

CSE 548: Analysis of Algorithms

Lecture 7 (Generating Functions)

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An Impossible Counting Problem

Suppose you went to a grocery store to buy some fruits. There are some constraints though:



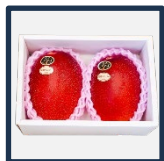
A. The store has only two **apples** left: one red and one green. So, you cannot take more than two.



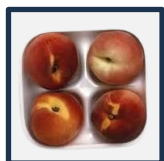
B. All but 3 **bananas** are rotten. You do not like rotten bananas.



F. **Figs** are sold 6 per pack. You can take as many packs as you want.



M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.



P. They sell 4 **peaches** per pack. Take as many packs as you want.

Question: in how many ways can you buy n fruits from the store?

Generating Functions

Generating functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series.

For example, one can represent a sequence s_0, s_1, s_2, \dots as:

$$S(z) = s_0 + s_1z + s_2z^2 + s_3z^3 + \dots + s_nz^n + \dots$$

So s_n is the coefficient of z^n in $S(z)$.

An Impossible Counting Problem

- A. The store has only two **apples** left: one red and one green.
So you cannot take more than 2 apples.



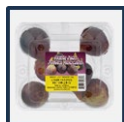
$$A(z) = 1 + 2z + z^2 = (1 + z)^2$$

- B. All but 3 **bananas** are rotten. You do not like rotten bananas.



$$B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}$$

- F. **Figs** are sold 6 per pack. You can take as many packs as you want.



$$F(z) = 1 + z^6 + z^{12} + z^{18} + \dots = \frac{1}{1 - z^6}$$

- M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.



$$M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}$$

- P. They sell 4 **peaches** per pack. Take as many packs as you want.



$$P(z) = 1 + z^4 + z^8 + z^{12} + \dots = \frac{1}{1 - z^4}$$

An Impossible Counting Problem

Suppose you can choose n fruits in s_n different ways.

Then the generating function for s_n is:

$$\begin{aligned} S(z) = A(z)B(z)F(z)M(z)P(z) &= (1+z)^2 \times \frac{1-z^4}{1-z} \times \frac{1}{1-z^6} \times \frac{1-z^6}{1-z^2} \times \frac{1}{1-z^4} \\ &= \frac{1+z}{(1-z)^2} \\ &= (1+z) \sum_{n=0}^{\infty} (n+1)z^n \\ &= \sum_{n=0}^{\infty} (2n+1)z^n \end{aligned}$$

Equating the coefficients of z^n from both sides:

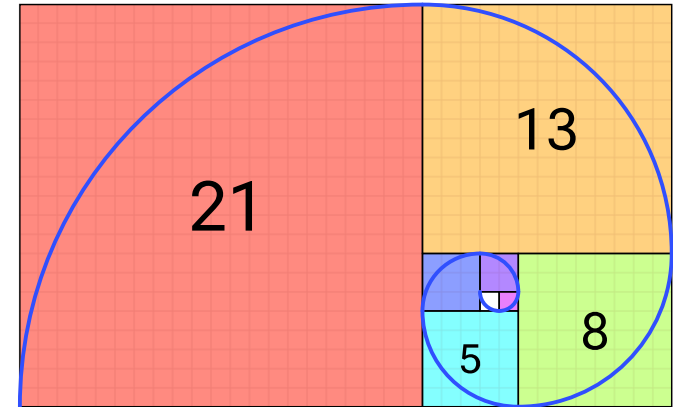
$$s_n = 2n + 1$$

Fibonacci Numbers

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f_{n-1} + f_{n-2} & \text{otherwise.} \end{cases}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2} + [n = 1]$$



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Let's define: $f_n = 0$ for $n < 0$

Generating function: $F(z) = f_0 + f_1z + f_2z^2 + f_3z^3 + \dots$

$$F(z) = \sum_n f_n z^n = \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n = 1] z^n$$

$$= \sum_n f_n z^{n+1} + \sum_n f_n z^{n+2} + z$$

$$= zF(z) + z^2F(z) + z$$

Fibonacci Numbers

$$F(z) = zF(z) + z^2F(z) + z$$

$$\Rightarrow F(z) = \frac{z}{1 - z - z^2}$$

$$= \frac{z}{(1 - \varphi z)(1 - \hat{\varphi} z)}, \text{ where } \varphi = \frac{1 + \sqrt{5}}{2} \text{ \& } \hat{\varphi} = \frac{1 - \sqrt{5}}{2}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi z} - \frac{1}{1 - \hat{\varphi} z} \right)$$

$$= \frac{1}{\sqrt{5}} \sum_n (\varphi^n - \hat{\varphi}^n) z^n$$

Equating the coefficients of z^n from both sides:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Average-case Analysis of Quicksort

Quicksort

Input: An array $A[1:n]$ of n distinct numbers.

Output: Numbers of $A[1:n]$ rearranged in increasing order of value.

Steps:

1. **Pivot Selection:** Select pivot $x = A[1]$.
2. **Partition:** Use a stable partitioning algorithm to rearrange the numbers of $A[1:n]$ such that $A[k] = x$ for some $k \in [1, n]$, each number in $A[1:k-1]$ is smaller than x , and each in $A[k+1:n]$ is larger than x .
3. **Recursion:** Recursively sort $A[1:k-1]$ and $A[k+1:n]$.
4. **Output:** Output $A[1:n]$.

Stable Partitioning: If two numbers p and q end up in the same partition and p appears before q in the input, then p must also appear before q in the resulting partition.

Average Number of Comparisons by Quicksort

We will average the number of comparisons performed by *Quicksort* on all possible arrangements of the numbers in the input array.

Let t_n = average #comparisons performed by *Quicksort* on n numbers.

Then

$$t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{1}{n} \sum_{k=1}^n (t_{k-1} + t_{n-k}) & \text{otherwise.} \end{cases}$$

The recurrence can be rewritten as follows.

$$t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise.} \end{cases}$$

Average Number of Comparisons by Quicksort

The recurrence: $t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise.} \end{cases}$

Let $T(z)$ be an ordinary generating function for t_n 's:

$$T(z) = t_0 + t_1z + t_2z^2 + \cdots + t_nz^n + \cdots$$

$$= t_0 + \sum_{n=1}^{\infty} t_n z^n$$

$$= t_0 + \sum_{n=1}^{\infty} \left(n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n$$

Average Number of Comparisons by Quicksort

We have:
$$T(z) = t_0 + \sum_{n=1}^{\infty} \left(n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n$$

Differentiating:

$$T'(z) = \sum_{n=1}^{\infty} \left(n(n-1) + 2 \sum_{k=0}^{n-1} t_k \right) z^{n-1}$$

$$= z \sum_{n=2}^{\infty} n(n-1) z^{n-2} + 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^n t_k \right) z^n$$

$$= z \frac{d^2}{dz^2} \left(\left(\sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left(t_n z^n \left(\sum_{k=0}^{\infty} z^k \right) \right)$$

Average Number of Comparisons by Quicksort

$$T'(z) = z \frac{d^2}{dz^2} \left(\left(\sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left(t_n z^n \left(\sum_{k=0}^{\infty} z^k \right) \right)$$

$$= z \frac{d^2}{dz^2} \left((1-z)^{-1} - 1 - z \right) + 2(1-z)^{-1} \sum_{n=0}^{\infty} t_n z^n$$

$$= \frac{2z}{(1-z)^3} + \frac{2}{1-z} T(z)$$

Rearranging: $(1-z)^2 T'(z) - 2(1-z)T(z) = \frac{2z}{1-z}$

$$\Rightarrow \frac{d}{dz} \left((1-z)^2 T(z) \right) = \frac{d}{dz} (-2 \ln(1-z) - 2z)$$

Integrating: $(1-z)^2 T(z) = -2 \ln(1-z) - 2z + c$ (c is a constant)

Average Number of Comparisons by Quicksort

We have, $(1 - z)^2 T(z) = -2 \ln(1 - z) - 2z + c$ (c is a constant)

Putting $z = 0$, $T(0) = c \Rightarrow t_0 = c \Rightarrow c = 0$

Hence, $(1 - z)^2 T(z) = -2 \ln(1 - z) - 2z$

$$\Rightarrow T(z) = 2(-\ln(1 - z) - z)(1 - z)^{-2}$$

$$= 2 \left(\sum_{k=2}^{\infty} \frac{z^k}{k} \right) \left(\sum_{j=0}^{\infty} (j + 1)z^j \right)$$

Equating coefficients of z^n from both sides,

$$t_n = 2 \left(\sum_{k=2}^n \frac{n + 1 - k}{k} \right) = 2(n + 1) \sum_{k=1}^n \frac{1}{k} - 4n = 2(n + 1)H_n - 4n,$$

where $H_n = \sum_{k=1}^n \left(\frac{1}{k} \right)$ is the n^{th} harmonic number.

Average Number of Comparisons by Quicksort

We have, $t_n = 2(n + 1)H_n - 4n$,

where $H_n = \sum_{k=1}^n \left(\frac{1}{k}\right)$ is the n^{th} harmonic number.

But we know, $H_n = \ln n + O(1)$ (prove it)

Hence, $t_n = 2(n + 1)(\ln n + O(1)) - 4n = \Theta(n \log n)$.