CSE 613: Parallel Programming

Lecture 9
(Divide-and-Conquer: Partitioning for Selection and Sorting)

Rezaul A. Chowdhury
Department of Computer Science
SUNY Stony Brook
Spring 2012

Parallel Partition

Input: An array $A[\,q : r\,]$ of distinct elements, and an element x from $A[\,q : r\,]$.

Output: Rearrange the elements of $A[\,q : r\,]$, and return an index $k \in [\,q, r\,]$, such that all elements in $A[\,q : k - 1\,]$ are smaller than x, all elements in $A[\,k + 1 : r\,]$ are larger than x, and $A[\,k\,] = x$.

```
Par-Partition ( A[ q : r ], x )
n ← r − q + 1
if n = 1 then return q
array  B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
parallel for i ← 0 to n − 1 do
     B[ i ] ← A[ q + i ]
     if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
     if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
lt [ 0: n − 1 ] ← Par-Prefix-Sum (  lt[ 0: n − 1 ],  + )
gt[ 0: n − 1 ] ← Par-Prefix-Sum (  gt[ 0: n − 1 ],  + )
k ← q + lt [ n − 1 ],  A[ k ] ← x
parallel for i ← 0 to n − 1 do
     if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
     else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
return k
```

Parallel Partition

***A*:**

9	5	7	11	1	3	8	14	4	21

***x* = 8**

Parallel Partition

$x = 8$

***A*:**	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
***B*:**	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
***lt*:**	0	1	1	0	1	1	0	0	1	0

	0	1	2	3	4	5	6	7	8	9
***gt*:**	1	0	0	1	0	0	0	1	0	1

Parallel Partition

***A*:** $x = 8$

9	5	7	11	1	3	8	14	4	21

***B*:**

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

***lt*:**

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

***lt*:**

0	1	2	2	3	4	4	4	5	5

prefix sum

***gt*:**

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

***gt*:**

1	1	1	2	2	2	2	3	3	4

prefix sum

Parallel Partition

***A*:**	9	5	7	11	1	3	8	14	4	21

***x* = 8**

	0	1	2	3	4	5	6	7	8	9
***B*:**	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
***lt*:**	0	1	1	0	1	1	0	0	1	0
***lt*:**	0	1	2	2	3	4	4	4	5	5

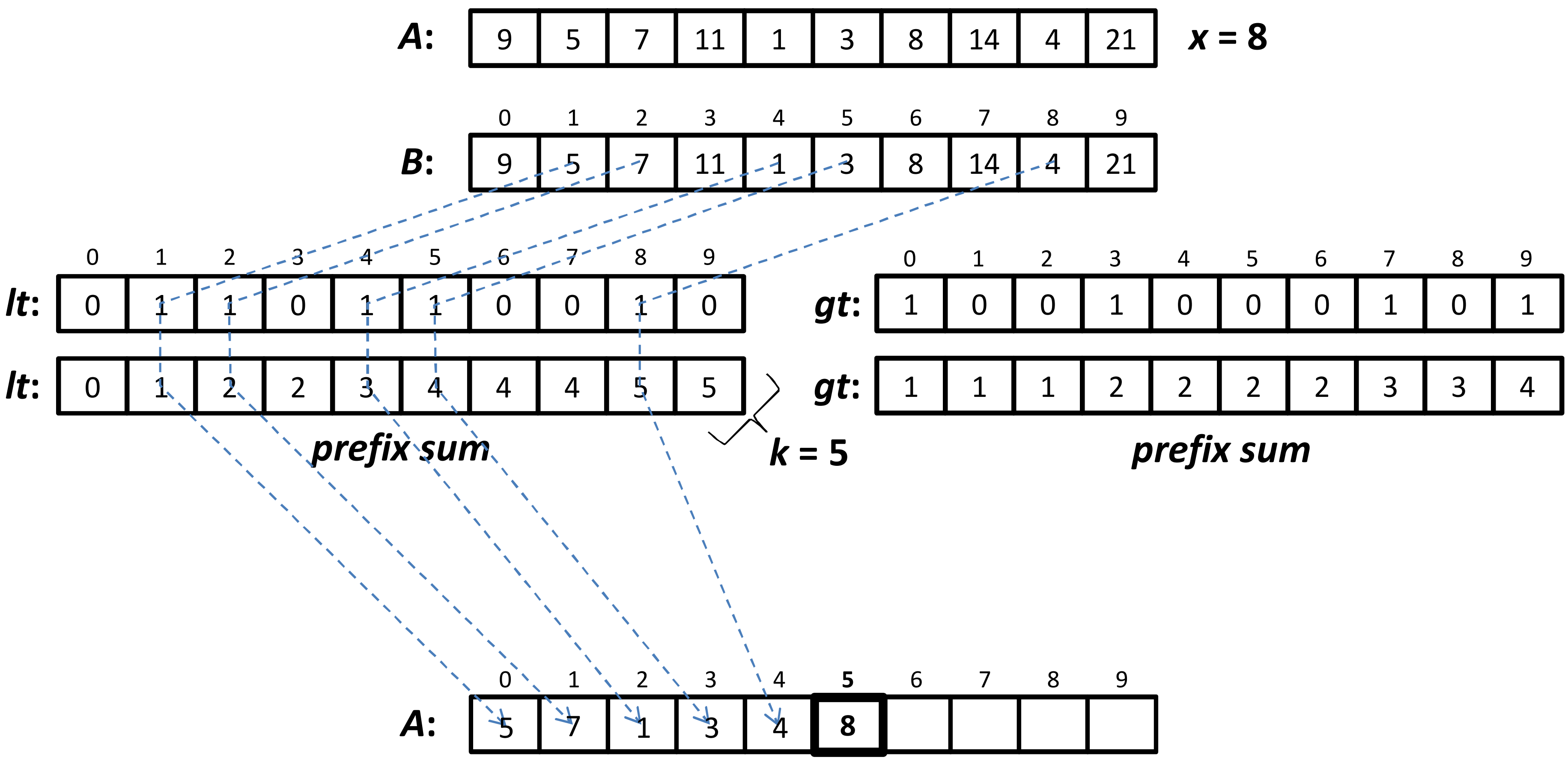
prefix sum

	0	1	2	3	4	5	6	7	8	9
***gt*:**	1	0	0	1	0	0	0	1	0	1
***gt*:**	1	1	1	2	2	2	2	3	3	4

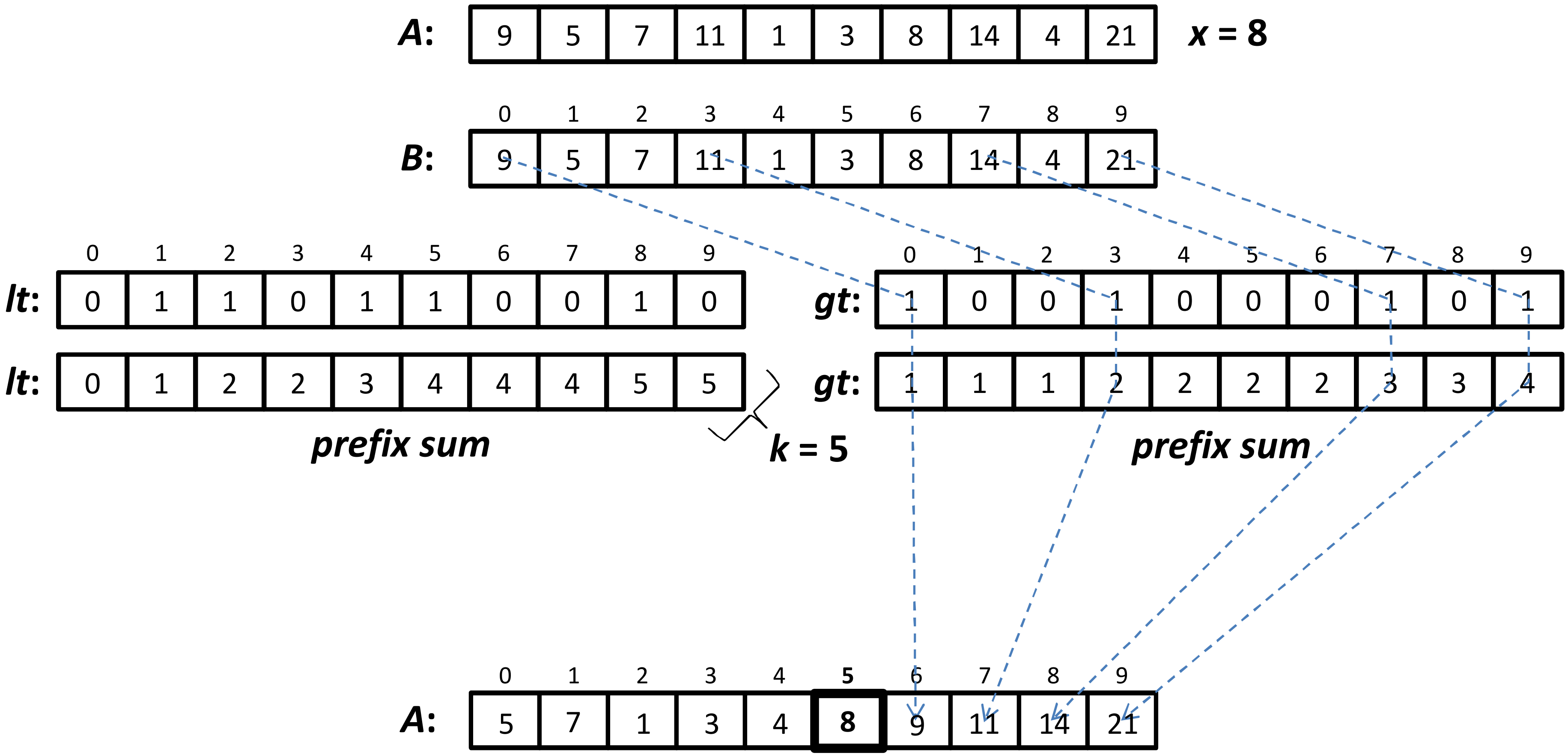
prefix sum

	0	1	2	3	4	5	6	7	8	9
***A*:**										

Parallel Partition



Parallel Partition



Parallel Partition

A: $x = 8$

9	5	7	11	1	3	8	14	4	21

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

lt:

0	1	2	2	3	4	4	4	5	5

prefix sum

$k = 5$

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4

prefix sum

A:

0	1	2	3	4	**5**	6	7	8	9
5	7	1	3	4	**8**	9	11	14	21

Parallel Partition: Analysis

```
Par-Partition ( A[ q : r ], x )
1.  n ← r − q + 1
2.  if n = 1 then return q
3.  array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
4.  parallel for i ← 0 to n − 1 do
5.      B[ i ] ← A[ q + i ]
6.      if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
7.      if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
8.  lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
9.  gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
10. k ← q + lt [ n − 1 ], A[ k ] ← x
11. parallel for i ← 0 to n − 1 do
12.     if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
13.     else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
14. return k
```

Work:

$$
\begin{aligned}
T_1(n) &= \Theta(n) && \text{[lines 1 – 7]}\\
&\quad + \Theta(n) && \text{[lines 8 – 9]}\\
&\quad + \Theta(n) && \text{[lines 10 – 14]}\\
&= \Theta(n)
\end{aligned}
$$

Span:

Assuming $\log n$ depth for *parallel for* loops:

$$
\begin{aligned}
T_\infty(n) &= \Theta(\log n) && \text{[lines 1 – 7]}\\
&\quad + \Theta(\log^2 n) && \text{[lines 8 – 9]}\\
&\quad + \Theta(\log n) && \text{[lines 10 – 14]}\\
&= \Theta(\log^2 n)
\end{aligned}
$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

Randomized Parallel QuickSort

Input: An array $A[\ q : r\ ]$ of distinct elements.

Output: Elements of $A[\ q : r\ ]$ sorted in increasing order of value.

```
Par-Randomized-QuickSort ( A[ q : r ] )
n ← r − q + 1
if n ≤ 30 then
    sort A[ q : r ] using any sorting algorithm
else
    select a random element x from A[ q : r ]
    k ← Par-Partition ( A[ q : r ], x )
    spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
    Par-Randomized-QuickSort ( A[ k + 1 : r ] )
    sync
```

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.      sort A[ q : r ] using any sorting algorithm
4. else
5.      select a random element x from A[ q : r ]
6.      k ← Par-Partition (  A[ q : r ],  x )
7.      spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.      Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.      sync
```

Lines 1—6 take $\Theta(\log^2 n)$ parallel time and perform $\Theta(n)$ work.

Also the recursive spawns in lines 7—8 work on disjoint parts of $A[\, q : r \,]$. So the upper bounds on the parallel time and the total work in each level of recursion are $\Theta(\log^2 n)$ and $\Theta(n)$, respectively.

Hence, if D is the *recursion depth* of the algorithm, then

$$T_1(n) = \mathrm{O}(nD) \text{ and } T_\infty(n) = \mathrm{O}(D \log^2 n)$$

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.       sort A[ q : r ] using any sorting algorithm
4. else
5.       select a random element x from A[ q : r ]
6.       k ← Par-Partition ( A[ q : r ], x )
7.       spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.       Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.       sync
```

We will show that w.h.p. recursion depth, $D = \mathrm{O}(\log n)$.

Hence, with high probability,

$$T_1(n) = \mathrm{O}(n \log n) \text{ and } T_\infty(n) = \mathrm{O}(\log^3 n)$$

Randomized Parallel QuickSort: Analysis

Approach: We will show the following

1. For any specific element v, the sizes of the partitions containing v in any two consecutive levels of recursion decrease by a constant factor with a certain probability.
2. With probability $1 - O\left(\frac{1}{n^7}\right)$, the partition containing v will be of size 30 or less after $O(\log n)$ levels of recursion.
3. With probability $1 - O\left(\frac{1}{n^6}\right)$, the partition containing every element will be of size 30 or less after $O(\log n)$ levels of recursion.

Randomized Parallel QuickSort: Analysis

Lemma 1: Let v be an arbitrary element of the original input array A of size $n = n_0$, and let n_j be the size of the partition containing v after partitioning at recursion depth $j \geq 1$. Then for any $j \geq 0$,

$$\Pr\left[n_{j+1} \geq \frac{7}{8}n_j\right] \leq \frac{1}{4}.$$

Proof: Suppose at recursion depth $j + 1 \geq 1$ element x was chosen as the pivot element.

One of the new partitions will have at least $\frac{7}{8}n_j$ elements provided x is among the smallest or largest $\frac{1}{8}n_j$ elements in the old partition.

The probability that x is among the smallest or largest $\frac{1}{8}n_j$ elements in the old partition is clearly $\leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Randomized Parallel QuickSort: Analysis

Lemma 2: In $20 \log n$ levels of recursion, the probability that an element goes through $20 \log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $\mathrm{O}(1/n^7)$.

[all logarithms are to the base 8/7]

Proof: The events consisting of the partitioning steps being successful can be modeled as *Bernoulli trials*.

Let X be a random variable denoting the number of unsuccessful partitioning steps among the $20 \log n$ steps. Then

$$\Pr[X > 20 \log n - \log(n/30)] \le \Pr[X > 19 \log n]$$

$$\le \sum_{j > 19 \log n} \binom{20 \log n}{j} \left(\frac{1}{4}\right)^{j} \left(\frac{3}{4}\right)^{20 \log n - j}$$

Randomized Parallel QuickSort: Analysis

Lemma 2: In $20 \log n$ levels of recursion, the probability that an element goes through $20 \log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $\mathrm{O}(1/n^7)$.

[all logarithms are to the base 8/7]

Proof: $\Pr[X > 20 \log n - \log(n/30)] \leq \Pr[X > 19 \log n]$

$$\leq \sum_{j > 19 \log n} \binom{20 \log n}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{20 \log n - j}$$

$$\leq \sum_{j > 19 \log n} \left(\frac{20e \log n}{j}\right)^j \left(\frac{1}{4}\right)^j = \sum_{j > 19 \log n} \left(\frac{5e \log n}{j}\right)^j$$

$$\leq \sum_{j > 19 \log n} \left(\frac{5e \log n}{19 \log n}\right)^j = \sum_{j > 19 \log n} \left(\frac{5e}{19}\right)^j = \mathrm{O}\left(\frac{1}{n^7}\right)$$

Randomized Parallel QuickSort: Analysis

Theorem 1: The recursion depth of *Par-Randomized-Quicksort* is $\leq 20\log_{8/7}n$ with probability $1 - \mathrm{O}(1/n^6)$.

Proof: The probability that one or more elements of *A* go through $20\log_{8/7}n - \log_{8/7}(n/30)$ unsuccessful partitioning steps is

$$\mathrm{O}\left(n \times \frac{1}{n^7}\right) = \mathrm{O}\left(\frac{1}{n^6}\right)$$ [using Lemma 2]

Hence, the probability that at least $\log_{8/7}(n/30)$ of the $20\log_{8/7}n$ partitioning steps is successful for all elements is $1 - \mathrm{O}(1/n^6)$.

After $t = \log_{8/7}(n/30)$ successful partitioning steps involving an element, the element belongs to a partition of size $\left(\frac{7}{8}\right)^t n = 30$.

Hence, *Par-Randomized-Quicksort* terminates in $\leq 20\log_{8/7}n$ levels of recursion with probability $1 - \mathrm{O}(1/n^6)$.

Parallel Selection

Input: A subarray $A[\ q : r\ ]$ of an array $A[\ 1 : n\ ]$ of n distinct elements, and a positive integer $k \in [1, r - q + 1]$.

Output: An element x of $A[\ q : r\ ]$ such that $rank(x, A[\ q : r\ ]) = k$.

```
Par-Selection ( A[ q : r ], n, k )
n´ ← r − q + 1
if n´ ≤ n / log n then
      sort A[ q : r ] using a parallel sorting algorithm and return A[ q + k - 1 ]
else
      partition A[ q : r ] into blocks B_i's each containing log n consecutive elements
      parallel for i ← 1 to ⌈ n´ / log n ⌉ do
            M[ i ] ← median of B_i using a sequential selection algorithm
      find the median m of M[ 1 : ⌈ n´ / log n ⌉ ] using a parallel sorting algorithm
      t ← Par-Partition ( A[ q : r ], m )
      if k = t − q + 1 then return A[ t ]
      else if k < t − q + 1 then return Par-Selection ( A[ q : t − 1 ], n, k )
           else return Par-Selection ( A[ t + 1 : r ], n, k − t + q − 1 )
```

Parallel Selection

Lemma 3: In *Par-Selection* (lines 11—12)

$$|A[q:t-1]| \leq \frac{3n'}{4} \text{ and } |A[t+1:r]| \leq \frac{3n'}{4}.$$

Proof: It suffices to show that $\frac{n'}{4} \leq rank(m, A[q:r]) \leq \frac{3n'}{4}$.

Since m is the median of $M[i]$'s, it is larger than one half of the $M[i]$'s. But each $M[i]$ is larger than $\frac{\log n}{2}$ elements in B_i.

Hence, $rank(m, A[q:r]) \geq \frac{n'}{2 \log n} \times \frac{\log n}{2} = \frac{n'}{4}$.

Similarly, one can show that $rank(m, A[q:r]) \leq \frac{3n'}{4}$.

Parallel Selection

Lemma 4: In *Par-Selection* $n' \leq \frac{n}{\log n}$ after at most $\log_{4/3} \log n$ levels of recursion.

Proof: It follows from Lemma 3 that $n' \leq \left(\frac{3}{4}\right)^k n$ after k levels of recursion.

Hence, for reaching $n' \leq \frac{n}{\log n}$, we need

$$\frac{n}{\log n} \geq \left(\frac{3}{4}\right)^k n \Rightarrow k \leq \log_{4/3} \log n.$$

Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
1.  n´ ← r − q + 1
2.  if n´ ≤ n / log n then
3.       sort A[ q : r ] using a parallel sorting algorithm and return A[ q + k - 1 ]
4.  else
5.       partition A[ q : r ] into blocks B_i's each containing log n consecutive elements
6.       parallel for i ← 1 to ⌈ n´ / log n ⌉ do
7.            M[ i ] ← median of B_i using a sequential selection algorithm
8.       find the median m of M[ 1 : ⌈ n´ / log n ⌉ ] using a parallel sorting algorithm
9.       t ← Par-Partition ( A[ q : r ], m )
10.      if k = t − q + 1 then return A[ t ]
11.      else if k < t − q + 1 then return Par-Selection ( A[ q : t − 1 ], n, k )
12.           else return Par-Selection ( A[ t + 1 : r ], n, k − t + q − 1 )
```

Step 7: Use a linear time (worst-case) sequential selection algorithm (see Section 9.3 of "Introduction to Algorithms", 3rd Ed. by Cormen et al.).

Steps 3 and 8: Use the parallel mergesort with parallel merge (see Lecture 8) that runs in $O(\log^3 n)$ parallel time and performs $O(n \log n)$ work in the worst case.

Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
1.  n′ ← r − q + 1
2.  if n′ ≤ n / log n then
3.      sort A[ q : r ] using a parallel sorting
        algorithm and return A[ q + k - 1 ]
4.  else
5.      partition A[ q : r ] into blocks B_i's each
        containing log n consecutive elements
6.      parallel for i ← 1 to ⌈ n′ / log n ⌉ do
7.          M[ i ] ← median of B_i using a
            sequential selection algorithm
8.      find the median m of M[ 1 : ⌈ n′ / log n ⌉ ]
        using a parallel sorting algorithm
9.      t ← Par-Partition ( A[ q : r ], m )
10.     if k = t − q + 1 then return A[ t ]
11.     else if k < t − q + 1 then
                return Par-Selection ( A[ q : t − 1 ], n, k )
12.         else return Par-Selection ( A[ t + 1 : r ],
                                   n, k − t + q − 1 )
```

Last Level of Recursion

Work: $O\left(\frac{n}{\log n}\log\left(\frac{n}{\log n}\right)\right) = O(n)$

Span: $O\left(\log^3\left(\frac{n}{\log n}\right)\right) = O(\log^3 n)$

Any Other Recursion Level (except last)

Work: $O\left(\frac{n'}{\log n} \times \log n\right)$ [lines 5 – 7]

$+ O\left(\frac{n'}{\log n}\log\left(\frac{n'}{\log n}\right)\right)$ [line 8]

$+ O(n')$ [line 9]

$= O(n')$

Span: $O\left(\log\left(\frac{n'}{\log n}\right) + \log n\right)$ [lines 5 – 7]

$+ O\left(\log^3\left(\frac{n'}{\log n}\right)\right)$ [line 8]

$+ O(\log^2 n')$ [line 9]

$= O(\log^3 n)$

Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
n′ ← r − q + 1
if n′ ≤ n / log n then
    sort A[ q : r ] using a parallel sorting
    algorithm and return A[ q + k - 1 ]
else
    partition A[ q : r ] into blocks B_i's each
    containing log n consecutive elements
    parallel for i ← 1 to ⌈ n′ / log n ⌉ do
        M[ i ] ← median of B_i using a
        sequential selection algorithm
    find the median m of M[ 1 : ⌈ n′ / log n ⌉ ]
    using a parallel sorting algorithm
    t ← Par-Partition ( A[ q : r ], m )
    if k = t − q + 1 then return A[ t ]
    else if k < t − q + 1 then
            return Par-Selection ( A[ q : t − 1 ], n, k )
        else return Par-Selection ( A[ t + 1 : r ],
                                    n, k − t + q − 1 )
```

Overall

Work:

$$T_1(n) = O\left(n + \sum_{i=0}^{\log_{4/3} \log n} \left(\frac{3}{4}\right)^i n\right) = O(n)$$

Span:

$$T_\infty(n) = O\left(\left(\log_{4/3} \log n\right) \log^3 n\right) = O(\log^3 n \log \log n)$$