CSE 548: Analysis of Algorithms

Lecture 8
(Amortized Analysis)

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A Binary Counter

counter value	counter	#bit flips	#bit resets ($1 \rightarrow 0$)	#bit sets ($0 \rightarrow 1$)
0	0 0 0 0 0 0 0 0			
1	0 0 0 0 0 0 0 1	1	0	1
2	0 0 0 0 0 0 1 0	2	1	1
3	0 0 0 0 0 0 1 1	1	0	1
4	0 0 0 0 0 1 0 0	3	2	1
5	0 0 0 0 0 1 0 1	1	0	1
6	0 0 0 0 0 1 1 0	2	1	1
7	0 0 0 0 0 1 1 1	1	0	1
8	0 0 0 0 1 0 0 0	4	3	1
9	0 0 0 0 1 0 0 1	1	0	1
10	0 0 0 0 1 0 1 0	2	1	1
11	0 0 0 0 1 0 1 1	1	0	1
12	0 0 0 0 1 1 0 0	3	2	1
13	0 0 0 0 1 1 0 1	1	0	1
14	0 0 0 0 1 1 1 0	2	1	1
15	0 0 0 0 1 1 1 1	1	0	1
16	0 0 0 1 0 0 0 0	5	4	1

A Binary Counter

Consider a k-bit counter initialized to 0 (i.e., all bits are 0's).

Suppose we increment the counter n times.

and cost of an increment = #bits flipped

Question: What is the worst-case total cost of n increments?

Worst-case cost of a single increment:

#bit sets ($0 \rightarrow 1$), $b_1 \leq 1$

#bit resets ($1 \rightarrow 0$), $b_0 \leq k - b_1$

#bit flips $= b_1 + b_0 \leq k$

Worst-case cost of n increments:

#bit flips $\leq nk$

This turns out to be a very loose upper bound!

Aggregate Analysis

A better upper bound can be obtained as follows.

Each increment sets ($0 \rightarrow 1$) at most one bit, i.e., $b_1 \leq 1$
So, total number of bits set by n increments, $B_1 = b_1 n \leq n$

Since at most n bits are set, there cannot be more than n bit resets ($1 \rightarrow 0$), i.e., $B_0 \leq B_1 \leq n$

So, total number of bit flips $= B_1 + B_0 \leq n + n = 2n$

Thus worst-case cost of a sequence of n increments, $T(n) \leq 2n$

Hence, in the worst case, average cost of an increment $= \frac{T(n)}{n} \leq 2$

This *worst-case average cost* is called the *amortized cost* of an increment in a sequence of n increments.

A Binary Counter

counter value	counter	#bit flips	#bit resets (1→ 0)	#bit sets (0 → 1)	total #bit flips
0	0 0 0 0 0 0 0 0				
1	0 0 0 0 0 0 0 1	1	0	1	1
2	0 0 0 0 0 0 1 0	2	1	1	3
3	0 0 0 0 0 0 1 1	1	0	1	4
4	0 0 0 0 0 1 0 0	3	2	1	7
5	0 0 0 0 0 1 0 1	1	0	1	8
6	0 0 0 0 0 1 1 0	2	1	1	10
7	0 0 0 0 0 1 1 1	1	0	1	11
8	0 0 0 0 1 0 0 0	4	3	1	15
9	0 0 0 0 1 0 0 1	1	0	1	16
10	0 0 0 0 1 0 1 0	2	1	1	18
11	0 0 0 0 1 0 1 1	1	0	1	19
12	0 0 0 0 1 1 0 0	3	2	1	22
13	0 0 0 0 1 1 0 1	1	0	1	23
14	0 0 0 0 1 1 1 0	2	1	1	25
15	0 0 0 0 1 1 1 1	1	0	1	26
16	0 0 0 1 0 0 0 0	5	4	1	31

Amortized Analysis

- often obtains a tighter worst-case upper bound on the cost of a sequence of operations on a data structure by reasoning about the interactions among those operations
- the actual cost of any given operation may be very high, but that operation may change the state of the data structure in such a way that similar high-cost operations cannot appear for a while
- tries to show that there must be enough low-cost operations in the sequence to average out the impact of high-cost operations
- unlike average case analysis proves a worst-case upper bound on the total cost of the sequence of operations
- unlike expected case analysis no probabilities are involved

Accounting Method (Banker's View)

Consider a k-bit counter initialized to 0 (i.e., all bits are 0's).

Worst-case cost of a single increment:

#bit sets ($0 \rightarrow 1$), $b_1 \leq 1$

#bit resets ($1 \rightarrow 0$), $b_0 \leq k - b_1$

#bit flips $= b_1 + b_0 \leq k$

Thus each increment is paying for the bit it sets (fair).

But also paying for resetting bits set by prior increments (unfair)!

A fairer cost accounting for each increment:

(1) Pay for the bit it sets.

(2) Pay in advance for resetting this bit (by some other increment) in the future. Store this advanced payment as a *credit* associated with that bit position.

(3) When resetting a bit use the credit stored in that bit position.

Accounting Method (Banker's View)

counter value	counter	actual cost (c_i)	amortized cost ($\hat{c}_i$)		$\sum c_i$	$\leq$	$\sum \hat{c}_i$
0	0 0 0 0 0 0 0 0						
1	0 0 0 0 0 0 0 1	1	2	(overcharged)	1	$\leq$	2
2	0 0 0 0 0 0 1 0	2	2		3	$\leq$	4
3	0 0 0 0 0 0 1 1	1	2	(overcharged)	4	$\leq$	6
4	0 0 0 0 0 1 0 0	3	2	(undercharged)	7	$\leq$	8
5	0 0 0 0 0 1 0 1	1	2	(overcharged)	8	$\leq$	10
6	0 0 0 0 0 1 1 0	2	2		10	$\leq$	12
7	0 0 0 0 0 1 1 1	1	2	(overcharged)	11	$\leq$	14
8	0 0 0 0 1 0 0 0	4	2	(undercharged)	15	$\leq$	16
9	0 0 0 0 1 0 0 1	1	2	(overcharged)	16	$\leq$	18

Accounting Method (Banker's View)

counter value	counter	actual cost (c_i)	amortized cost ($\hat{c}_i$)		$\sum c_i \le \sum \hat{c}_i$
0	0 0 0 0 0 0 0 0				
1	0 0 0 0 0 0 0 1	1	2	(overcharged)	$1 \le 2$
2	0 0 0 0 0 0 1 0	2	2		$3 \le 4$
3	0 0 0 0 0 0 1 1	1	2	(overcharged)	$4 \le 6$
4	0 0 0 0 0 1 0 0	3	2	(undercharged)	$7 \le 8$

Total credits remaining after n increments, $\Delta_n = \sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i$

We must make sure that for all n, $\Delta_n \ge 0$

$$\Rightarrow \sum_{i=1}^{n} \hat{c}_i \ge \sum_{i=1}^{n} c_i$$

This will ensure that *the total amortized cost is always an upper bound on the total actual cost*.

Potential Method (Physicist's View)

Banker's View: Store prepaid work as credit with specific objects in the data structure.

Physicist's View: Represent total remaining credit in the data structure as a single potential function.

Suppose: state of the initial data structure $= D_0$

state of the data structure after the i-th operation $= D_i$

potential associated with D_i is $= \Phi(D_i)$

Then amortized cost of the i-th operation,

$$\hat{c}_i = \text{actual cost} + \text{potential change due to that operation}$$

$$= c_i + \Phi(D_i) - \Phi(D_{i-1})$$

Potential Method (Physicist's View)

Then amortized cost of the i-th operation,

$$\hat{c}_i = \text{actual cost} + \text{potential change due to that operation}$$
$$= c_i + \Phi(D_i) - \Phi(D_{i-1})$$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

Since we do not know n in advance, if we make sure that for all n, $\Phi(D_n) \geq \Phi(D_0)$, we ensure that always $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$.

In other words, in that case, *the total amortized cost will always be an upper bound on the total actual cost*.

One way of achieving that is to find a Φ such that $\Phi(D_0) = 0$ and for all n, $\Phi(D_n) \geq 0$.

Potential Method (Physicist's View)

For the binary counter,

$$\Phi(D_i) = \text{number of set bits (i.e., 1 bits) after the } i\text{-th operation}$$

counter value	counter	actual cost (c_i)	$\Phi(D_i)$	amortized cost ($\hat{c}_i$)		$\sum c_i \leq \sum \hat{c}_i$
0	0 0 0 0 0 0 0 0		0			
1	0 0 0 0 0 0 0 1	1	1	2	(overcharged)	$1 \leq 2$
2	0 0 0 0 0 0 1 0	2	1	2		$3 \leq 4$
3	0 0 0 0 0 0 1 1	1	2	2	(overcharged)	$4 \leq 6$
4	0 0 0 0 0 1 0 0	3	1	2	(undercharged)	$7 \leq 8$
5	0 0 0 0 0 1 0 1	1	2	2	(overcharged)	$8 \leq 10$
6	0 0 0 0 0 1 1 0	2	2	2		$10 \leq 12$
7	0 0 0 0 0 1 1 1	1	3	2	(overcharged)	$11 \leq 14$
8	0 0 0 0 1 0 0 0	4	1	2	(undercharged)	$15 \leq 16$