CSE 548 / AMS 542: Analysis of Algorithms

Prerequisites Review 4
( Linear Recurrences with Constant Coefficients )

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**Linear Homogeneous Recurrence**

A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real constants, and $c_k \neq 0$.

For constant $r$, $a_n = r^n$ is a solution of the recurrence relation iff:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

The equation above is called the *characteristic equation* of the recurrence, and its roots are called *characteristic roots*. 
Linear Homogeneous Recurrence

Recurrence: \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \),

Characteristic Equation: \( r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0 \)

If the characteristic equation has \( k \) distinct roots \( r_1, r_2, \ldots, r_k \), then a sequence \( \{a_n\} \) is a solution of the recurrence relation iff

\[
a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n \quad \text{for integers } n \geq 0,
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are constants.
Linear Homogeneous Recurrence

Recurrence: \( a_n = c_1 a_{n-1} + c_2 a_{n-2} \)

Characteristic Equation: \( r^2 - c_1 r - c_2 = 0 \)

\( a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \Rightarrow \{a_n\} \) is a solution to the recurrence:

\( r_1^2 = c_1 r_1 + c_2 \) and \( r_2^2 = c_1 r_2 + c_2 \)

\[ c_1 a_{n-1} + c_2 a_{n-2} = c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \]
\[ = \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2) \]
\[ = \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \]
\[ = \alpha_1 r_1^n + \alpha_2 r_2^n \]
\[ = a_n \]
Linear Homogeneous Recurrence

Recurrence:  \( a_n = c_1a_{n-1} + c_2a_{n-2} \)

Characteristic Equation:  \( r^2 - c_1r - c_2 = 0 \)

\( \{a_n\} \text{ is a solution to the recurrence } \Rightarrow a_n = \alpha_1r_1^n + \alpha_2r_2^n : \)

Assume initial conditions:  \( a_0 = C_0 \) and  \( a_1 = C_1 \)

\[
\begin{align*}
a_0 &= C_0 = \alpha_1 + \alpha_2 \\
a_1 &= C_1 = \alpha_1r_1 + \alpha_2r_2
\end{align*}
\]

Solving:  \( \alpha_1 = \frac{c_1-C_0r_2}{r_1-r_2} \) and  \( \alpha_2 = \frac{C_0r_1-c_1}{r_1-r_2} \)

Since the initial conditions uniquely determine the sequence, it follows that  \( a_n = \alpha_1r_1^n + \alpha_2r_2^n \).
Linear Homogeneous Recurrence

Recurrence for *Fibonacci numbers*:

\[
f_n = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
f_{n-1} + f_{n-2} & \text{otherwise}.
\end{cases}
\]

Characteristic equation: \( r^2 - r - 1 = 0 \)

Characteristic roots: \( r_1 = \frac{1+\sqrt{5}}{2} \) and \( r_2 = \frac{1-\sqrt{5}}{2} \)

Then for constants \( \alpha_1 \) and \( \alpha_2 \):

\[
f_n = \alpha_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1-\sqrt{5}}{2} \right)^n
\]

Initial conditions:

\[
f_0 = \alpha_1 + \alpha_2 = 0
\]

\[
f_1 = \alpha_1 \left( \frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1-\sqrt{5}}{2} \right) = 1
\]

Constants: \( \alpha_1 = \frac{1}{\sqrt{5}} \) and \( \alpha_2 = -\frac{1}{\sqrt{5}} \)

Solution:

\[
f_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n
\]
Linear Homogeneous Recurrence

Recurrence: \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \)

Characteristic Equation: \( r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0 \)

If the characteristic equation has \( t \) distinct roots \( r_1, r_2, \ldots, r_t \) with multiplicities \( m_1, m_2, \ldots, m_t \), respectively, so that all \( m_i \)’s are positive and \( \sum_{1 \leq i \leq t} m_i = k \), then a sequence \( \{a_n\} \) is a solution of the recurrence relation iff

\[
a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\
+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\
+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \quad \text{for integers } n \geq 0,
\]

where \( \alpha_{i,j} \) are constants for \( 1 \leq i \leq t \) and \( 0 \leq j \leq m_i - 1 \).
Linear Homogeneous Recurrence

\[ a_n = \begin{cases} 
1 & \text{if } n = 0, \\
6 & \text{if } n = 1, \\
6a_{n-1} - 9a_{n-2} & \text{otherwise.}
\end{cases} \]

Characteristic equation: \[ r^2 - 6r + 9 = 0 \]

Characteristic root: \[ r = 3 \]

Then for constants \( \alpha_1 \) and \( \alpha_2 \): \[ a_n = \alpha_1 3^n + \alpha_2 n3^n \]

Initial conditions: \[ a_0 = \alpha_1 = 1 \]
\[ a_1 = 3\alpha_1 + 3\alpha_2 = 6 \]

Constants: \( \alpha_1 = 1 \) and \( \alpha_2 = 1 \)

Solution: \[ a_n = 3^n(n + 1) \]
Linear Homogeneous Recurrence

\[ a_n = \begin{cases} 
2 & \text{if } n = 0, \\
7 & \text{if } n = 1, \\
a_{n-1} + 2a_{n-2} & \text{otherwise.} 
\end{cases} \]

\[ = 3 \cdot 2^n - (-1)^n \]

\[ a_n = \begin{cases} 
2 & \text{if } n = 0, \\
5 & \text{if } n = 1, \\
15 & \text{if } n = 2, \\
6a_{n-1} - 11a_{n-2} + 6a_{n-3} & \text{otherwise.} 
\end{cases} \]

\[ = 1 - 2^n + 2 \cdot 3^n \]

\[ a_n = \begin{cases} 
1 & \text{if } n = 0, \\
-2 & \text{if } n = 1, \\
-1 & \text{if } n = 2, \\
-3a_{n-1} - 3a_{n-2} - a_{n-3} & \text{otherwise.} 
\end{cases} \]

\[ = (1 + 3n - 2n^2)(-1)^n \]
A linear nonhomogeneous recurrence relation of degree \( k \) with constant coefficients is a recurrence relation of the form:

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),
\]

where \( c_1, c_2, \ldots, c_k \) are real constants, \( c_k \neq 0 \), and \( F(n) \) is a function not identically zero depending only on \( n \).

The recurrence relation

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}
\]

is called the associated homogeneous recurrence relation.
Linear Nonhomogeneous Recurrence

Recurrence: \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n), \)

Suppose \( \{a_n^{(p)}\} \) is a particular solution of the recurrence above, and \( \{a_n^{(h)}\} \) is a solution of the associated homogeneous recurrence.

Then every solution of the given nonhomogeneous recurrence is of the form \( \{a_n^{(p)} + a_n^{(h)}\} \).
Linear Nonhomogeneous Recurrence

Recurrence: \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \),

Suppose \( F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0)s^n \),

where \( b_0, b_1, ..., b_t \) and \( s \) are real numbers.

If \( s \) is not a solution of the characteristic equation of the associated homogeneous recurrence, then there is an \( a_n^{(p)} \) of the form:

\[ (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n. \]

If \( s \) is a solution of the characteristic equation and its multiplicity is \( m \), then there is an \( a_n^{(p)} \) of the form:

\[ n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n. \]
Linear Nonhomogeneous Recurrence

\[
a_n = \begin{cases} 
3 & \text{if } n = 1, \\
3a_{n-1} + 2n & \text{otherwise.}
\end{cases}
\]

Associated homogeneous equation: \( a_n = 3a_{n-1} \)

Homogeneous solution: \( a_n^{(h)} = \alpha 3^n \)

Particular solution of nonhomogeneous recurrence: \( a_n^{(p)} = p_1 n + p_0 \)

Then \( p_1 n + p_0 = 3(p_1(n - 1) + p_0) + 2n \)

\[ (2 + 2p_1)n + (2p_0 - 3p_1) = 0 \Rightarrow p_1 = -1, p_0 = -\frac{3}{2} \]

Solution: \( a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n \)

\[ a_1 = 3 \Rightarrow \alpha = \frac{11}{6} \]

Hence \( a_n = -n - \frac{3}{2} + \frac{11}{6} \cdot 3^n \)