CSE 613: Parallel Programming

Lecture 6
(Basic Parallel Algorithmic Techniques)

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Some Basic Techniques

1. Divide-and-Conquer
   - Recursive
   - Non-recursive
   - Contraction
2. Pointer Techniques
   - Pointer Jumping
   - Graph Contraction
3. Randomization
   - Sampling
   - Symmetry Breaking

Divide-and-Conquer

1. **Divide:** divide the original problem into smaller subproblems that are easier are to solve
2. **Conquer:** solve the smaller subproblems (perhaps recursively)
3. **Merge:** combine the solutions to the smaller subproblems to obtain a solution for the original problem

Divide-and-Conquer

- The divide-and-conquer paradigm improves program modularity, and often leads to simple and efficient algorithms
- Since the subproblems created in the divide step are often independent, they can be solved in parallel
- If the subproblems are solved recursively, each recursive divide step generates even more independent subproblems to be solved in parallel
- In order to obtain a highly parallel algorithm it is often necessary to parallelize the divide and merge steps, too

Recursive D&C: Parallel Merge Sort

```
Merge-Sort ( A, p, r )        { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    Merge-Sort ( A, p, q )
    Merge-Sort ( A, q + 1, r )
    Merge ( A, p, q, r )
```

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Merge ( A, p, q, r )
```

Recursive D&C: Parallel Merge Sort

Par-Merge-Sort (A, p, r) { *sort the elements in* $A[\, p \dots r \,]$ }

1. *if* $p < r$ *then*
2. $q \leftarrow \lfloor (p + r) / 2 \rfloor$
3. *spawn* *Merge-Sort* (A, p, q)
4. *Merge-Sort* (A, $q + 1$, r)
5. *sync*
6. *Merge* (A, p, q, r)

Work: $$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n \log n)$$

Span: $$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n)$$

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\log n)$$

Too small!
Must parallelize the *Merge* routine.

Non-Recursive D&C: Parallel Sample Sort

Task: Sort an array $A[1, \ldots, n]$ of n distinct keys using $p \leq n$ processors.

Steps (without oversampling):

1. **Pivot Selection:** Select (uniformly at random) and sort $m = p - 1$ pivot elements $e_1, e_2, \ldots, e_m$. These elements define $m + 1 = p$ buckets: $(-\infty, e_1), (e_1, e_2), \ldots, (e_{m-1}, e_m), (e_m, +\infty)$
2. **Local Sort:** Divide A into p segments of equal size, assign each segment to different processor, and sort locally.
3. **Local Bucketing:** If $m \leq \frac{n}{p}$, each processor inserts the pivot elements into its local sorted sequence using binary search, otherwise inserts its local elements into the sorted pivot elements. Thus the keys are divided among $m + 1 = p$ buckets.
4. **Merge Local Buckets:** Processor i $(1 \leq i \leq p)$ merges the contents of bucket i from all processors through a local sort.
5. **Final Result:** Each processor copies its bucket to a global output array so that bucket i $(1 \leq i \leq p - 1)$ precedes bucket $i + 1$ in the output.

Non-Recursive D&C: Parallel Sample Sort

Steps (without oversampling):

1. **Pivot Selection:** $\mathrm{O}(m\log(m)) = \mathrm{O}(p\log p)$ [worst case]
2. **Local Sort:** $\mathrm{O}\left(\frac{n}{p}\log\frac{n}{p}\right)$ [worst case]
3. **Local Bucketing:**

$$\mathrm{O}\left(min\left(m\log\frac{n}{p},\frac{n}{p}\log m\right)\right) = \mathrm{O}\left(\frac{n}{p}\log\frac{n}{p}\right)$$ [worst case]

4. **Merge Local Buckets:** $\mathrm{O}\left(\frac{n}{m}\log\frac{n}{m}\right) = \mathrm{O}\left(\frac{n}{p}\log\frac{n}{p}\right)$ [expected]

   (not quite correct as the largest bucket can have $\Theta\left(\frac{n}{m}\log m\right)$ keys with significant probability)

5. **Final Result:** $\mathrm{O}\left(\frac{n}{m}\right) = \mathrm{O}\left(\frac{n}{p}\right)$ [expected]

Overall: $\mathrm{O}\left(\frac{n}{p}\log\frac{n}{p} + p\log p\right)$ [expected]

Contraction

1. **Reduce:** reduce the original problem to a smaller problem
2. **Conquer:** solve the smaller problem (often recursively)
3. **Expand:** use the solution to the smaller problem
   to obtain a solution for the original larger problem

Contraction: Prefix Sums

Input: A sequence of n elements $\{x_1, x_2, \ldots, x_n\}$ drawn from a set S with a binary associative operation, denoted by $\oplus$.

Output: A sequence of n partial sums $\{s_1, s_2, \ldots, s_n\}$, where

$$s_i = x_1 \oplus x_2 \oplus \ldots \oplus x_i \text{ for } 1 \le i \le n.$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
5	3	7	1	3	6	2	4

$\oplus$ = binary addition

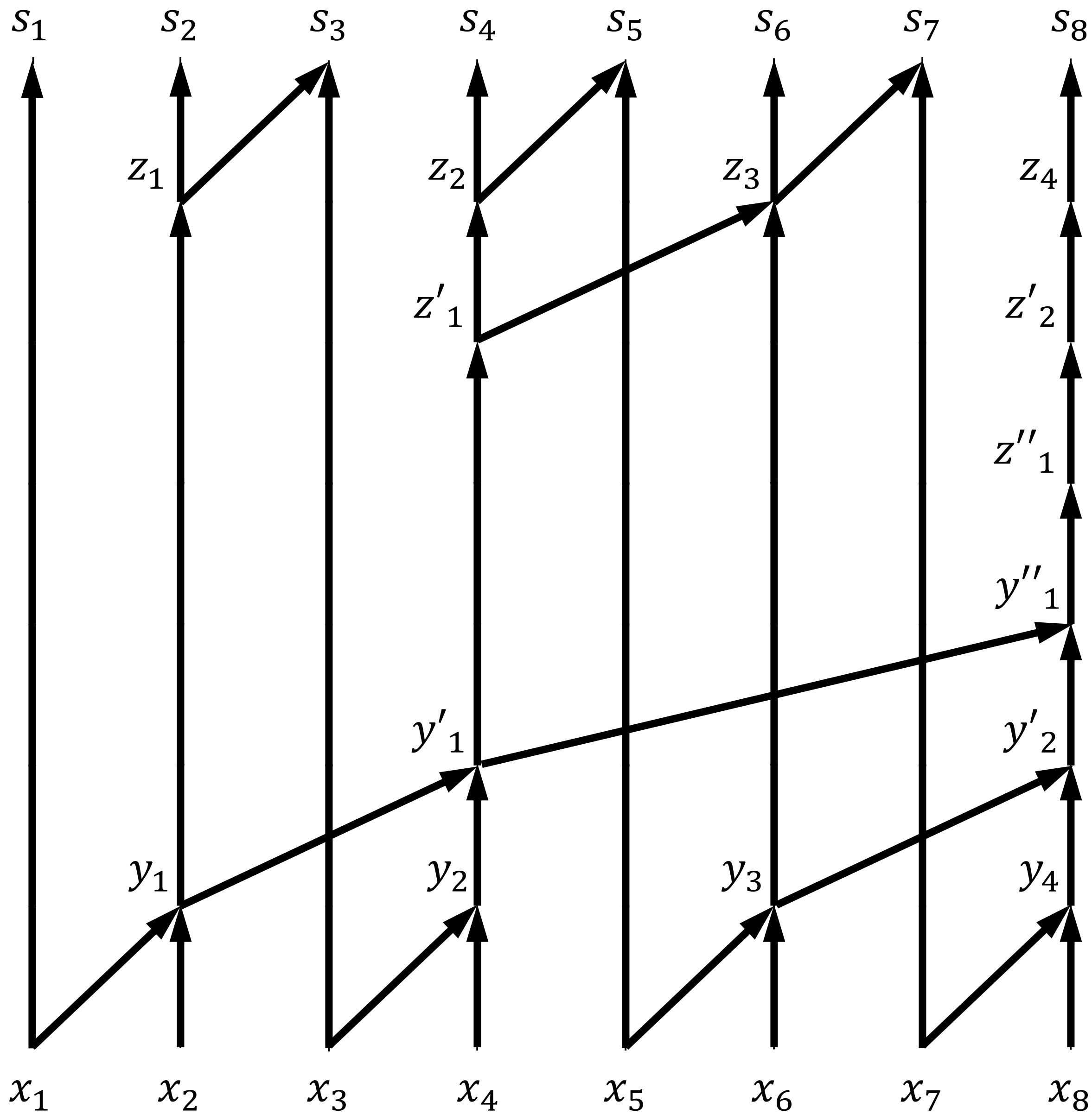
5	8	15	16	19	25	27	31
s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8

Contraction: Prefix Sums

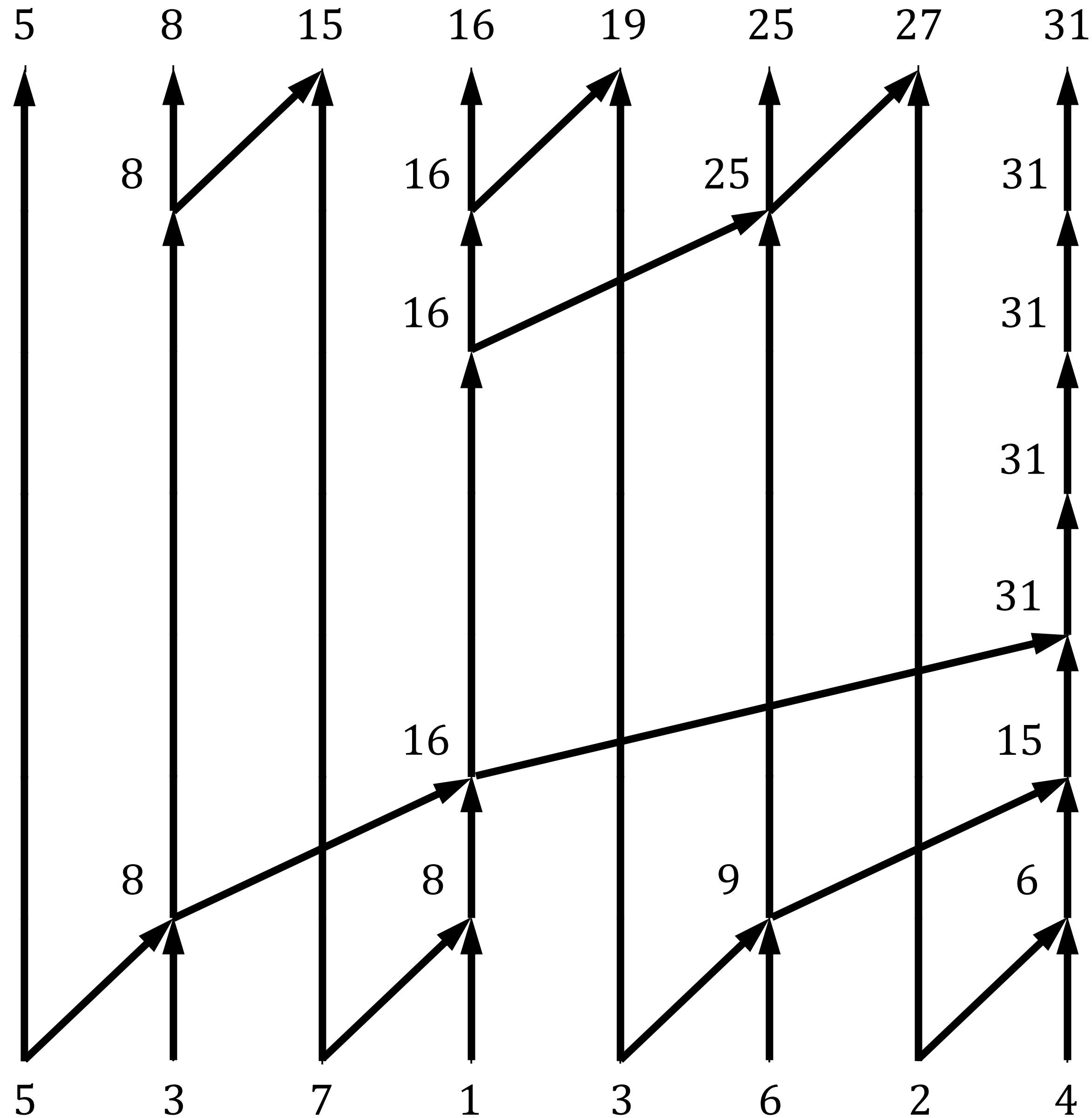
Prefix-Sum ($\langle x_1, x_2, \ldots, x_n \rangle, \oplus$) { $n = 2^k$ *for some* $k \geq 0$. *Return prefix sums* $\langle s_1, s_2, \ldots, s_n \rangle$ }

if $n = 1$ *then*
    $s_1 \leftarrow x_1$
else
    parallel for $i \leftarrow 1$ *to* $n/2$ *do*
        $y_i \leftarrow x_{2i-1} \oplus x_{2i}$
    $\langle z_1, z_2, \ldots, z_{n/2} \rangle \leftarrow$ *Prefix-Sum*($\langle y_1, y_2, \ldots, y_{n/2} \rangle, \oplus$)
    parallel for $i \leftarrow 1$ *to* n *do*
        if $i = 1$ *then* $s_1 \leftarrow x_1$
        else if $i = even$ *then* $s_i \leftarrow z_{i/2}$
            else $s_i \leftarrow z_{(i-1)/2} \oplus x_i$
return $\langle s_1, s_2, \ldots, s_n \rangle$

Contraction: Prefix Sums



Contraction: Prefix Sums



Contraction: Prefix Sums

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    $\langle z_1, z_2, \dots, z_{n/2} \rangle \leftarrow$ *Prefix-Sum*($\langle y_1, y_2, \dots, y_{n/2} \rangle, \oplus$)
    parallel for $i \leftarrow 1$ *to* n *do*
        if $i = 1$ *then* $s_1 \leftarrow x_1$
        else if $i = even$ *then* $s_i \leftarrow z_{i/2}$
            else $s_i \leftarrow z_{(i-1)/2} \oplus x_i$
return $\langle s_1, s_2, \dots, s_n \rangle$

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n)$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(\log n)$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log n}\right)$

Observe that we have assumed here that a *parallel for loop* can be executed in $\Theta(1)$ time. But recall that *cilk_for* is implemented using divide-and-conquer, and so in practice, it will take $\Theta(\log n)$ time. In that case, we will have $T_\infty(n) = \Theta(\log^2 n)$, and parallelism $= \Theta\left(\frac{n}{\log^2 n}\right)$.

Pointer Techniques: Pointer Jumping

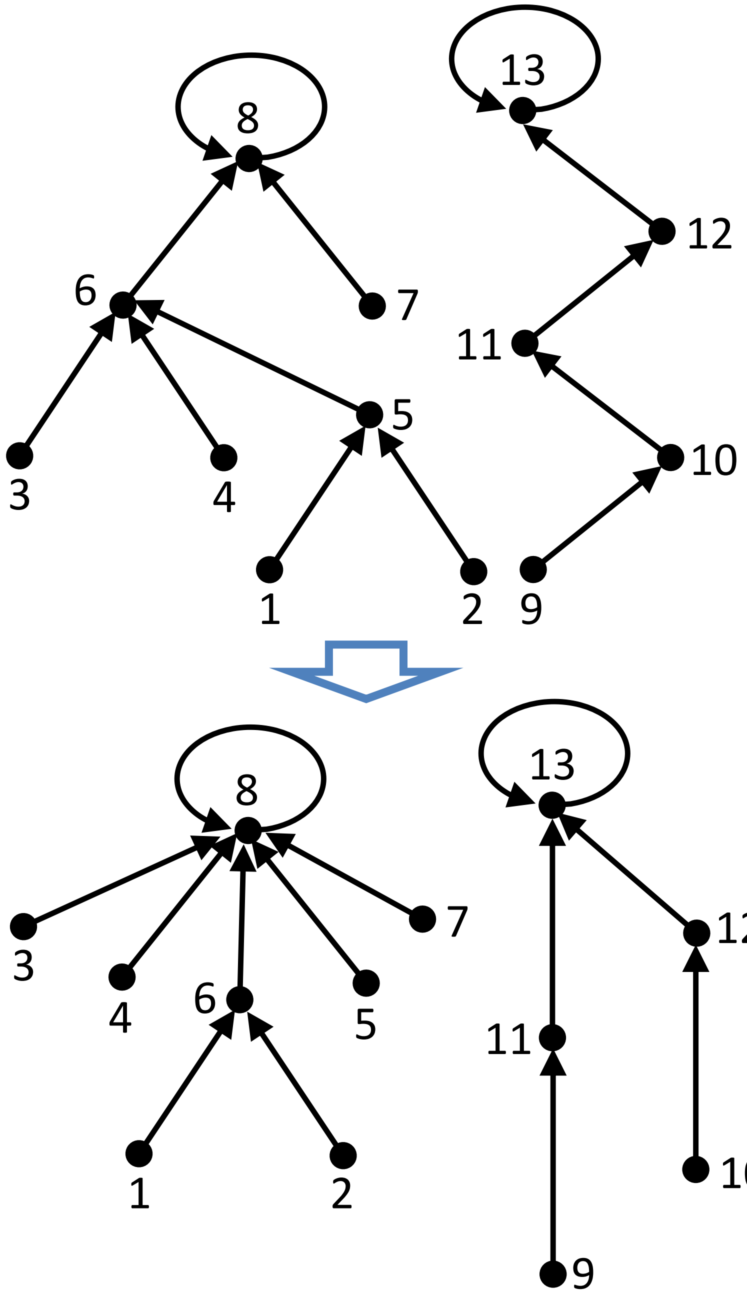
The *pointer jumping* (or *path doubling*) technique allows fast processing of data stored in the form of a set of rooted directed trees.

For every node v in the set pointer jumping involves replacing $v \rightarrow next$ with $v \rightarrow next \rightarrow next$ at every step.

Some Applications

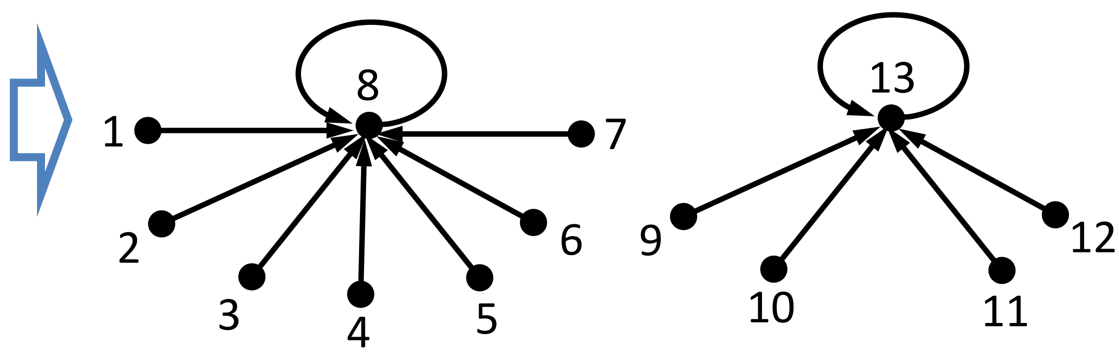
- Finding the roots of a forest of directed trees
- Parallel prefix on rooted directed trees
- List ranking

Pointer Jumping: Roots of a Forest of Directed Trees



Find-Roots (n, P, S) { ***Input:*** *A forest of rooted directed trees, each with a self-loop at its root, such that each edge is specified by* $\langle v, P(v) \rangle$ *for* $1 \le v \le n$. ***Output:*** *For each* v, *the root* $S(v)$ *of the tree containing* v. }

parallel for $v \leftarrow 1$ *to* n *do*
 $S(v) \leftarrow P(v)$
$flag \leftarrow true$
while $flag = true$ *do*
 $flag \leftarrow false$
 parallel for $v \leftarrow 1$ *to* n *do*
  $S(v) \leftarrow S(S(v))$
  if $S(v) \neq S(S(v))$ *then* $flag \leftarrow true$



Pointer Jumping: Roots of a Forest of Directed Trees

Let h be the maximum height of any tree in the forest.

Observe that the distance between v and $S(v)$ doubles after each iteration until $S\big(S(v)\big)$ is the root of the tree containing v.

Find-Roots (n, P, S) { ***Input:*** *A forest of rooted directed trees, each with a self-loop at its root, such that each edge is specified by* $\langle v, P(v) \rangle$ *for* $1 \le v \le n$.
Output: *For each* v, *the root* $S(v)$ *of the tree containing* v. }

```
parallel for v ← 1 to n do
    S(v) ← P(v)
flag ← true
while flag = true do
    flag ← false
    parallel for v ← 1 to n do
        S(v) ← S(S(v))
        if S(v) ≠ S(S(v)) then flag ← true
```

Hence, the number of iterations is $\log h$. Thus (assuming that each parallel for loop takes $\Theta(1)$ time to execute),

Work: $T_1(n) = O(n \log h)$ and **Span:** $T_\infty(n) = \Theta(\log h)$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = O(n)$

Pointer Techniques: Graph Contraction

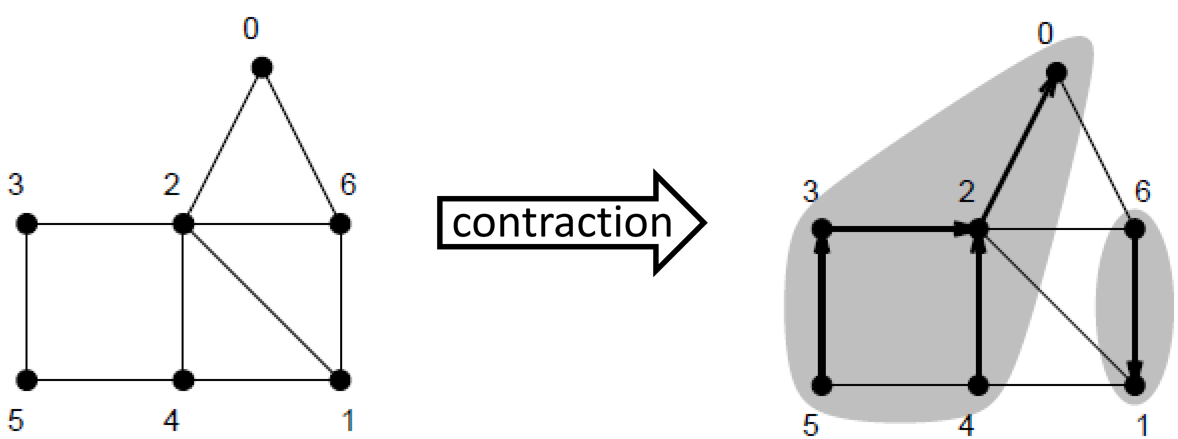
1. **Contract:** the graph is reduced in size while maintaining some of its original properties (depending on the problem)
2. **Conquer:** solve the problem on the contracted graph (often recursively)
3. **Expand:** use the solution to the contracted graph to obtain a solution for the original graph

Some Applications

- Finding connected components of a graph
- Minimum spanning trees

Graph Contraction: Connected Components (CC)

1. Direct the edges to form a forest of rooted directed trees
2. Use pointer jumping to contract each such tree to a single vertex
3. Recursively find the CCs of the contracted graph
4. Expand those CCs to label the vertices of the original graph with CC numbers



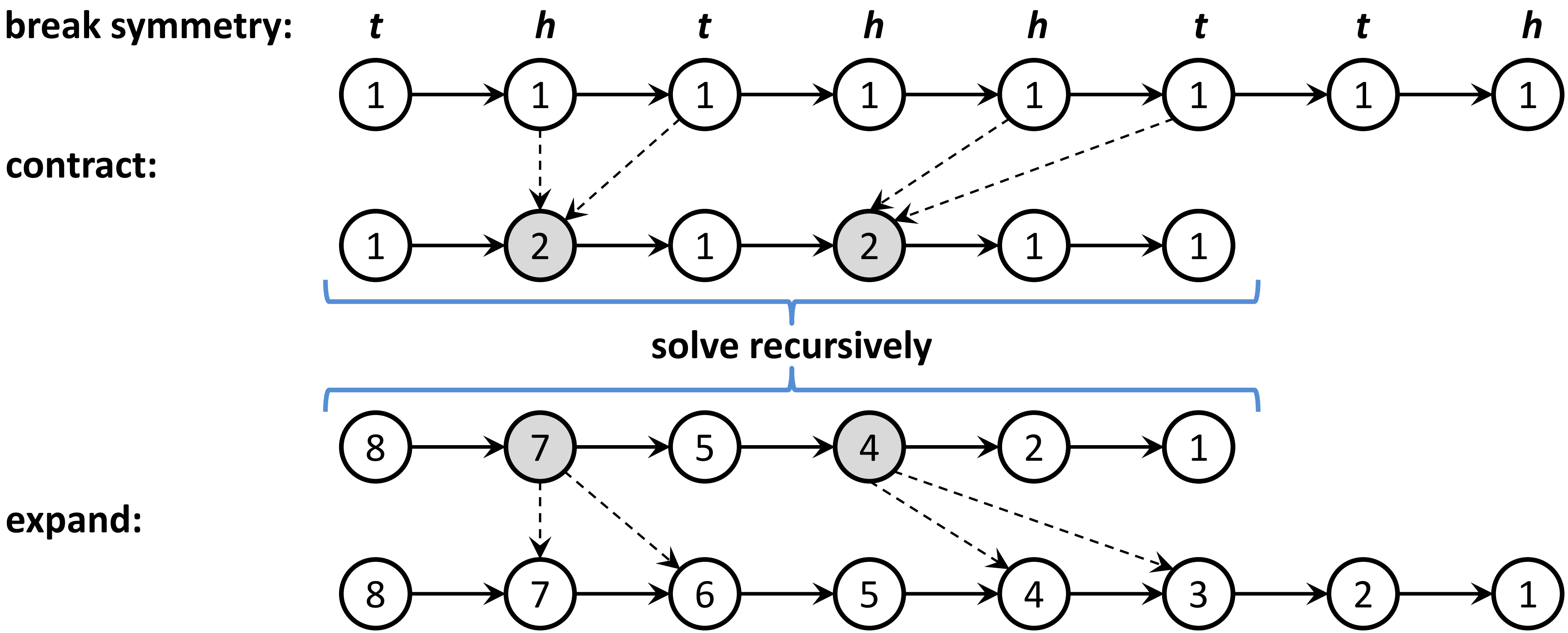
Randomization: Symmetry Breaking

A technique to break symmetry in a structure, e.g., a graph which can locally look the same to all vertices.

Some Applications

- Prefix sums in a linked list (list ranking)
- Selecting a large independent set from a graph
- Graph contraction

Symmetry Breaking: List Ranking



1. Flip a coin for each list node
2. If a node u points to a node v, and u got a head while v got a tail, combine u and v
3. Recursively solve the problem on the contracted list
4. Project this solution back to the original list

Symmetry Breaking: List Ranking

In every iteration a node gets removed with probability $\frac{1}{4}$

(as a node gets head with probability $\frac{1}{2}$ and the next node gets tail with probability $\frac{1}{2}$).

Hence, a quarter of the nodes get removed in each iteration (expected number).

Thus the expected number of iterations is $\Theta(\log n)$.

In fact, it can be shown that with high probability,

$$T_1(n) = \mathrm{O}(n) \text{ and } T_\infty(n) = \mathrm{O}(\log n)$$