CSE 638: Advanced Algorithms

Lectures 20 & 21
(Cache-oblivious Priority Queue with Decrease-Keys)

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Cache-Oblivious Buffer Heap

		Amortized I/O Bounds	
	Priority Queue	***Delete / Delete-Min***	***Decrease-Key***
Cache-oblivious	Buffer Heap	$O\left(\frac{1}{B}\log_2\frac{N}{B}\right)$	
Cache-aware	Tournament Tree		
Internal Memory	Binary Heap (worst-case)	$O\left(\log_2 N\right)$	
	Fibonacci Heap	$O\left(\log_2 N\right)$	$O\left(1\right)$

Cache-Oblivious Buffer Heap: Structure

Consists of $r = 1 + \lceil \log_2 N \rceil$ levels, where N = total number of elements.

For $0 \leq i \leq r - 1$, level i contains two buffers:

- ***element buffer B_i***

  contains elements of the form (x, k_x), where x is the element id, and k_x is its key

- ***update buffer U_i***

  contains updates (*Delete*, *Decrease-Key* and *Sink*), each augmented with a time-stamp.

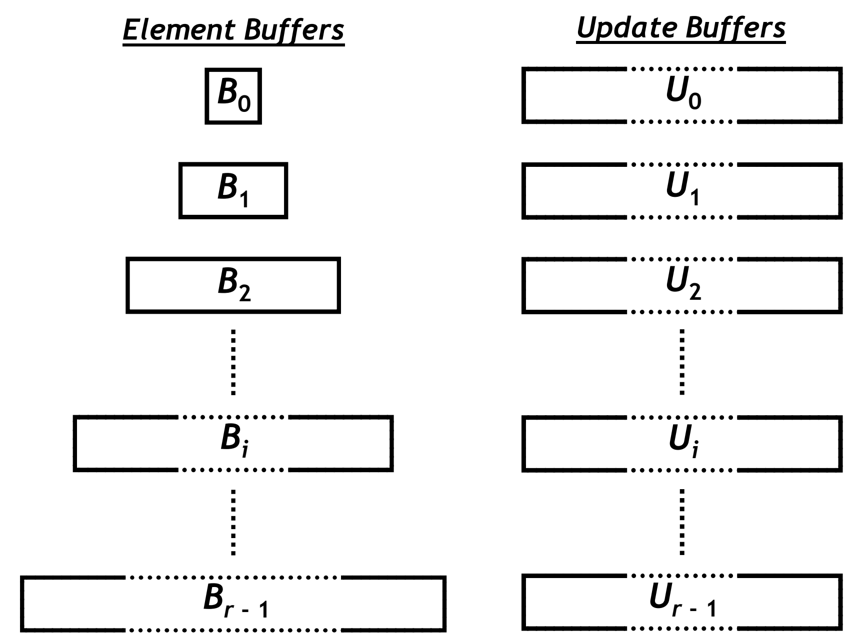


Fig: The Buffer Heap

Cache-Oblivious Buffer Heap: Invariants

Invariant 1: $|B_i| \leq 2^i$

Invariant 2:

(a) No key in B_i is larger than any key in B_{i+1}

(b) For each element x in B_i, all updates yet to be applied on x reside in $U_0, U_1, \ldots, U_i$

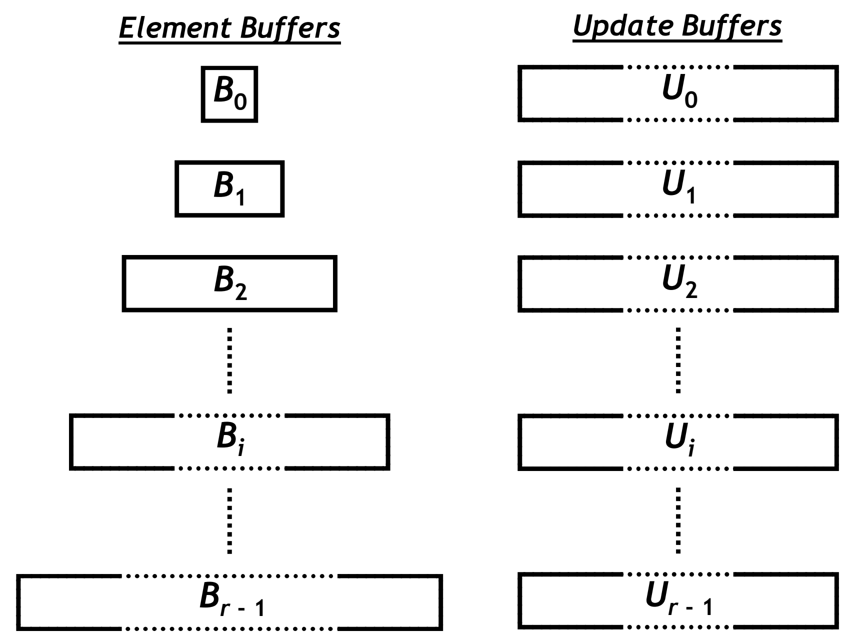


Fig: The Buffer Heap

Invariant 3:

(a) Each B_i is kept sorted by element id

(b) Each U_i (except U_0) is kept (coarsely) sorted by element id and time-stamp

Cache-Oblivious Buffer Heap: Operations

The following operations are supported:

– ***Delete(x)***:

Deletes the element x from the queue.

– ***Delete-Min( )***:

Extracts an element with minimum key from queue.

– ***Decrease-Key(x, k_x)***: (weak *Decrease-Key*)

If x already exists in the queue, replaces key k'_x of x with $\min(k_x, k'_x)$, otherwise inserts x with key k_x into the queue.

A new element x with key k_x can be inserted into queue by *Decrease-Key*(x, k_x).

Cache-Oblivious Buffer Heap: Operations

Decrease-Key(x, k_x) :

Insert the operation into U_0 augmented with current time-stamp.

Delete(x) :

Insert the operation into U_0 augmented with current time-stamp.

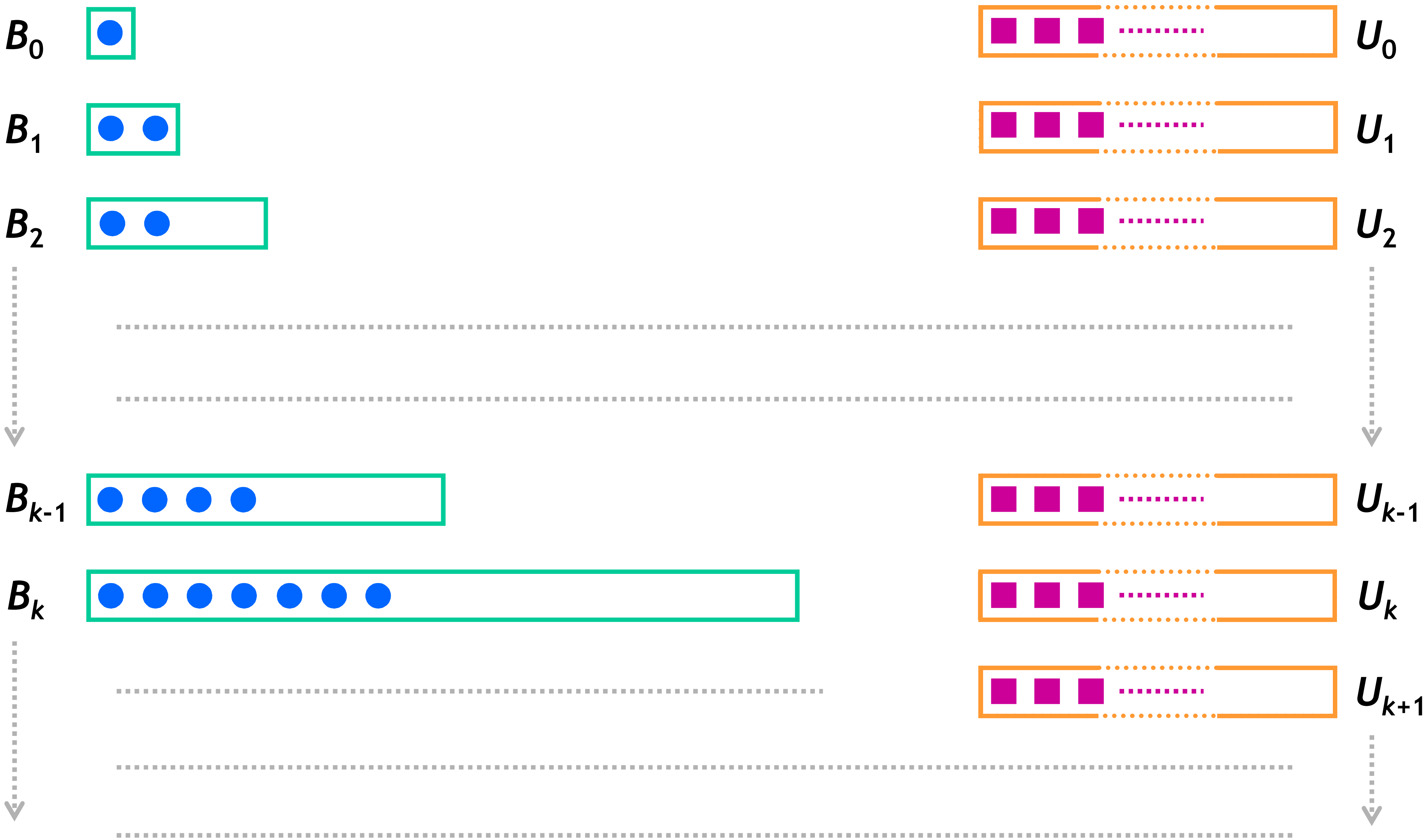
Delete-Min( ) :

Two phases:

- Descending Phase (Apply Updates)
- Ascending Phase (Redistribute Elements)

Cache-Oblivious Buffer Heap: Delete-Min

Delete-Min( ) - Descending Phase (Apply Updates) :



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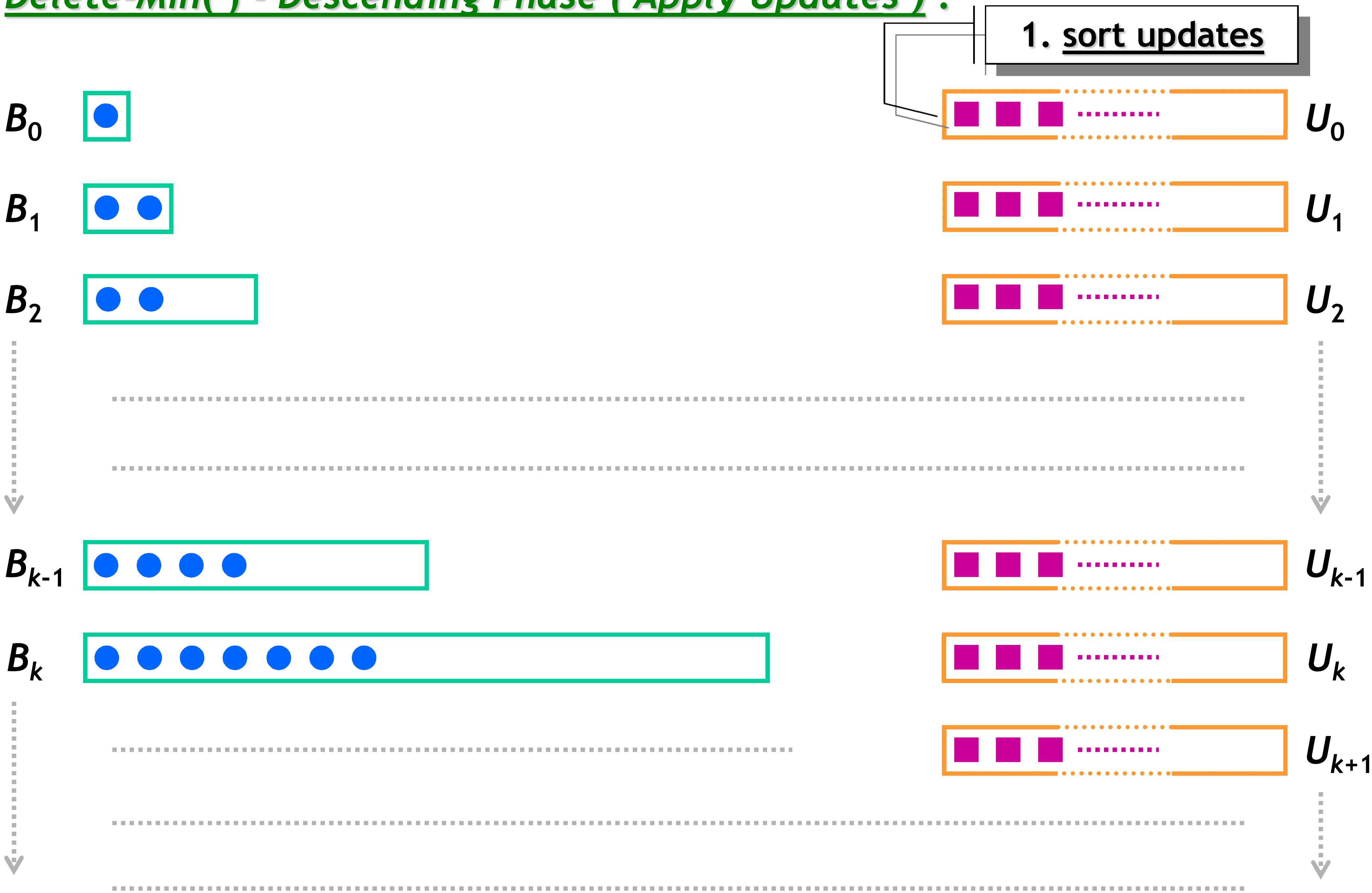

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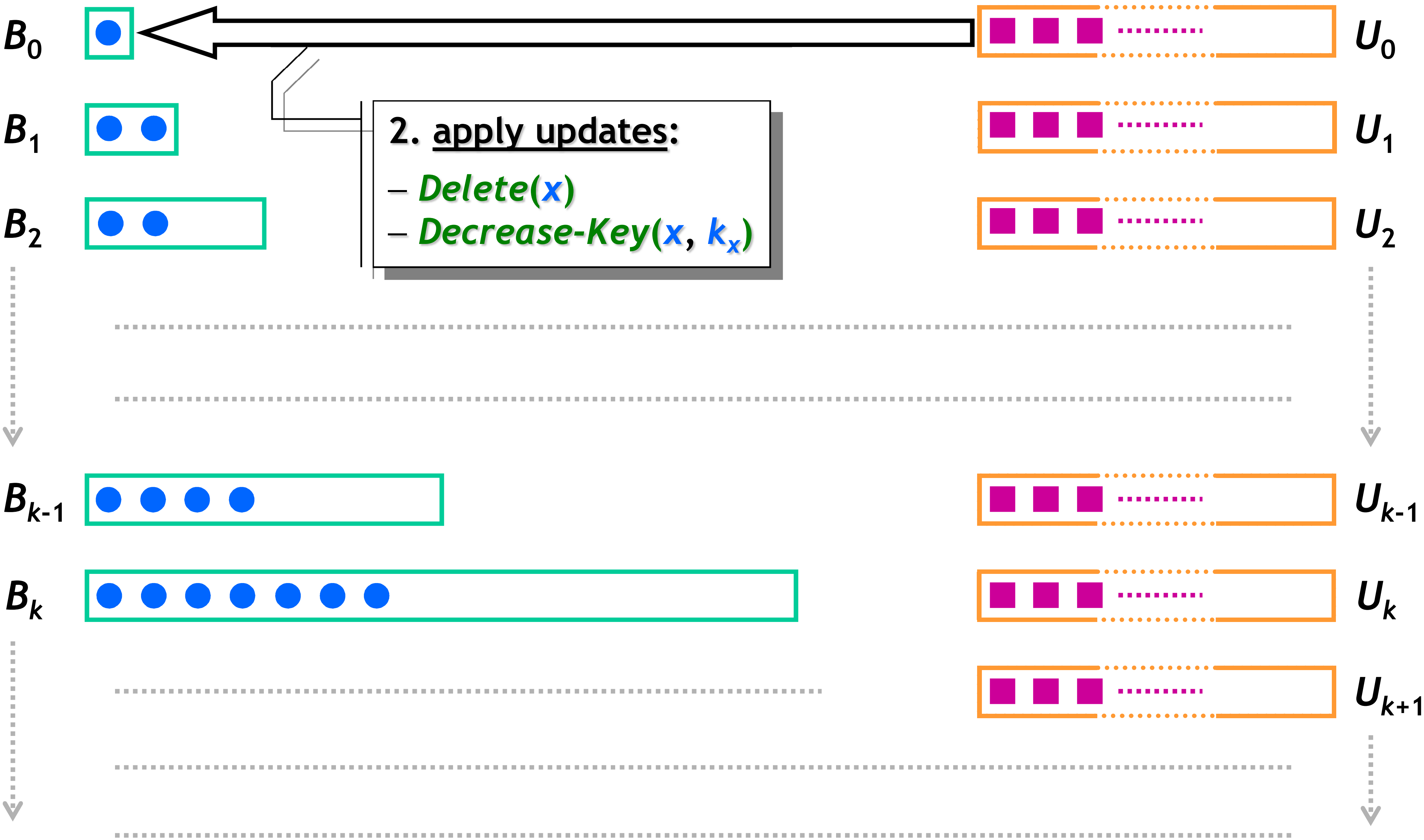

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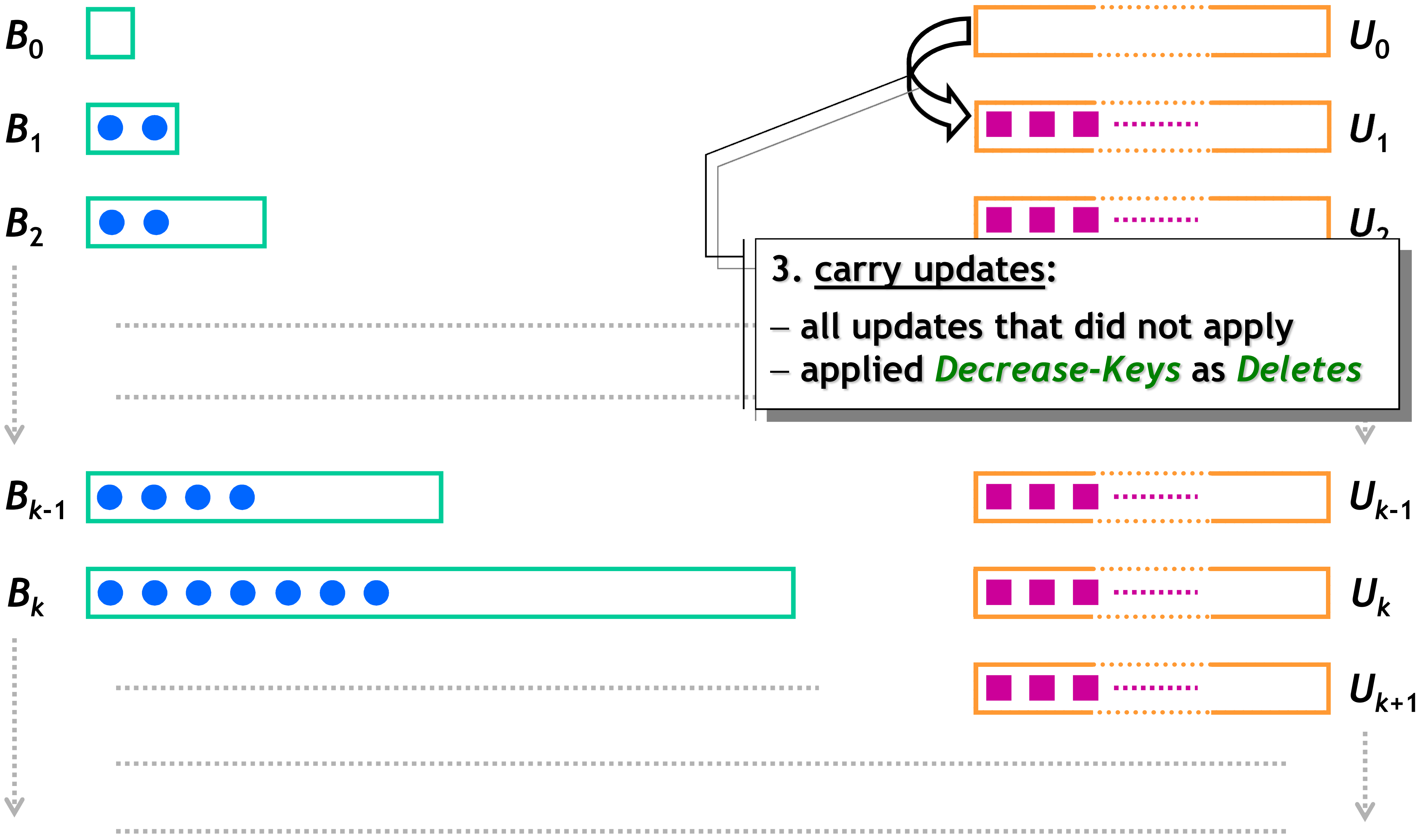

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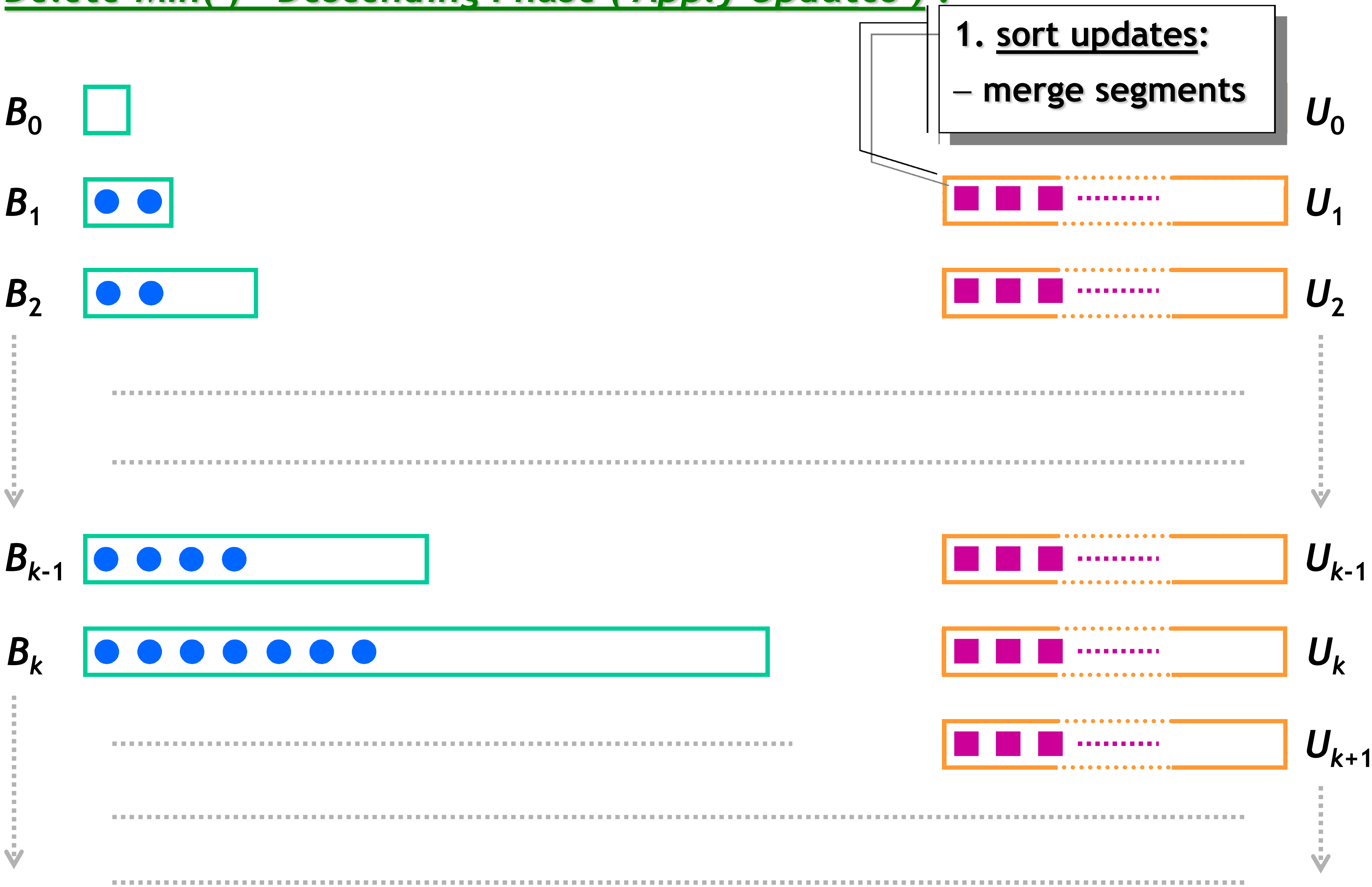

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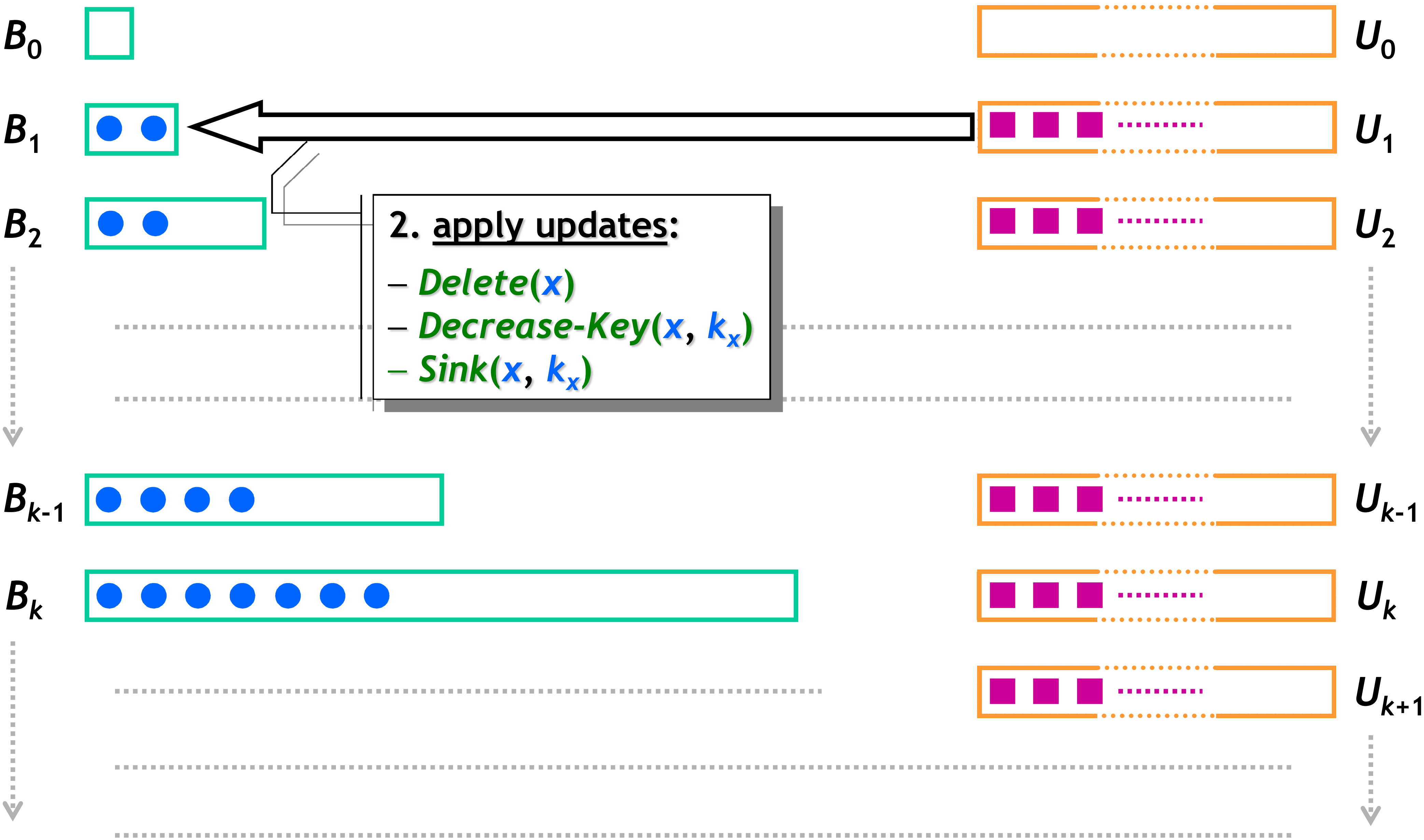

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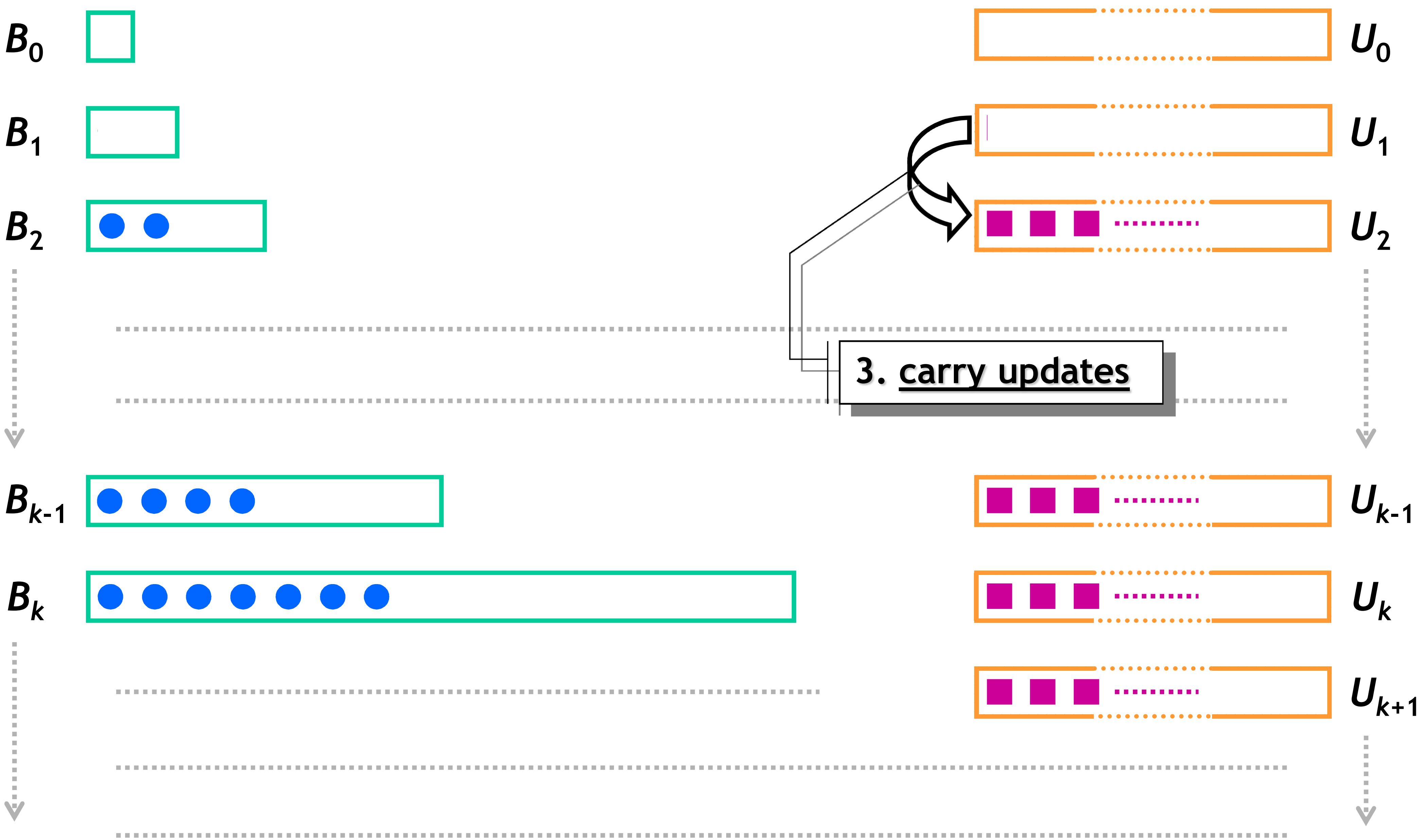

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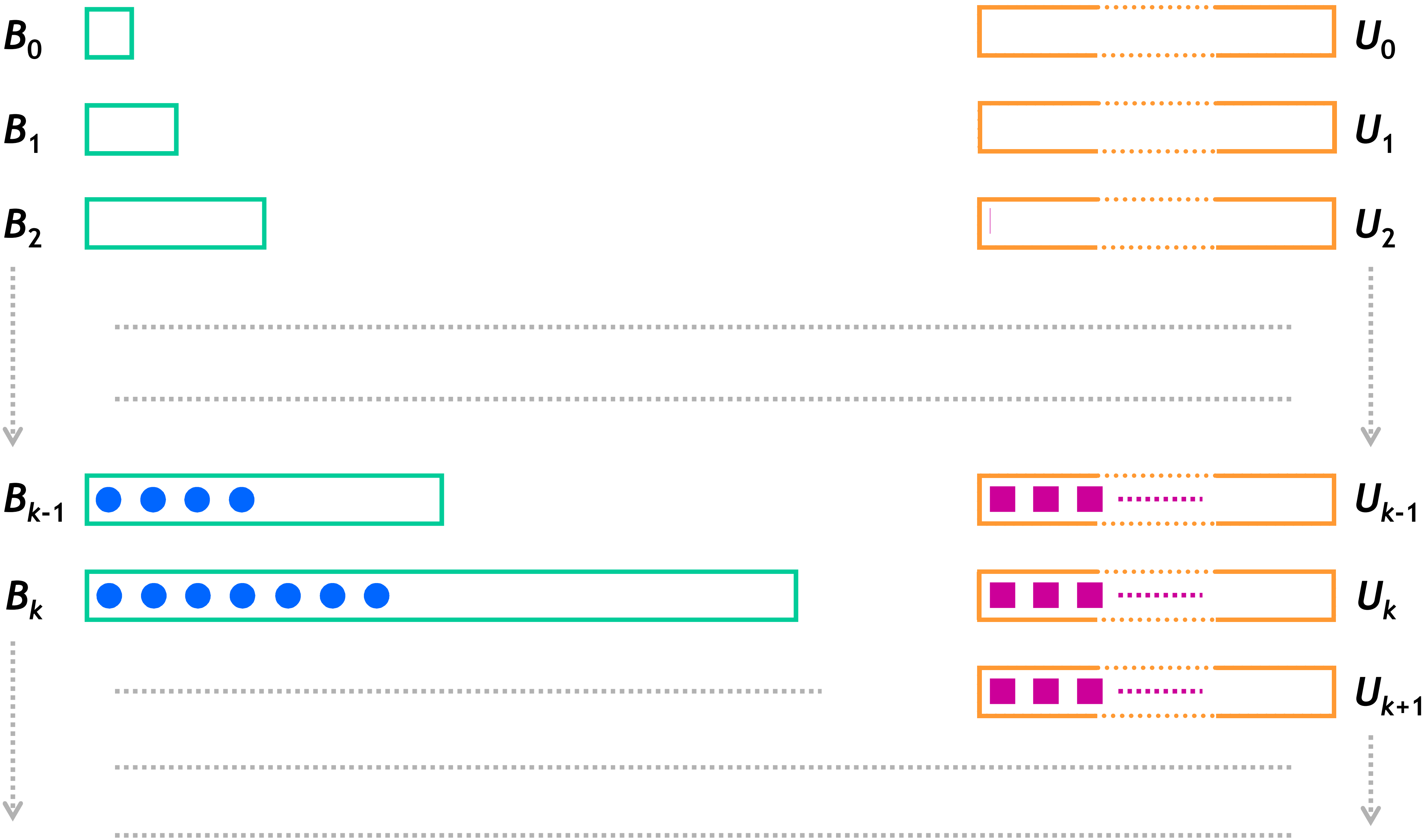

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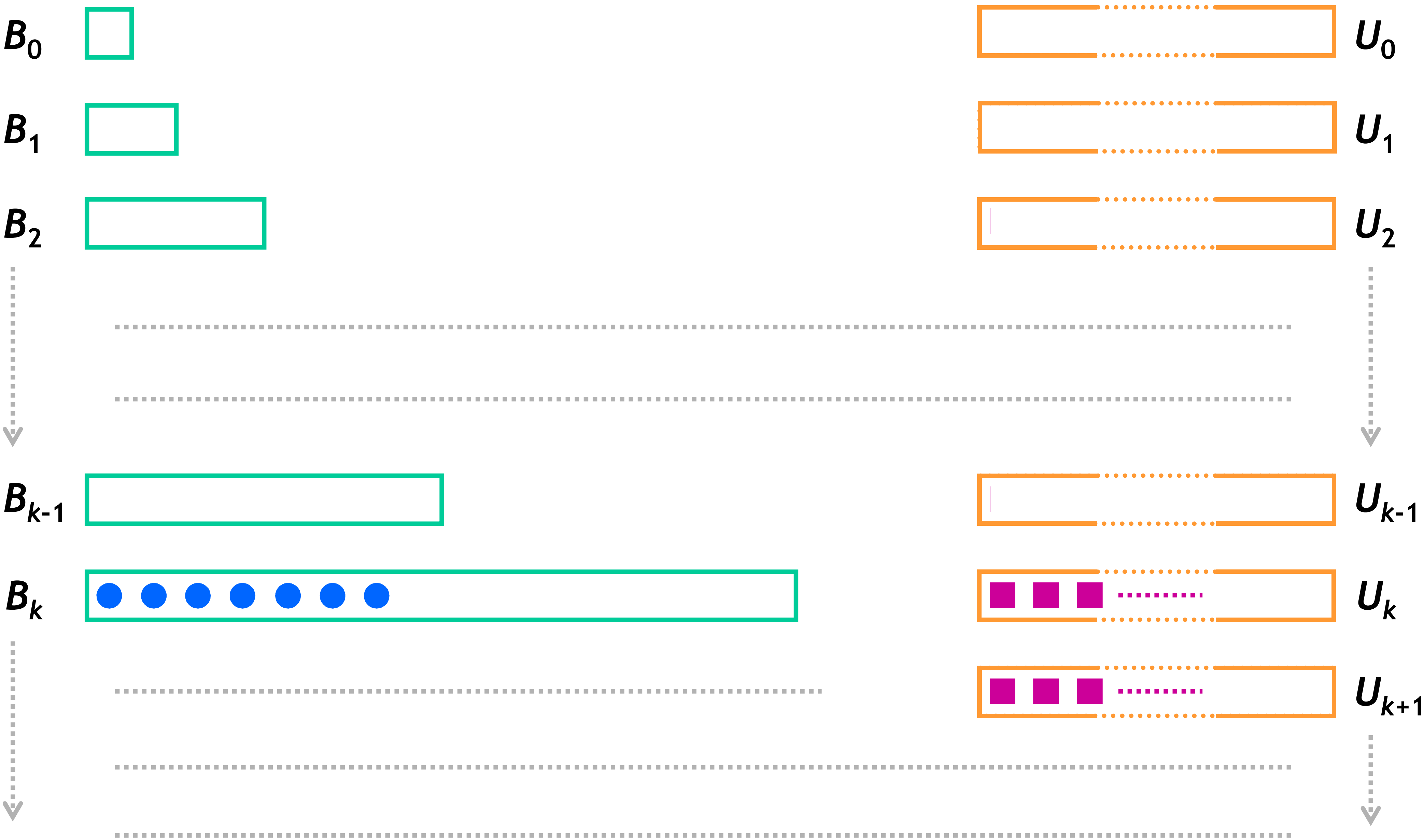

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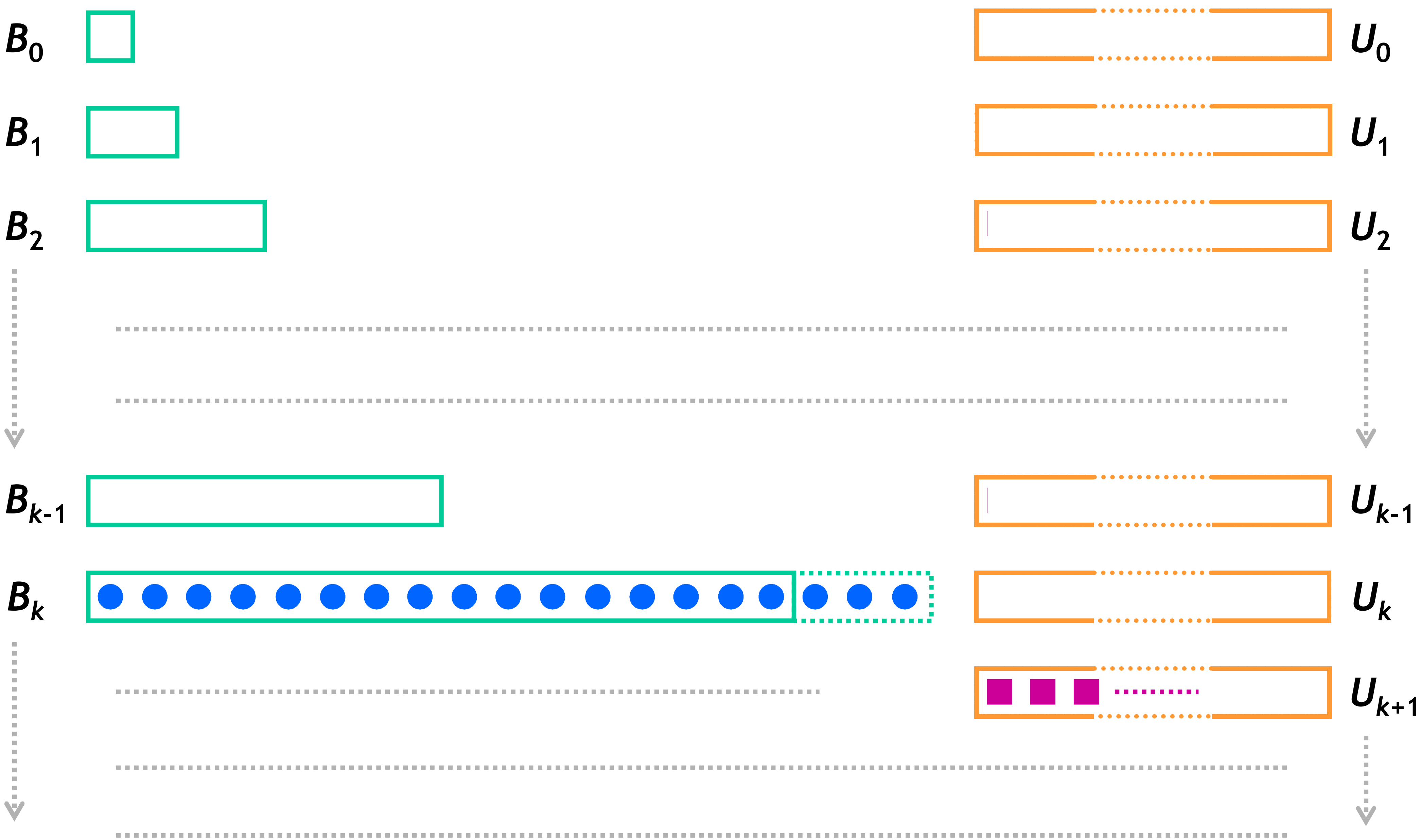

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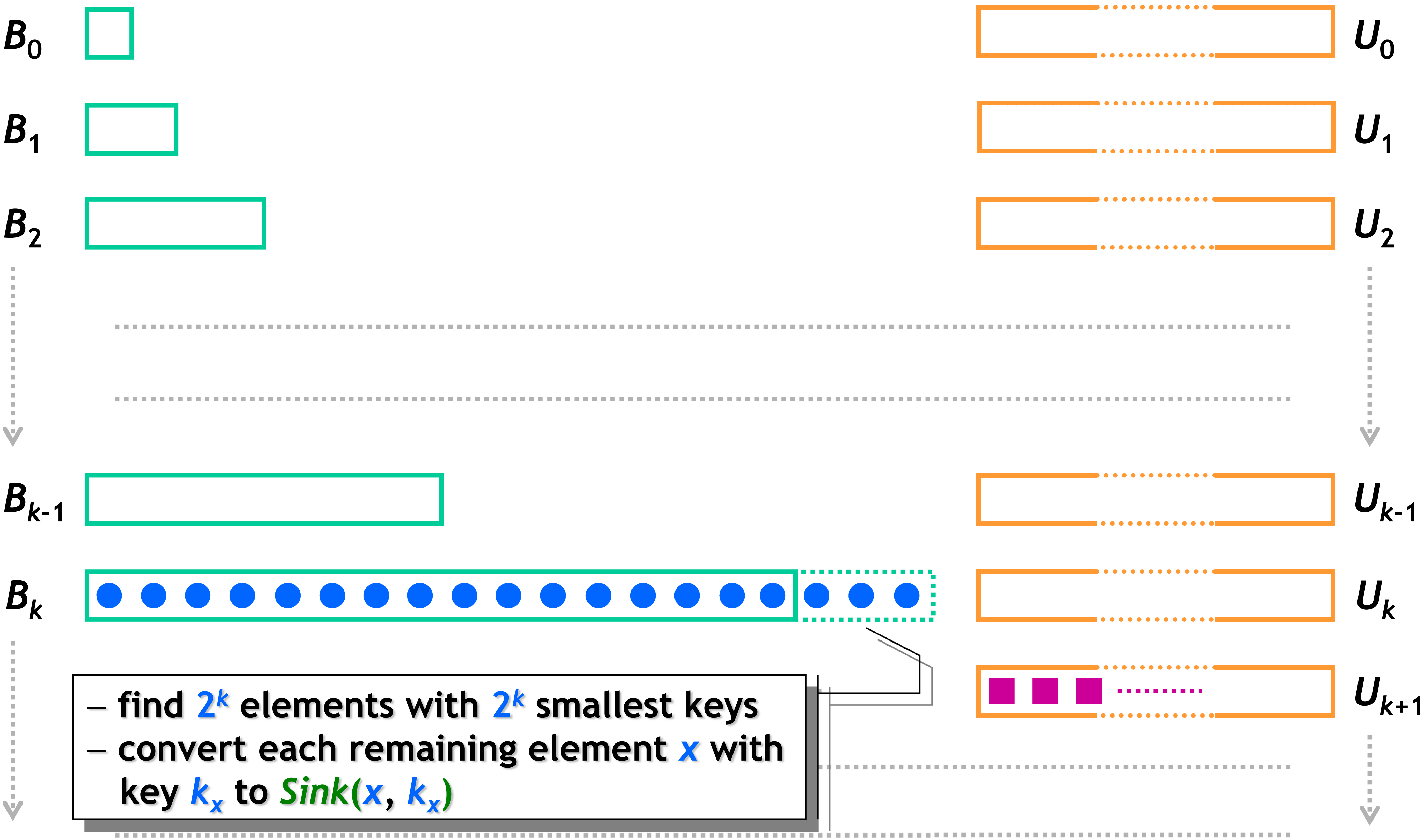
Delete-Min( ) - Ascending Phase (Redistribute Elements) :



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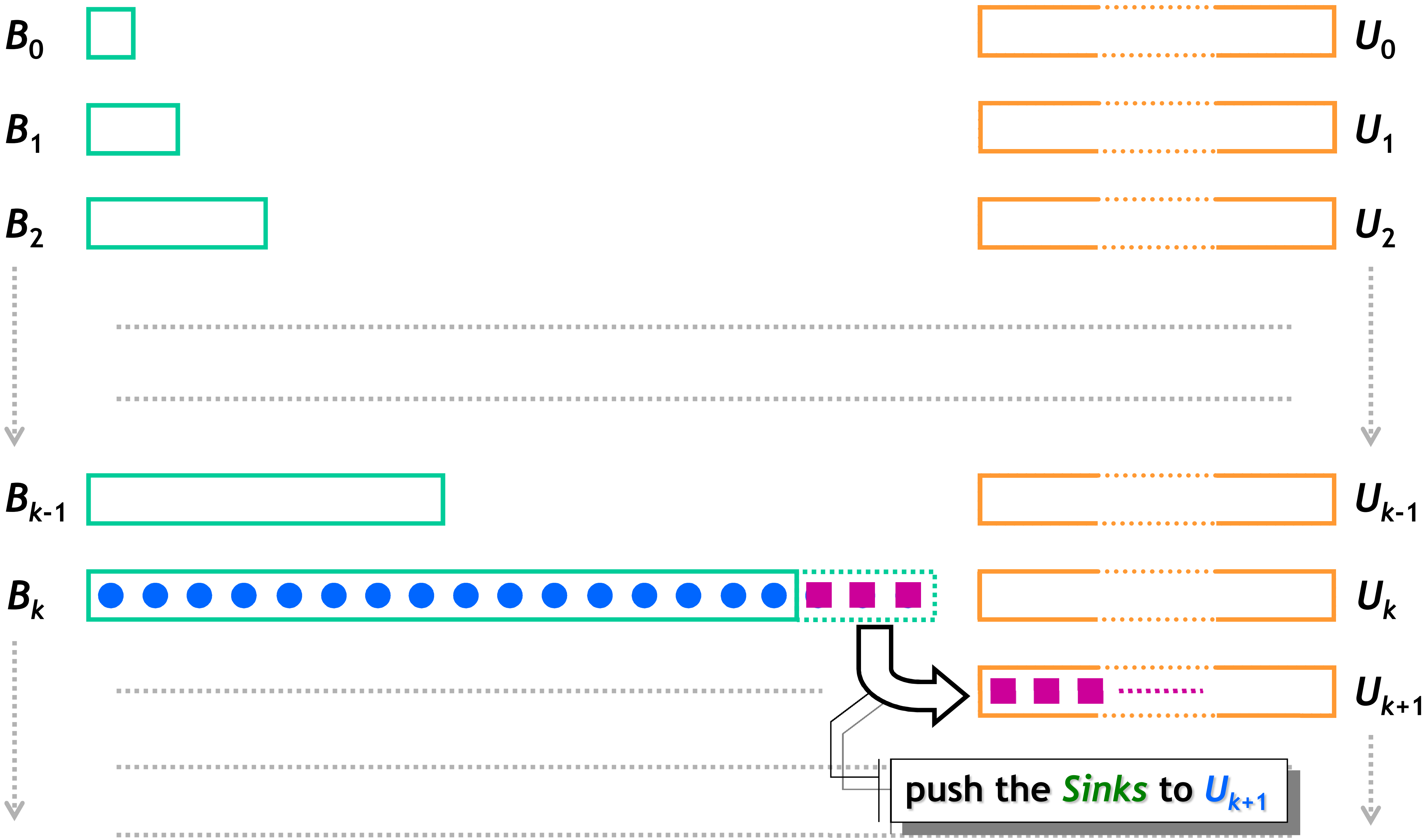

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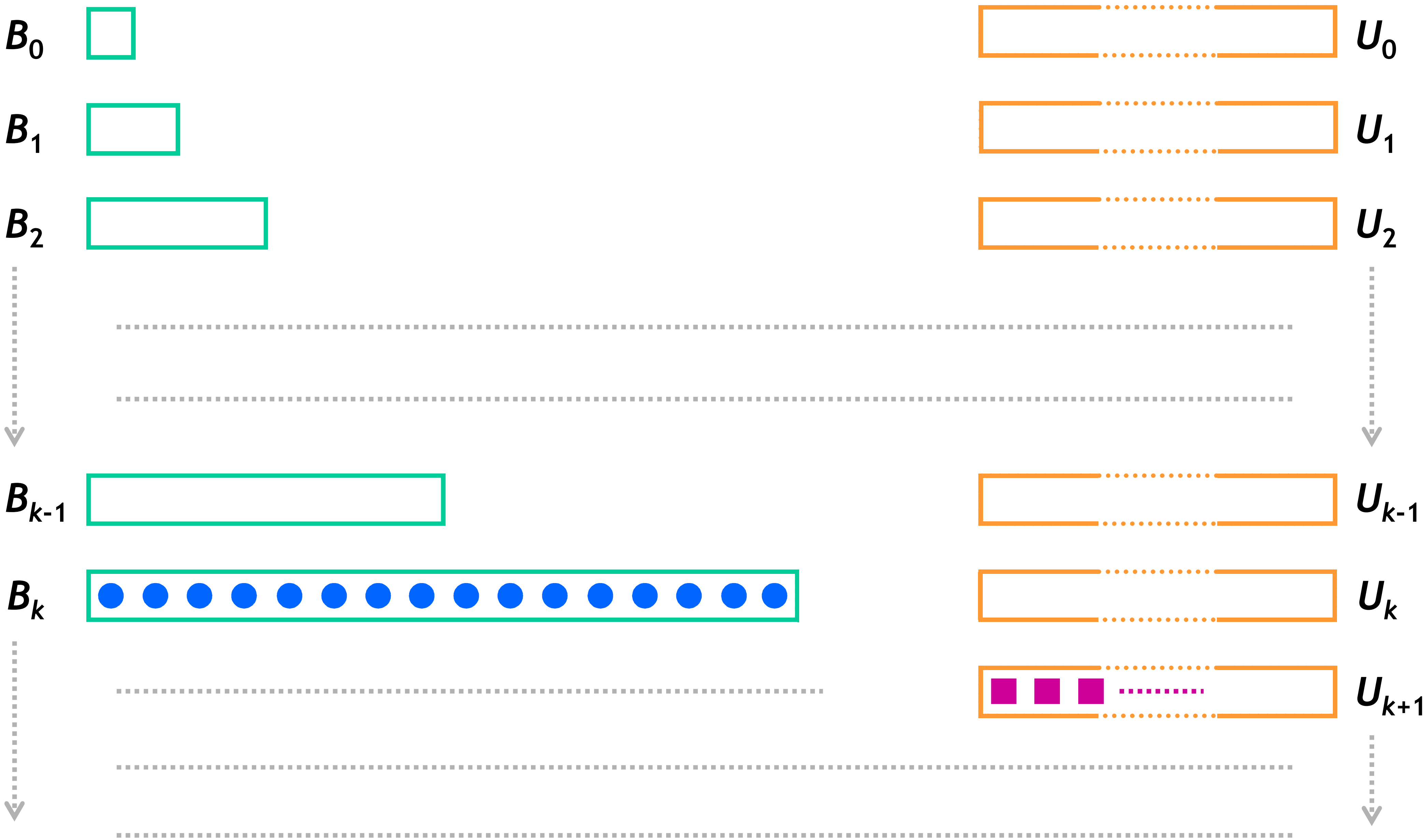

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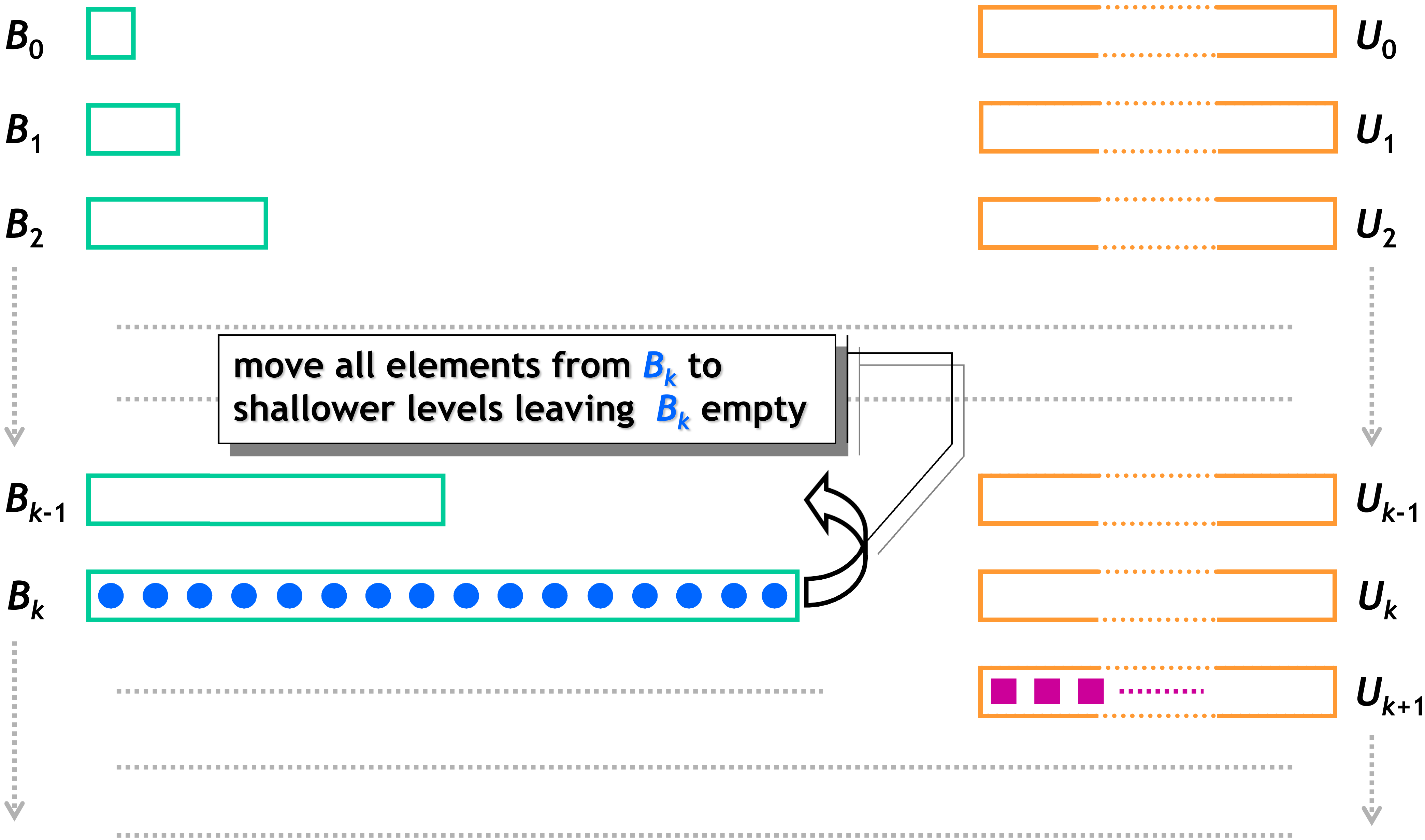

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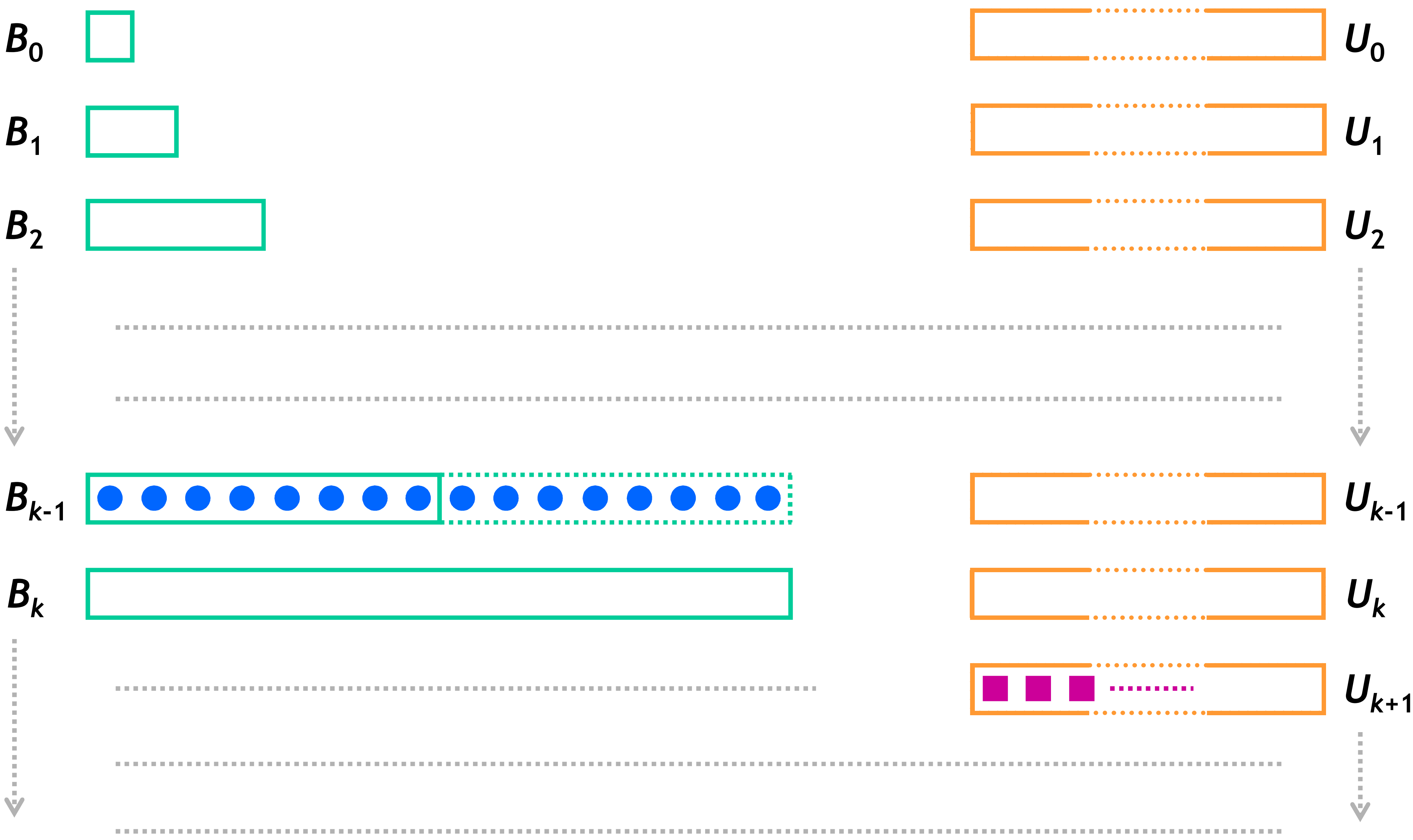

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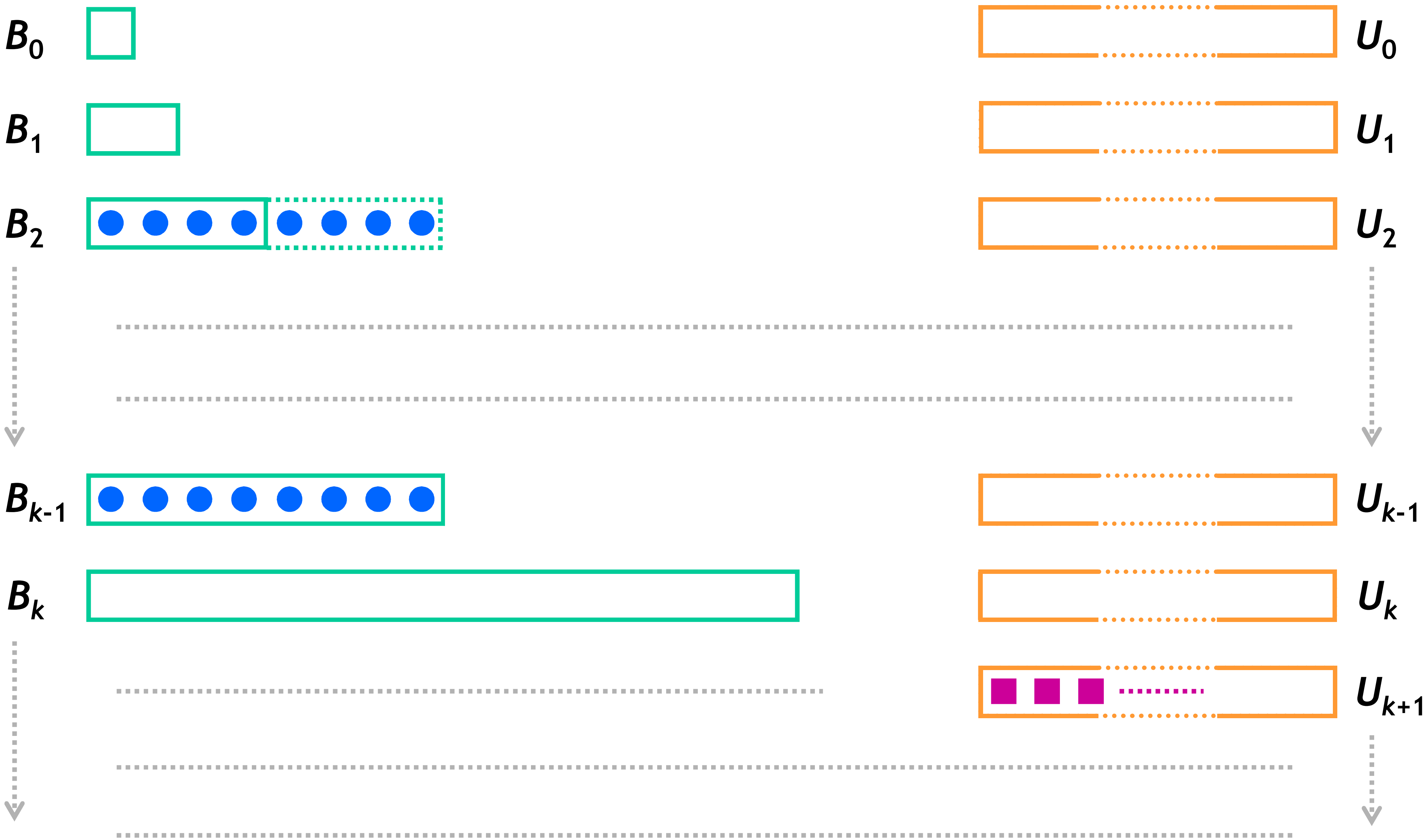

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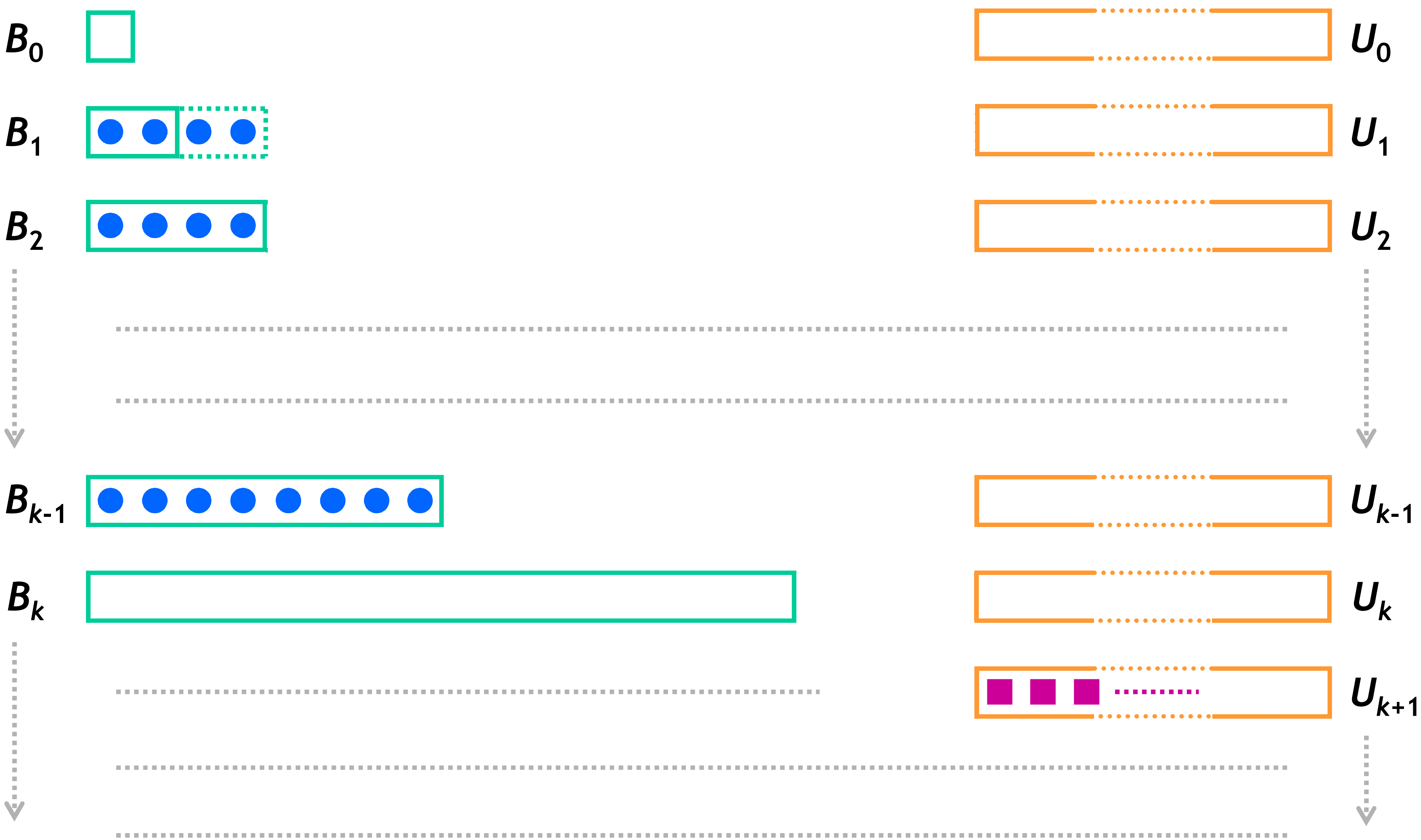

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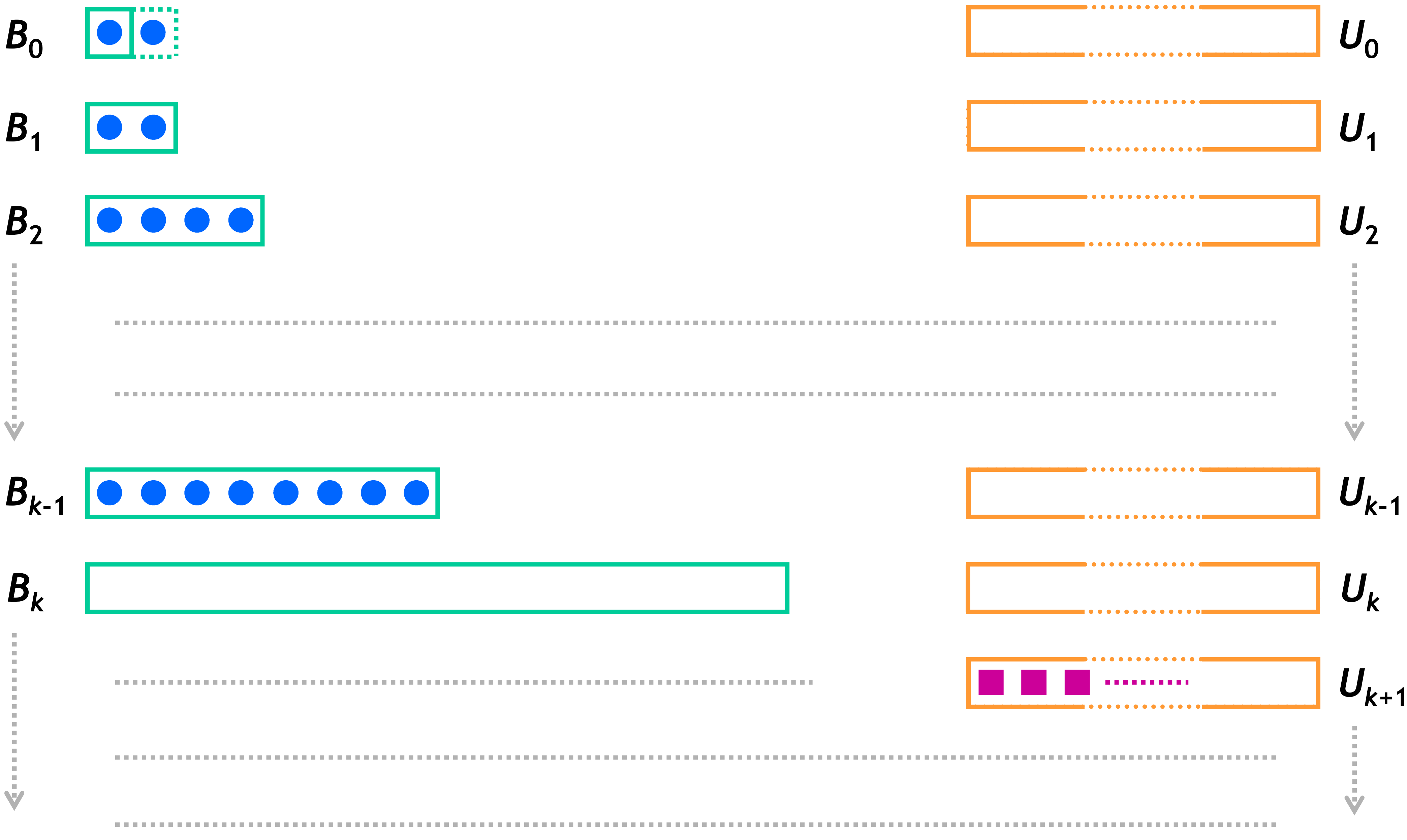

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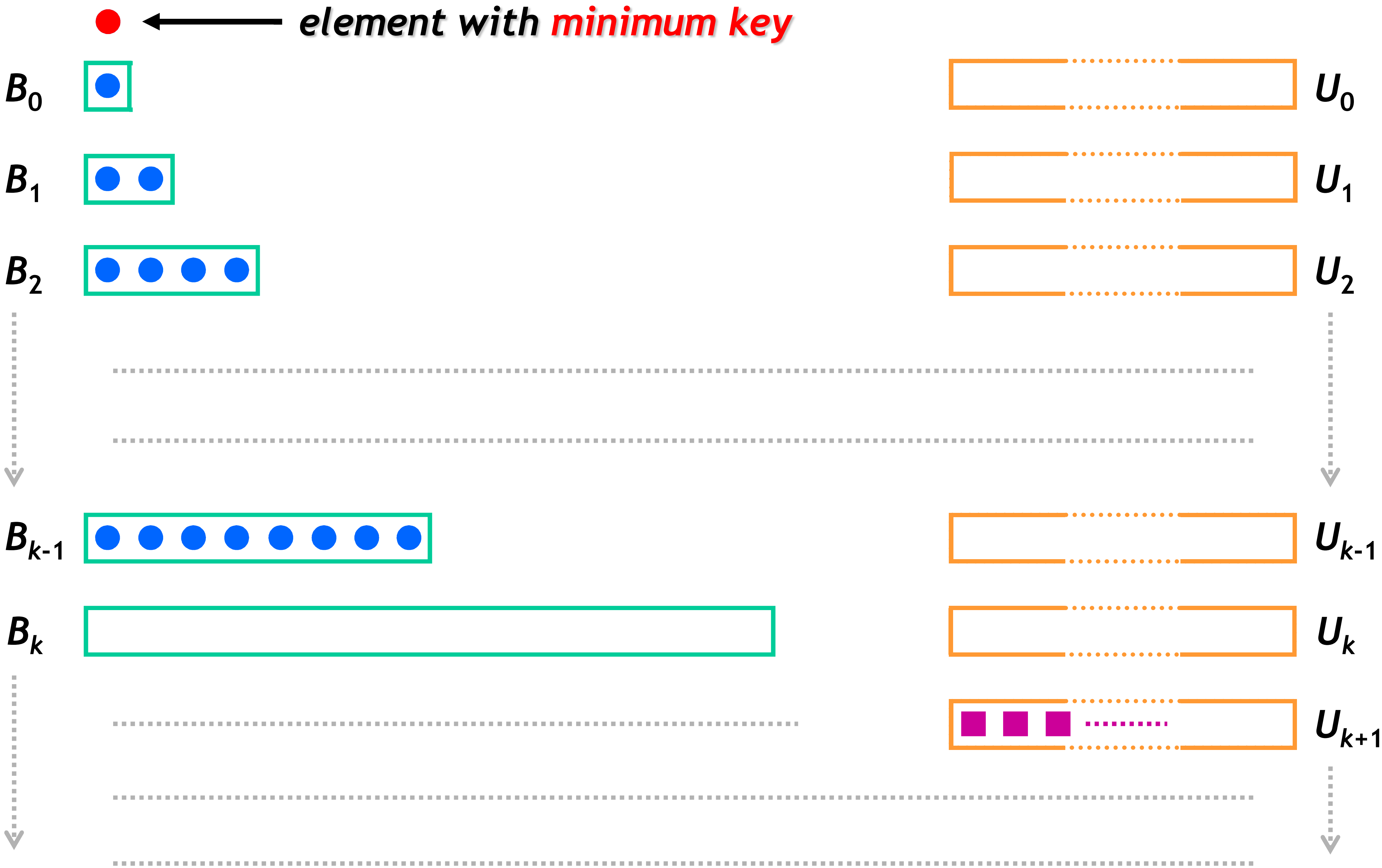

Cache-Oblivious Buffer Heap: I/O Complexity

Potential Function: $\Phi(H) = \frac{1}{B}\left(4r\left|U_0\right| + \sum_{i=1}^{r-1}(3r-i)\left|U_i\right| + \sum_{i=0}^{r-1}(i+1)\left|B_i\right|\right)$

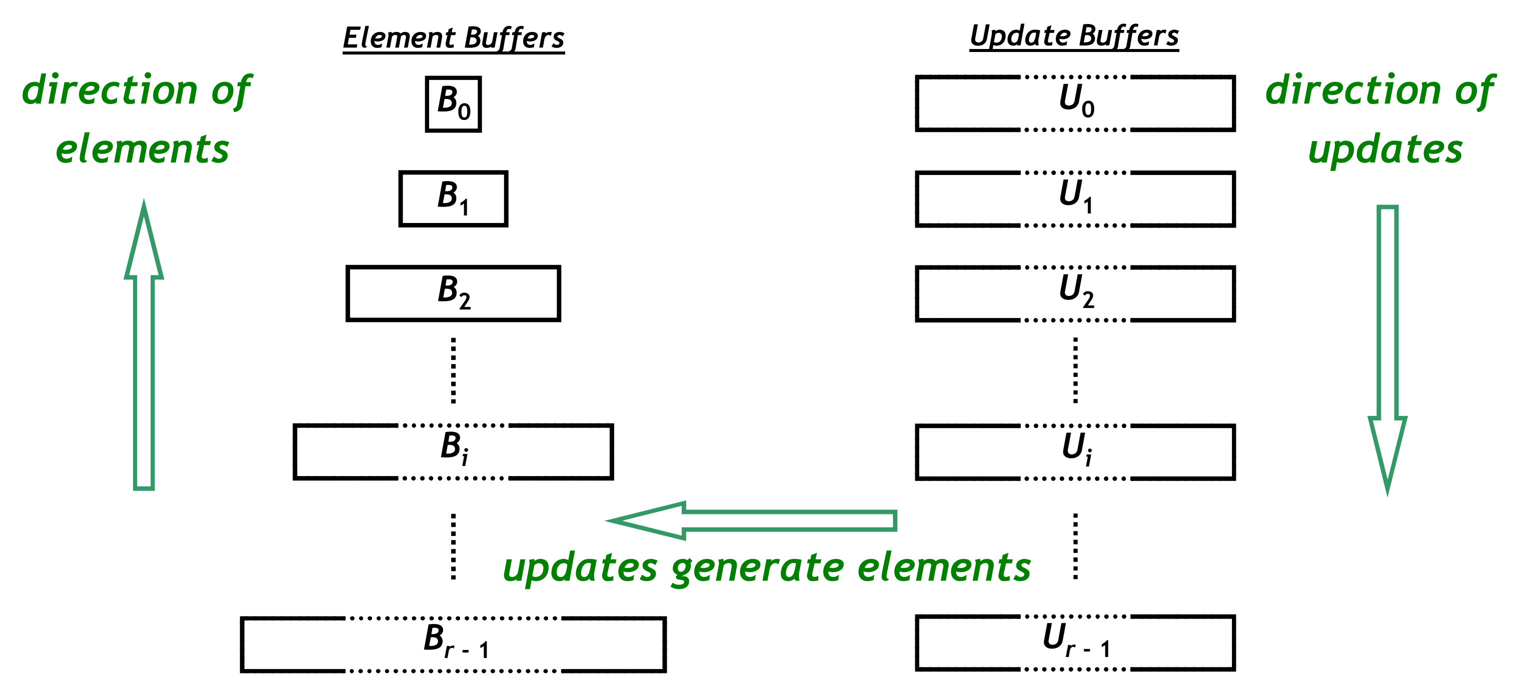

Fig: Major Data Flow Paths in the Buffer Heap

Lemma: A *Buffer Heap* on N elements supports *Delete*, *Delete-Min* and *Decrease-Key* operations cache-obliviously in $O\left(\frac{1}{B}\log_2 N\right)$ amortized I/Os each using $O(N)$ space.

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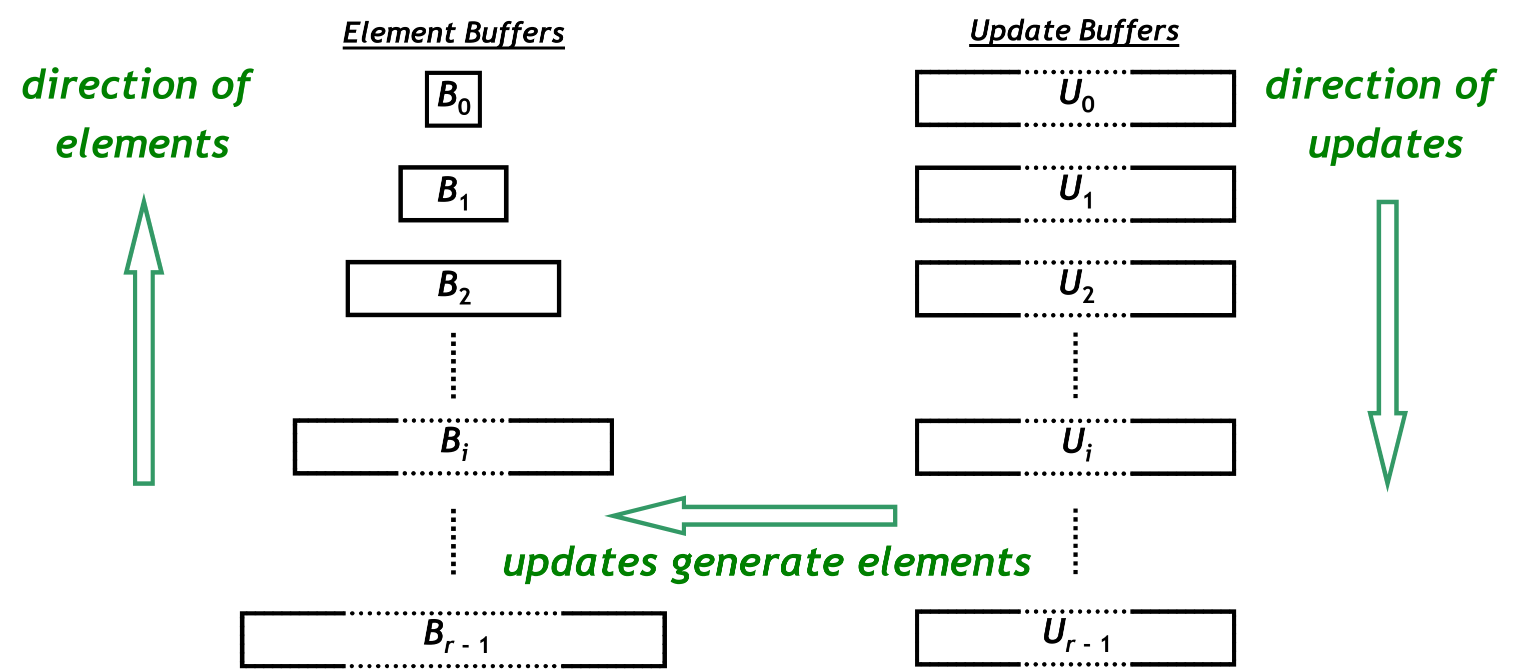

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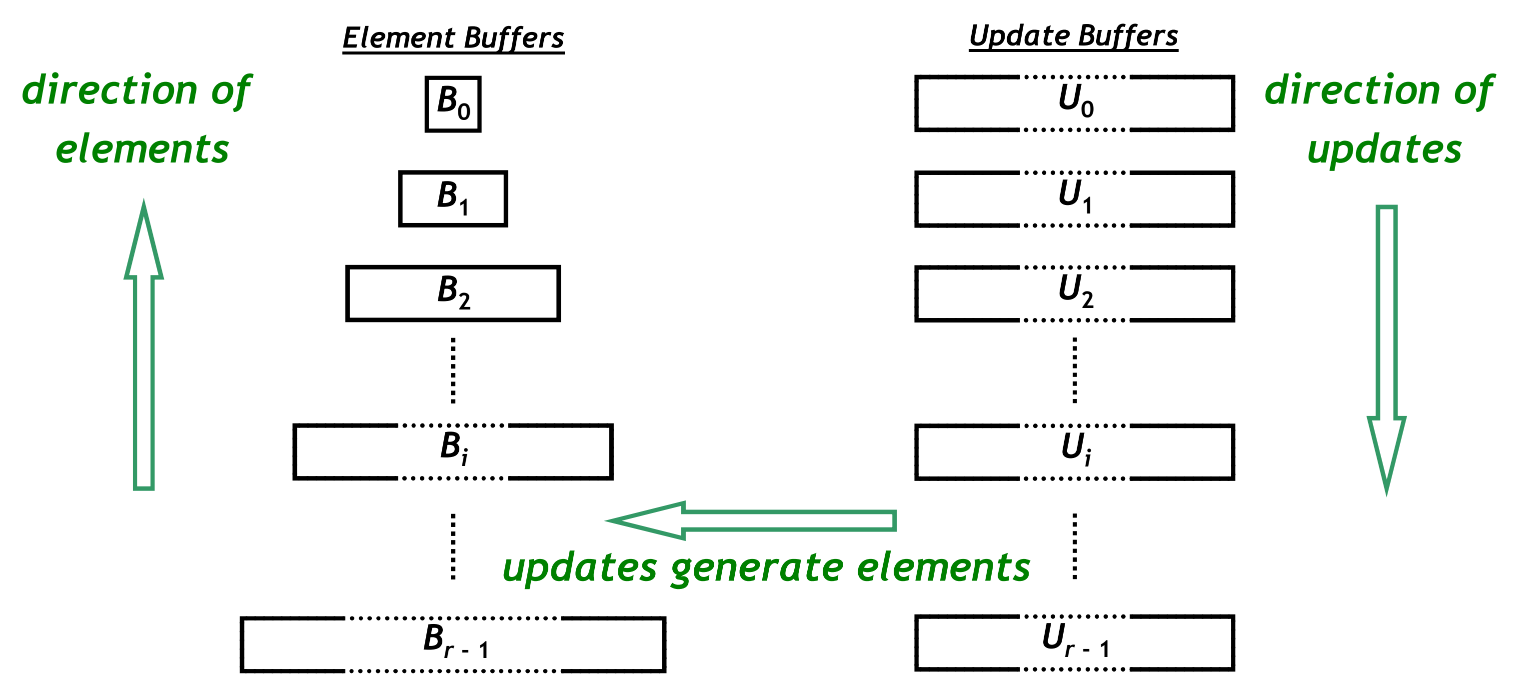

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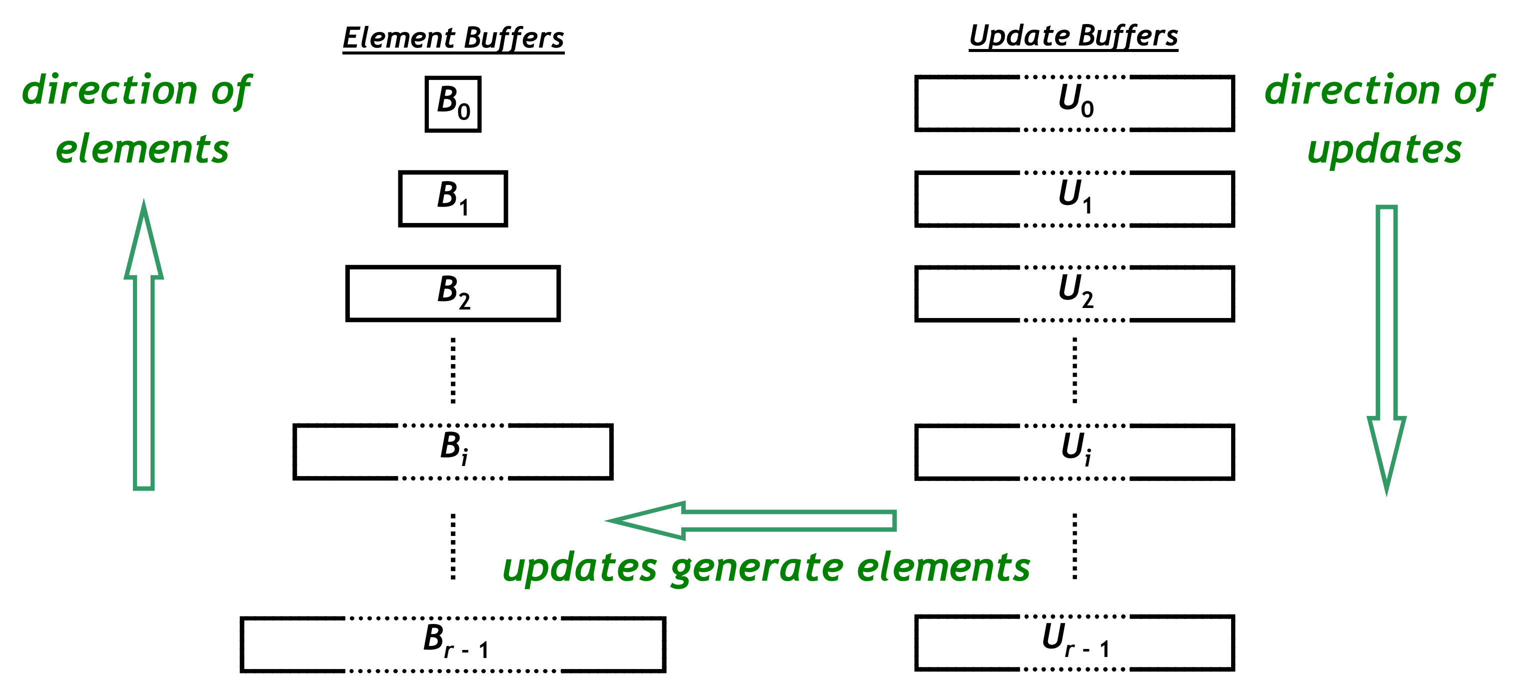

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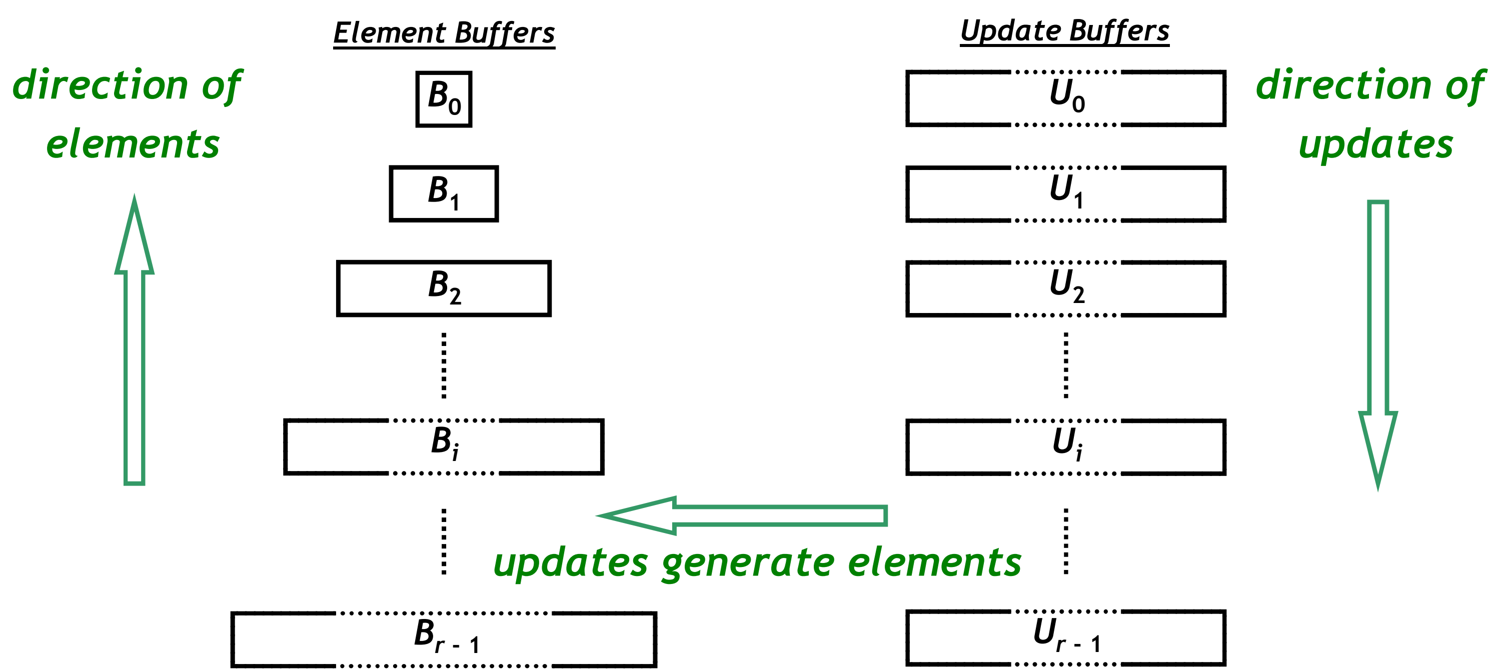

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Cache-Oblivious Buffer Heap: Layout

- All B_i's are kept in a stack S_B.
- All U_i's are kept in a stack S_U.
- An array A_s is maintained in a stack S_A.

  for $0 \le i \le r - 1$ $A_s[i]$ contains:
  - $|B_i|$
  - number of segments in U_i
  - number of updates in each segment of U_i

In both stacks lower level buffers are placed above higher level buffers. The left to right order of the elements in any buffer are maintained top to bottom in the stack.

Cache-Oblivious Buffer Heap: Reconstruction

After each operation check whether $\sum |U_i| \geq \sum |B_i|$, and if so,

Step 1: Sort the elements in S_B by element id and level number.

Step 2: Sort the updates in S_U by element id and time-stamp.

Step 3: Scan S_B and S_u simultaneously, and apply the updates in S_u on the elements of S_B.

Step 4: Reconstruct the data structure by filling the shallowest levels with the current elements in S_B, and emptying S_u.

Cache-Oblivious Buffer Heap: I/O Complexity

Lemma: A *BH* supports *Delete*, *Delete-Min*, and *Decrease-Key* operations in $O((1 / B) \log_2(N / B))$ amortized I/Os each assuming a tall cache.

Potential Method:

Associates credit with the entire data structure instead of specific objects.

- D_0 = initial state of the data structure
- D_i = state of data structure after i-th operation, $i = 1, 2, \ldots, n$
- Φ = a *potential function* mapping each D_i to a real number $\Phi(D_i)$
- c_i / a_i = actual / amortized cost of the i-th operation, $i = 1, 2, \ldots, n$

Then $a_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \Rightarrow \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$

- define Φ so that $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i.

Then $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i - \Phi(D_n) \leq \sum_{i=1}^{n} a_i$

Thus the total amortized cost is an *upper bound* on the total actual cost.

Cache-Oblivious Buffer Heap: I/O Complexity

Lemma: A *BH* supports *Delete*, *Delete-Min*, and *Decrease-Key* operations in $O((1 / B) \log_2(N / B))$ amortized I/Os each assuming a tall cache.

Proof: We will use the *Potential Method*.

- Each *Decrease-Key* inserted into U_0 will be treated as a pair of operations: ⟨*Decrease-Key*, *Dummy*⟩.
- Each component of the actual cost will have a $\Theta(1 / B)$ factor associated with it which we will drop for simplicity.
- For $0 \le i \le r - 1$, let $u_i = |U_i|$, and $b_i = |B_i|$.
- If H is the current state of *BH*, we define *potential* of H as:

$$\boxed{\Phi(H) = 4ru_0 + \sum_{i=1}^{r-1} (3r - i) u_i + \sum_{i=0}^{r-1} (i+1) b_i}$$

Cache-Oblivious Buffer Heap: I/O Complexity

$$\Phi(H) = 4ru_0 + \sum_{i=1}^{r-1}(3r-i)u_i + \sum_{i=0}^{r-1}(i+1)b_i$$

Amortized Cost of *Reconstruction*:

- Starts with $\sum_{i=0}^{r-1} u_i \geq \sum_{i=0}^{r-1} b_i$. Thus cost of steps 1 to 4 is $\leq 2r\sum_{i=0}^{r-1} u_i$.
- Total potential drop is $\geq 2r\sum u_i$.
- Amortized cost of reconstruction is $\leq 2r\sum_{i=0}^{r-1} u_i - 2r\sum_{i=0}^{r-1} u_i = 0$.

After each operation check whether $\sum u_i \geq \sum b_i$, and if so,

Step 1: Sort the elements in S_B by element id and level number.

Step 2: Sort the updates in S_U by element id and time-stamp.

Step 3: Scan S_B and S_u simultaneously, and apply the updates in S_u on the elements of S_B.

Step 4: Reconstruct the data structure by filling the shallowest levels with the current elements in S_B, and emptying S_u.

I/O cost of sorting X elements = $O\left(\frac{X}{B}\log_2 X\right)$	$r = O(\log_2 N)$

Cache-Oblivious Buffer Heap: I/O Complexity

$$\Phi(H) = 4ru_0 + \sum_{i=1}^{r-1}(3r-i)u_i + \sum_{i=0}^{r-1}(i+1)b_i$$

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Amortized Cost of *Delete*:

- Actual cost $= 1$.
- Increase in potential $= 4r$.
- Amortized cost $= 1 + 4r = O(\log_2 N)$.

Delete(x) :

Insert the operation into U_0 augmented with current time-stamp.

[Stack *Push*/*Pop* requires $O(1 / B)$ amortized I/Os each]

Cache-Oblivious Buffer Heap: I/O Complexity

$$\Phi(H) = 4ru_0 + \sum_{i=1}^{r-1}(3r-i)u_i + \sum_{i=0}^{r-1}(i+1)b_i$$

Amortized Cost of *Reconstruction*:

– Starts with $\sum_{i=0}^{r-1} u_i \geq \sum_{i=0}^{r-1} b_i$. Thus cost of steps 1 to 4 is $\leq 2r\sum_{i=0}^{r-1} u_i$.

– Total potential drop is $\geq 2r\sum_{i=0}^{r-1} u_i$.

– Amortized cost of reconstruction is $\leq 2r\sum_{i=0}^{r-1} u_i - 2r\sum_{i=0}^{r-1} u_i = 0$.

Amortized Cost of *Delete*:

– Actual cost = 1.

– Increase in pot

– Amortized cost

Decrease-Key(x, k_x) :

Insert the operation into U_0 augmented with current time-stamp.

[Each *Decrease-Key* is considered as two operations]

Amortized Cost of *Decrease-Key*:

- Actual cost = 2×1.
- Increase in potential = $2 \times 4r$.

– Amortized cost = $2 + 8r = O(\log_2 N)$.

Cache-Oblivious Buffer Heap: I/O Complexity

I/O cost of sorting X elements = $O\left(\frac{X}{B}\log_2 X\right)$

Amortized Cost of *Delete-Min*

Actual Cost:

- Cost of sorting U_0 is $\leq ru_0$.
- Cost of examining the updates in U_0, U_1, …, U_k is $\sum_{i=0}^{k}(k-i+1)u_i$.
- Cost of examining the elements in B_0, B_1, …, B_{k-1} is $\sum_{i=0}^{k-1} b_i$.

– Let b_k and b'_k be the number of elements in B_k before and after the updates, respectively

Then the total cost of exa

after updates is $\max(b_k$

I/O cost of scanning X elements = $O\left(\frac{X}{B}\right)$

k is the smallest level with non-empty B_k after updates.

– Cost of accessing A_s is $\leq r$.

Thus actual cost, $c \leq ru_0 + \sum_{i=0}^{k}(k-i+1)u_i + \sum_{i=0}^{k-1} b_i + \max(b_k, b'_k) + r$

Cache-Oblivious Buffer Heap: I/O Complexity

Amortized Cost of *Delete-Min*:

Actual Cost:

– Cost of sorting U_0 is $\leq r u_0$.

– Cost of examining th

I/O cost of selection from X elements = $O\left(\frac{X}{B}\right)$

– Cost of examining the elements in B_0, B_1, …, B_{k-1} is $\sum_{i=0} b_i$.

- Let b_k and b'_k be the number of ele

A_s stores information on each buffer and has length = $O(r)$

  the updates, respectively.
  Then the total cost of examining B_k during updates and selection after updates is $\max(b_k, b'_k)$.

- Cost of accessing A_s is $\leq r$.

Thus actual cost, $c \leq r u_0 + \sum_{i=0}^{k} (k - i + 1) u_i + \sum_{i=0}^{k-1} b_i + \max(b_k, b'_k) + r$

Cache-Oblivious Buffer Heap: I/O Complexity

$$\Phi(H) = 4ru_0 + \sum_{i=1}^{r-1}(3r-i)u_i + \sum_{i=0}^{r-1}(i+1)b_i$$

Amortized Cost of *Delete-Min*:

Potential Drop:

- Potential drop due to changes in update buffers is

$$\geq 4ru_0 + \sum_{i=1}^{k}(3r-i)u_i - (3r-k-1)\sum_{i=0}^{k}u_i$$

- Potential drop due to changes in element buffers is

$$\geq \sum_{i=0}^{k-1}b_i + \max(b_k, b'_k)$$

Thus, total drop in potential is

$$\geq ru_0 + \sum_{i=0}^{k}(k-i+1)u_i + \sum_{i=0}^{k-1}b_i + \max(b_k, b'_k)$$

Therefore, amortized cost of *Delete-Min* is

$$\hat{c} = c + \left(\Phi(H_{after}) - \Phi(H_{before})\right) \leq r = O(\log_2 N)$$

Cache-Oblivious Buffer Heap: I/O Complexity

We assume $N \gg M = \Omega\left(B^{1+\varepsilon}\right)$ for some $\varepsilon > 0$

$$\Rightarrow \log_2 N = \mathrm{O}\left(\log_2 \frac{N}{B}\right)$$

Therefore, under the tall cache assumption,

Amortized I/O cost of each operation

(*Delete*, *Delete-Min* and *Decrease-Key*) is = $\mathrm{O}\left(\frac{1}{B}\log_2 \frac{N}{B}\right)$

Removing the "Tall Cache" Assumption

- Restrict the size of each update buffer: $|U_i| \leq 2^i$
- Now all buffers (U_i and B_i) of the first $\log_2 B$ levels occupy only $O(B)$ blocks.
- No external I/O is required to access the first $\log_2 B$ levels.
- Thus amortized I/O cost of each operation is

$$= O\left(\frac{1}{B}\left(\log_2 N - \log_2 B\right)\right) = O\left(\frac{1}{B}\log_2 \frac{N}{B}\right)$$