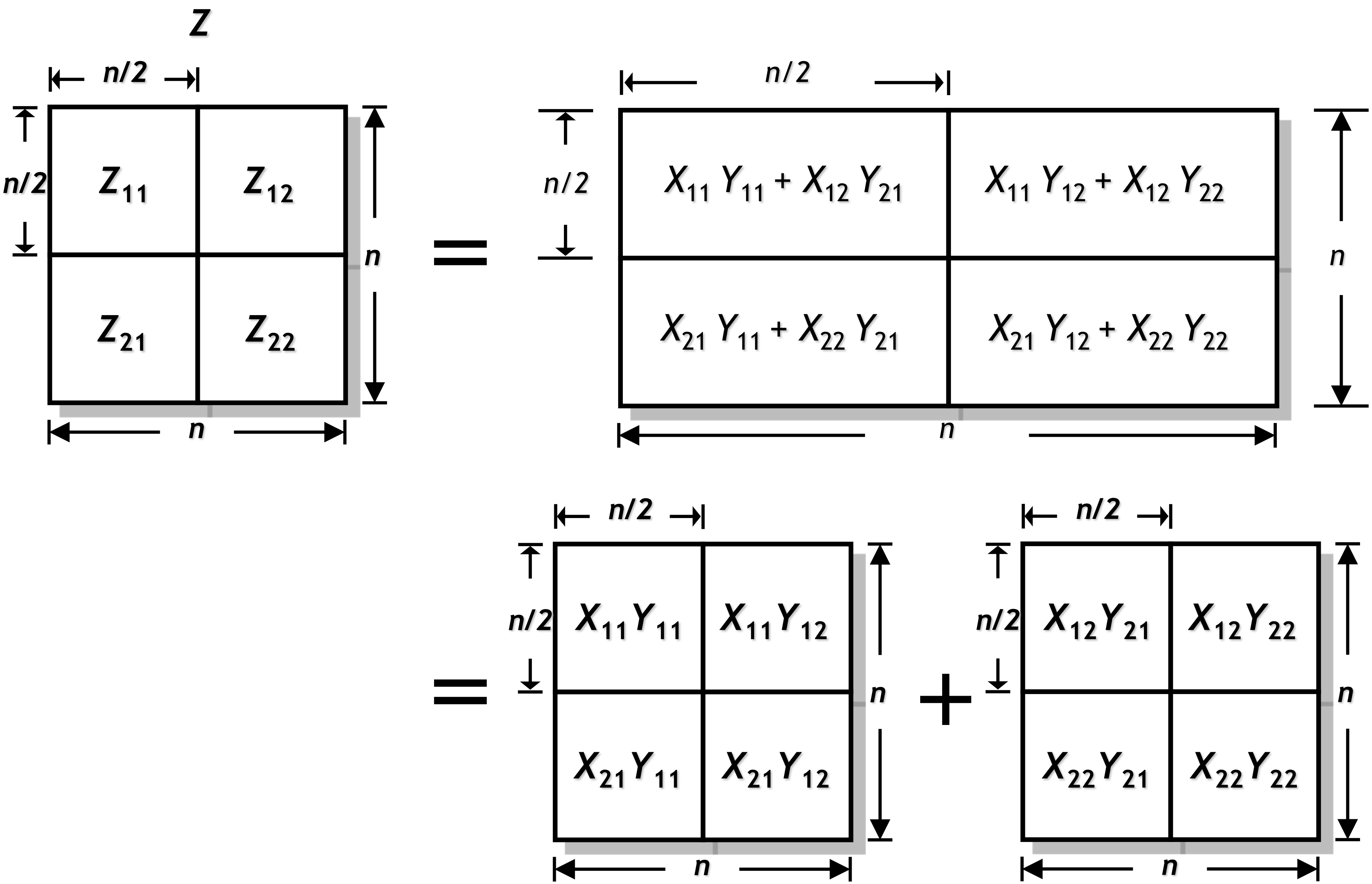
$$= \Theta(n) \qquad \text{[MT Case 1]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(n^2)$

Additional Space:

$$s_\infty(n) = \Theta(1)$$

Recursive MM with More Parallelism



Recursive MM with More Parallelism

```
Par-Rec-MM2 ( Z, X, Y )      { X, Y, Z are n × n matrices,
                               where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else          { T is a temporary n × n matrix }
    spawn Par-Rec-MM2 ( Z11, X11, Y11 )
    spawn Par-Rec-MM2 ( Z12, X11, Y12 )
    spawn Par-Rec-MM2 ( Z21, X21, Y11 )
    spawn Par-Rec-MM2 ( Z22, X21, Y12 )
    spawn Par-Rec-MM2 ( T11, X12, Y21 )
    spawn Par-Rec-MM2 ( T12, X12, Y22 )
    spawn Par-Rec-MM2 ( T21, X22, Y21 )
          Par-Rec-MM2 ( T22, X22, Y22 )
    sync
    parallel for i ← 1 to n do
        parallel for j ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + T[ i ][ j ]
endif
```

Recursive MM with More Parallelism

```
Par-Rec-MM2 ( Z, X, Y )      { X, Y, Z are n × n matrices,
                               where n = 2^k for integer k ≥ 0 }
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    sync
    parallel for i ← 1 to n do
       parallel for j ← 1 to n do
          Z[ i ][ j ] ← Z[ i ][ j ] + T[ i ][ j ]
endif
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta(n^2), & \text{otherwise.} \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log n), & \text{otherwise.} \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n^3}{\log^2 n}\right)$

Additional Space:

$$s_\infty(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ 8s_\infty\left(\frac{n}{2}\right) + \Theta(n^2), & \text{otherwise.} \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

Multithreaded
Merge Sort

Parallel Merge Sort

```
Merge-Sort ( A, p, r )        { sort the elements in A[ p … r ] }

1. if p < r then
2.      q ← ⌊ ( p + r ) / 2 ⌋
3.      Merge-Sort ( A, p, q )
4.      Merge-Sort ( A, q + 1, r )
5.      Merge ( A, p, q, r )
```

⇓

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

1. if p < r then
2.      q ← ⌊ ( p + r ) / 2 ⌋
3.      spawn Merge-Sort ( A, p, q )
4.            Merge-Sort ( A, q + 1, r )
5.      sync
6.      Merge ( A, p, q, r )
```

Parallel Merge Sort

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p ... r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Merge ( A, p, q, r )
```

Work: $$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n \log n) \qquad \text{[MT Case 2]}$$

Span: $$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n) \qquad \text{[MT Case 3]}$$

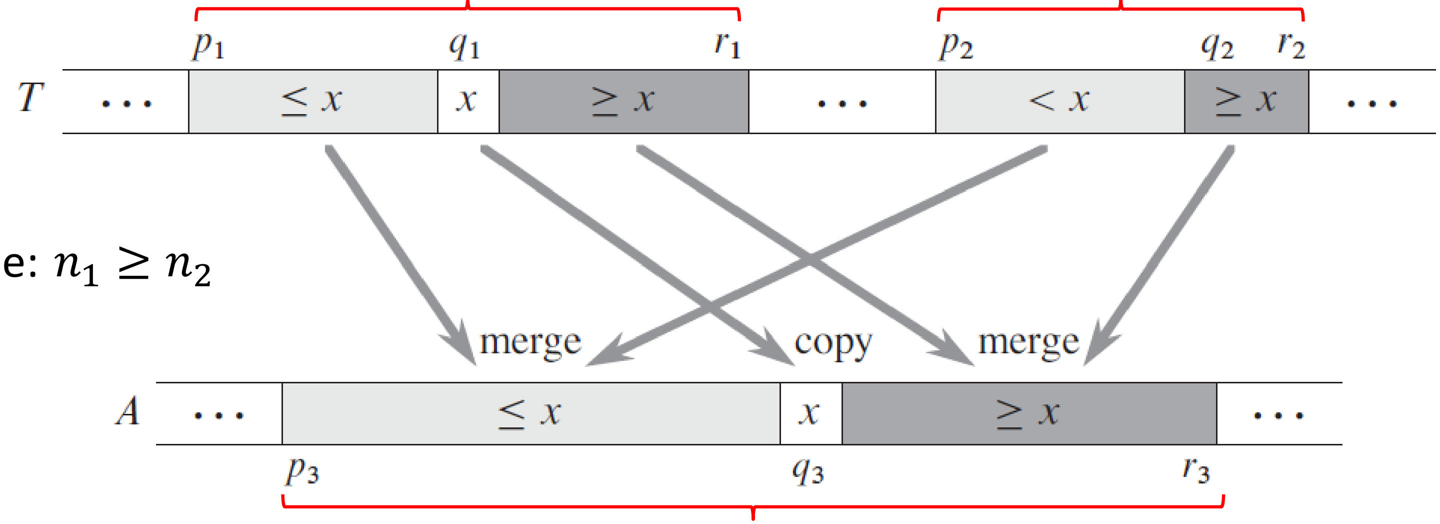
Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\log n)$$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1 .. r_1]$ $\quad$ $T[p_2 .. r_2]$

suppose: $n_1 \geq n_2$



merged output: $A[p_3 .. r_3]$

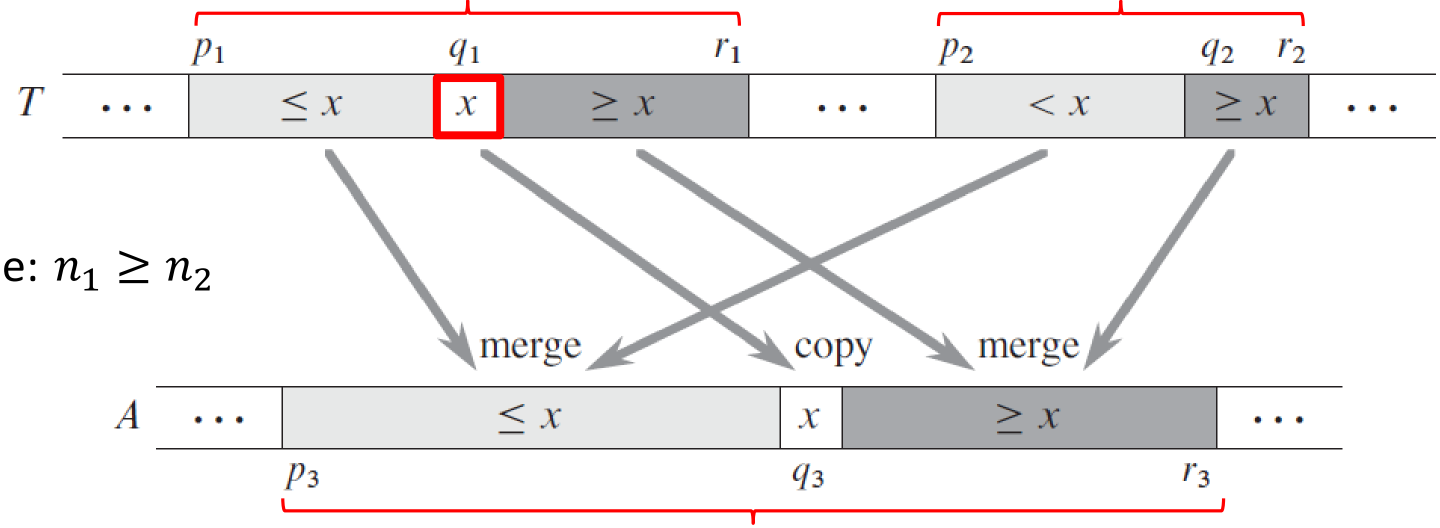
$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., “Introduction to Algorithms”, 3rd Edition

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1 \,..\, r_1]$ $\quad$ $T[p_2 \,..\, r_2]$



suppose: $n_1 \ge n_2$

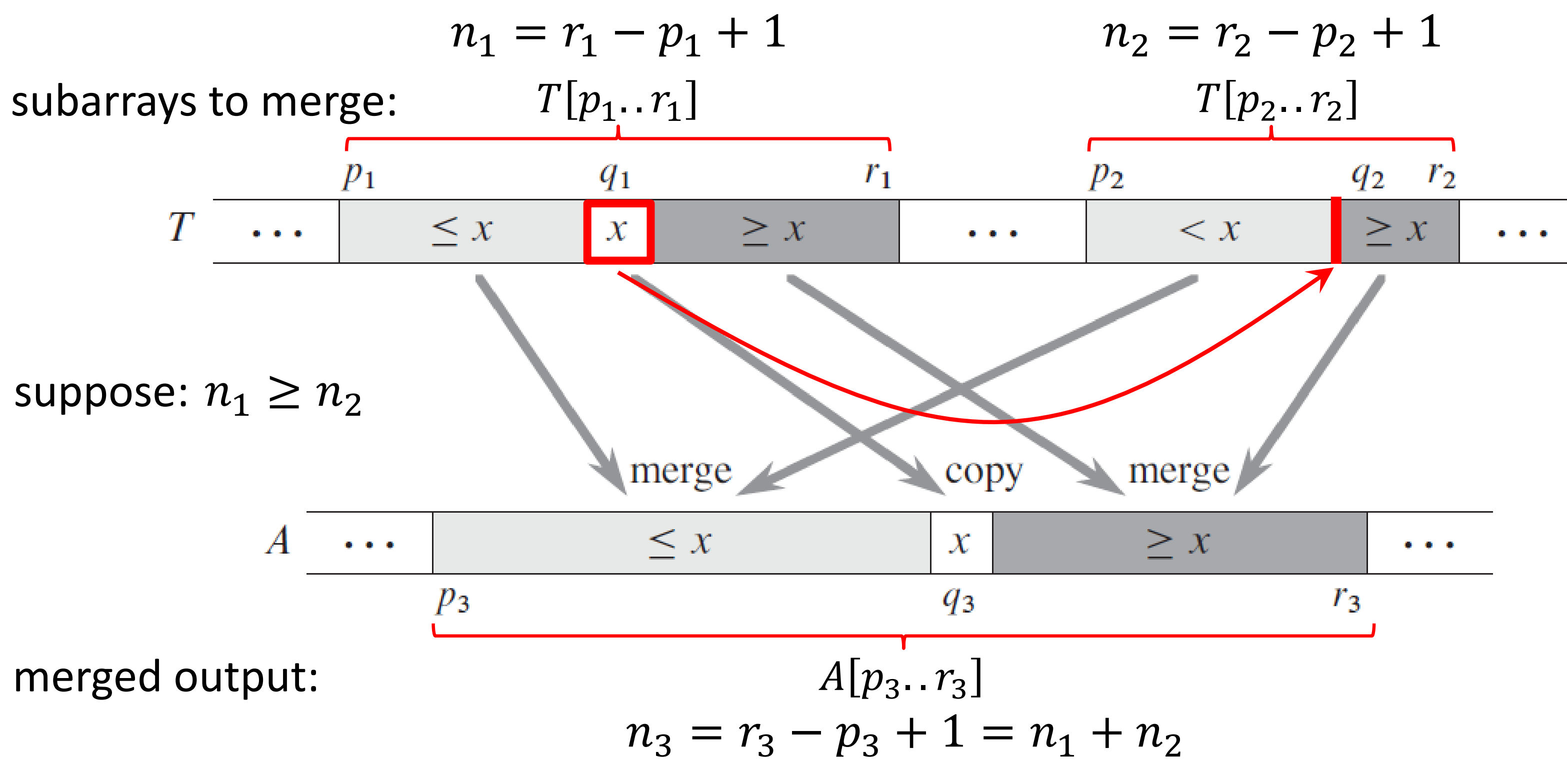
merged output: $A[p_3 \,..\, r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Step 1: Find $x = T[q_1]$, where q_1 is the midpoint of $T[p_1 \,..\, r_1]$

Parallel Merge



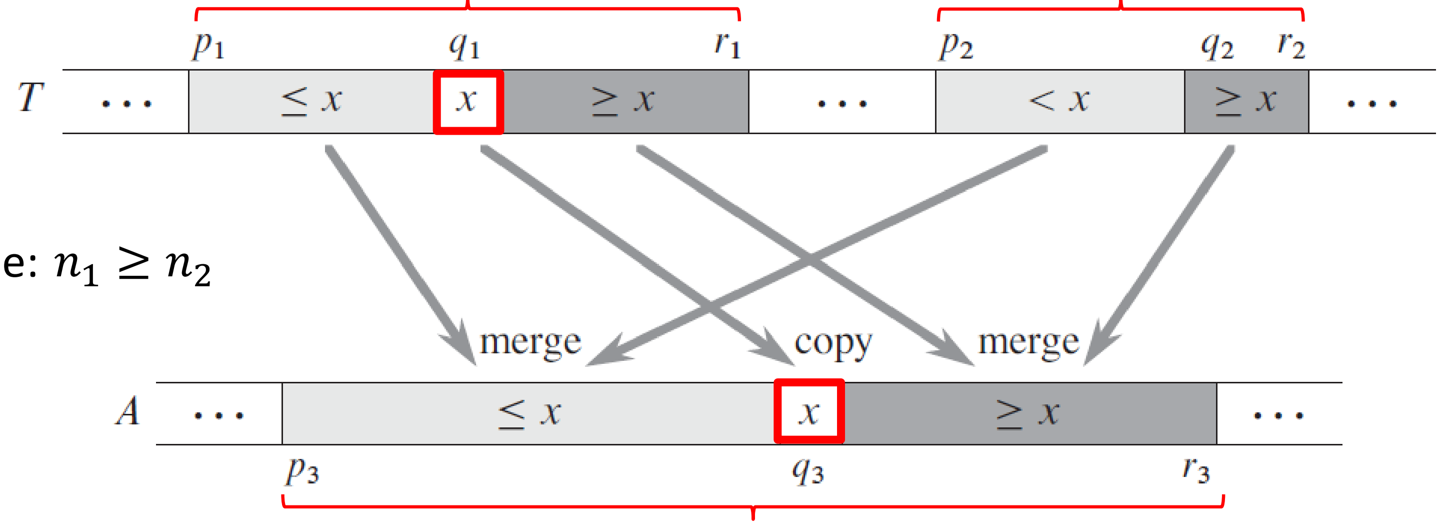
Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Step 2: Use binary search to find the index q_2 in subarray $T[p_2..r_2]$ so that the subarray would still be sorted if we insert x between $T[q_2 - 1]$ and $T[q_2]$

Parallel Merge

$n_1 = r_1 - p_1 + 1$ $\qquad$ $n_2 = r_2 - p_2 + 1$

subarrays to merge: $T[p_1..r_1]$ $\qquad$ $T[p_2..r_2]$



suppose: $n_1 \geq n_2$

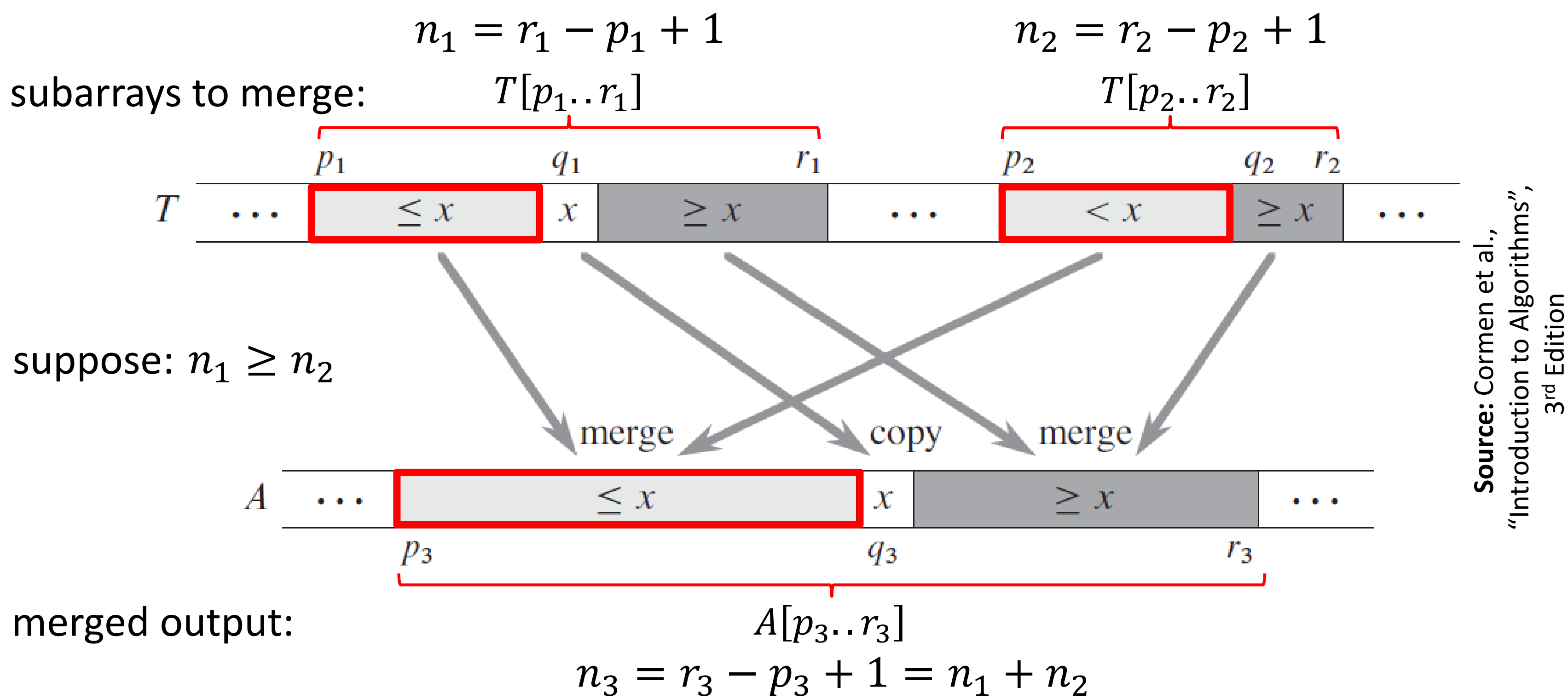
merged output: $A[p_3..r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Step 3: Copy x to $A[q_3]$, where $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$

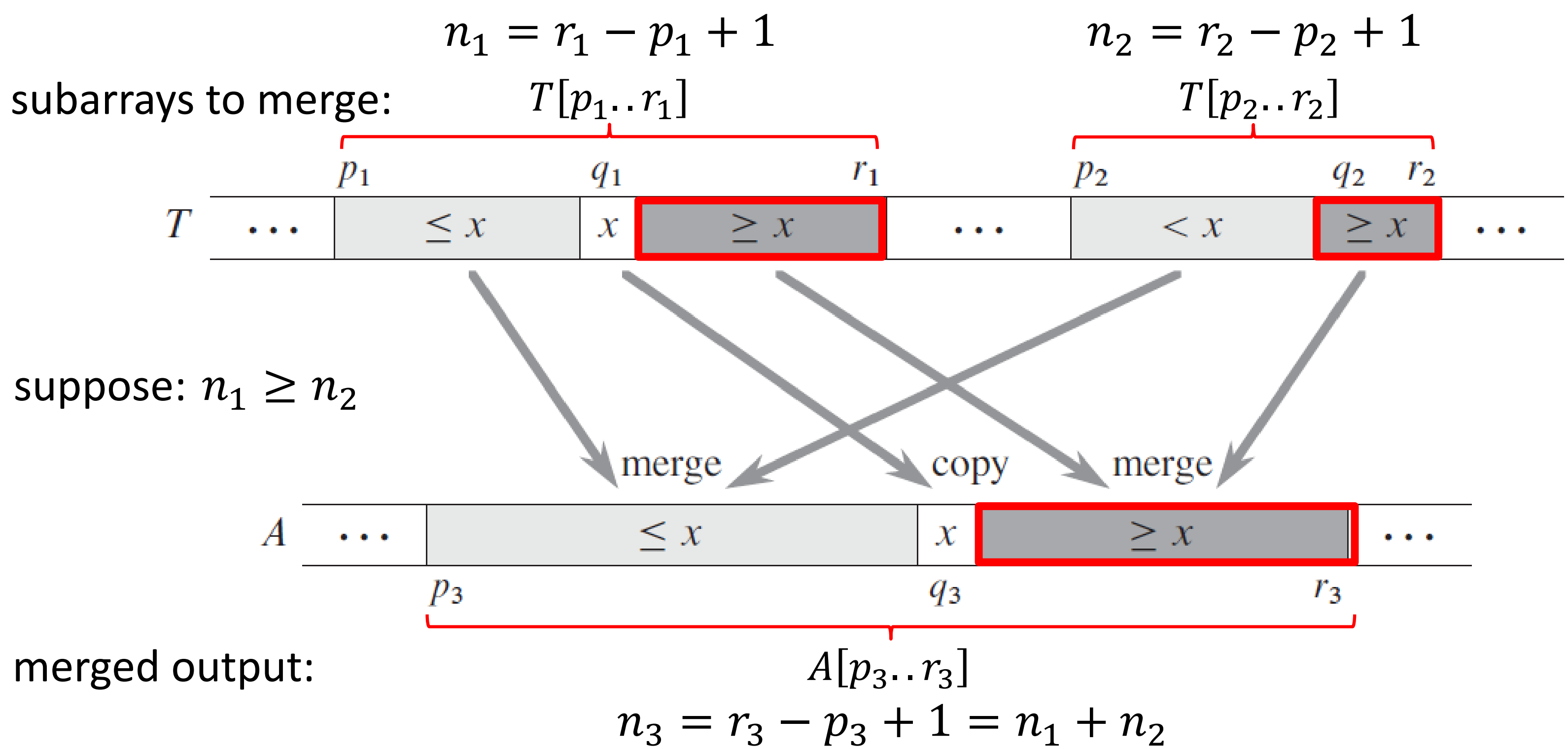
Parallel Merge



Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1 \ldots q_1 - 1]$ with $T[p_2 \ldots q_2 - 1]$, and place the result into $A[p_3 \ldots q_3 - 1]$

Parallel Merge



Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1 .. q_1 - 1]$ with $T[p_2 .. q_2 - 1]$, and place the result into $A[p_3 .. q_3 - 1]$

Step 4(b): Recursively merge $T[q_1 + 1 .. r_1]$ with $T[q_2 + 1 .. r_2]$, and place the result into $A[q_3 + 1 .. r_3]$

Parallel Merge

Par-Merge ($T, p_1, r_1, p_2, r_2, A, p_3$)

$n_1 \leftarrow r_1 - p_1 + 1, \quad n_2 \leftarrow r_2 - p_2 + 1$

if $n_1 < n_2$ *then*

    $p_1 \leftrightarrow p_2, \quad r_1 \leftrightarrow r_2, \quad n_1 \leftrightarrow n_2$

if $n_1 = 0$ *then return*

else

    $q_1 \leftarrow \lfloor (p_1 + r_1) / 2 \rfloor$

    $q_2 \leftarrow$ *Binary-Search* ($T[q_1], T, p_2, r_2$)

    $q_3 \leftarrow p_3 + (q_1 - p_1) + (q_2 - p_2)$

    $A[q_3] \leftarrow T[q_1]$

    spawn *Par-Merge* ($T, p_1, q_1-1, p_2, q_2-1, A, p_3$)

        Par-Merge ($T, q_1+1, r_1, q_2+1, r_2, A, q_3+1$)

    sync

Parallel Merge

```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
1.  n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
2.  if n1 < n2 then
3.      p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
4.  if n1 = 0 then return
5.  else
6.      q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
7.      q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
8.      q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
9.      A[ q3 ] ← T[ q1 ]
10.     spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
11.           Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
12.     sync
```

We have,

$n_2 \leq n_1 \Rightarrow 2n_2 \leq n_1 + n_2 = n$

In the worst case, a recursive call in lines 9-10 merges half the elements of $T[p_1..r_1]$ with all elements of $T[p_2..r_2]$.

Hence, #elements involved in such a call:

$$\left\lfloor \frac{n_1}{2} \right\rfloor + n_2 \leq \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_2}{2} = \frac{n_1 + n_2}{2} + \frac{2n_2}{4} \leq \frac{n}{2} + \frac{n}{4} = \frac{3n}{4}$$

Parallel Merge

```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
    n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
    if n1 < n2 then
        p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
    if n1 = 0 then return
    else
        q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
        q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
        q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
        A[ q3 ] ← T[ q1 ]
        spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
              Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
        sync
```

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{3n}{4}\right) + \Theta(\log n), & otherwise. \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Work:

Clearly, $T_1(n) = \Omega(n)$

We show below that, $T_1(n) = O(n)$

For some $\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have the following recurrence,

$$T_1(n) = T_1(\alpha n) + T_1\big((1-\alpha)n\big) + O(\log n)$$

Assuming $T_1(n) \le c_1 n - c_2 \log n$ for positive constants c_1 and c_2, and substituting on the right hand side of the above recurrence gives us: $T_1(n) \le c_1 n - c_2 \log n \ = O(n)$.

Hence, $T_1(n) = \Theta(n)$.

Parallel Merge Sort with Parallel Merge

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p ... r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Par-Merge ( A, p, q, r )
```

Work: $$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n \log n) \qquad \text{[MT Case 2]}$$

Span: $$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log^2 n), & otherwise. \end{cases}$$

$$= \Theta(\log^3 n) \qquad \text{[MT Case 2]}$$

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$$