# CSE 548: Analysis of Algorithms

## Lecture 3
## ( Divide-and-Conquer Algorithms: Matrix Multiplication )

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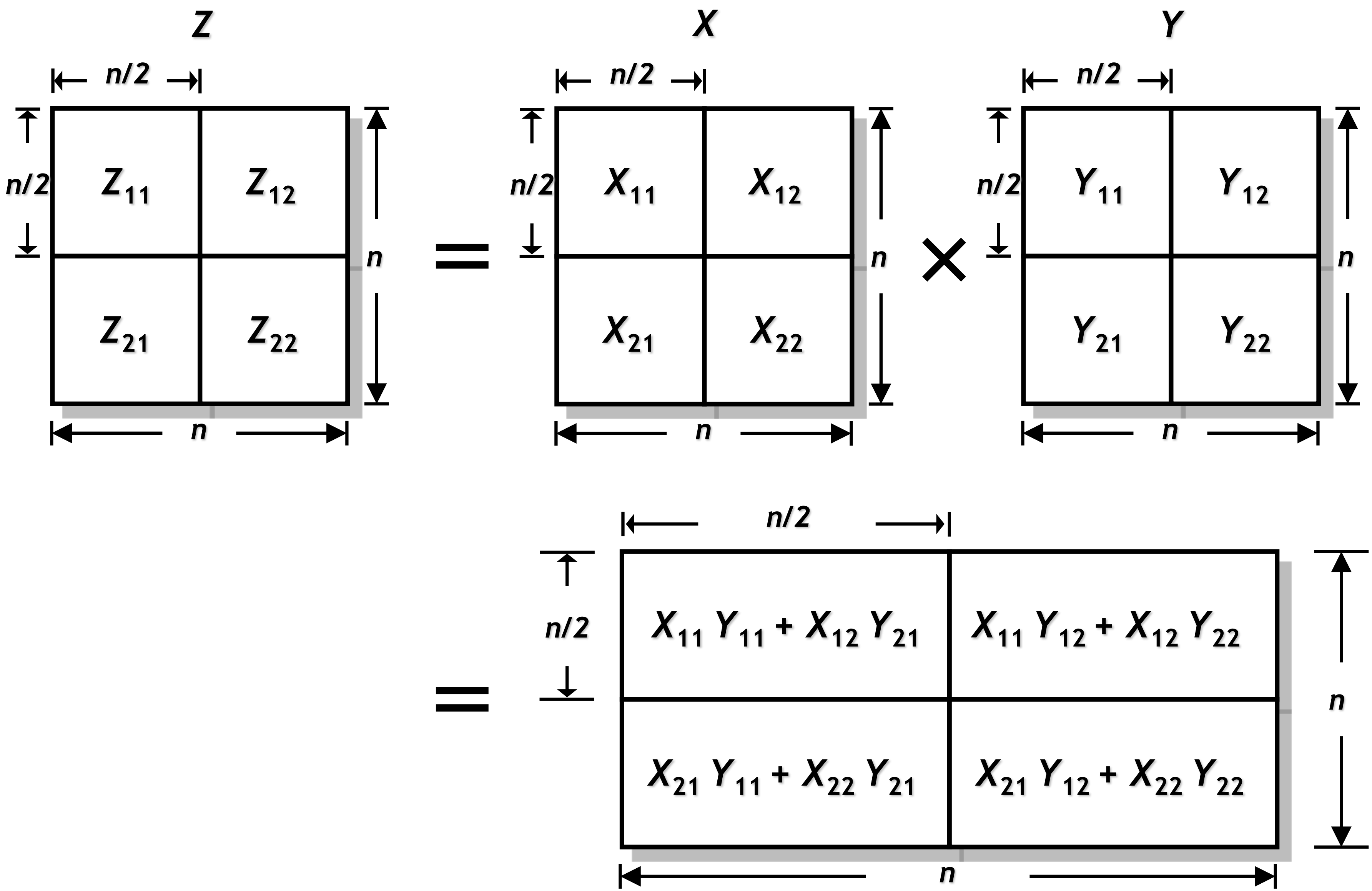
# Iterative Matrix Multiplication

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$$

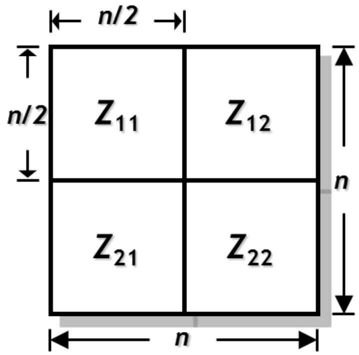
$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \times \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}$$

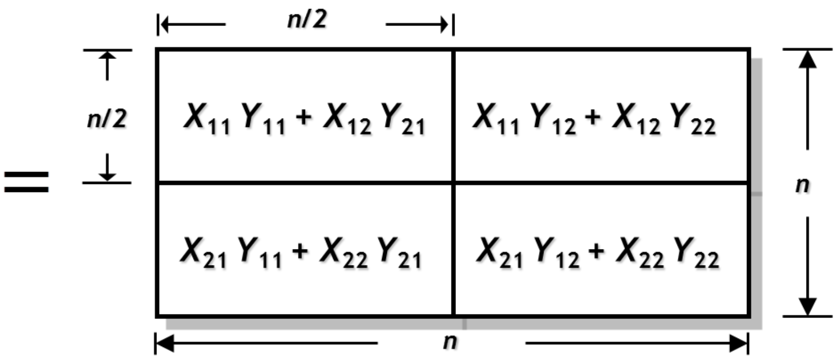
```
Iter-MM ( Z, X, Y )        { X, Y, Z are n × n matrices,
                             where n is a positive integer }
for i ← 1 to n do
    for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

# Recursive ( Divide & Conquer ) Matrix Multiplication



# Recursive ( Divide & Conquer ) Matrix Multiplication




```
Rec-MM ( X, Y )    { X and Y are n × n matrices,
                     where n = 2^k for integer k ≥ 0 }

Let Z be a new n × n matrix
if n = 1 then
    Z ← X · Y
else
    Z11 ← Rec-MM ( X11, Y11 ) + Rec-MM ( X12, Y21 )
    Z12 ← Rec-MM ( X11, Y12 ) + Rec-MM ( X12, Y22 )
    Z21 ← Rec-MM ( X21, Y11 ) + Rec-MM ( X22, Y21 )
    Z22 ← Rec-MM ( X21, Y12 ) + Rec-MM ( X22, Y22 )
endif
return Z
```

# recursive matrix products: 8

# matrix sums: 4

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta(n^3)$$

# Strassen's Algorithms for Matrix Multiplication ( MM )

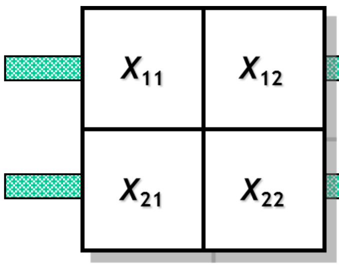
In 1968 Volker Strassen came up with a recursive MM algorithm that runs asymptotically faster than the classical $\Theta(n^3)$ algorithm.

In each level of recursion the algorithm uses:

7 recursive matrix multiplications ( instead of 8 ), and

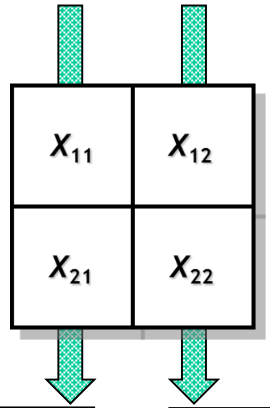
18 matrix additions ( instead of 4 ).

# Strassen's MM: 10 Matrix Additions/Subtractions



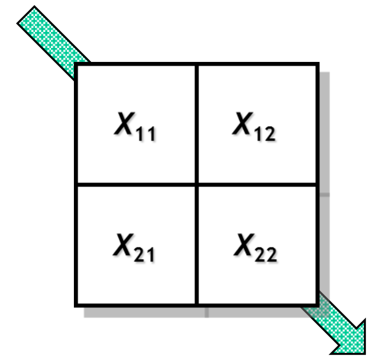

$X_{r1} = X_{11} + X_{12}$

$X_{r2} = X_{21} + X_{22}$

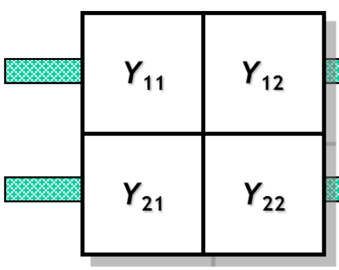



$X_{c1} = X_{11} - X_{21}$

$X_{c2} = X_{12} - X_{22}$

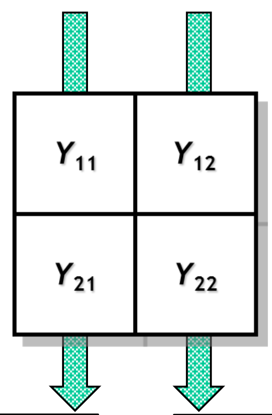



$X_{d1} = X_{11} + X_{22}$



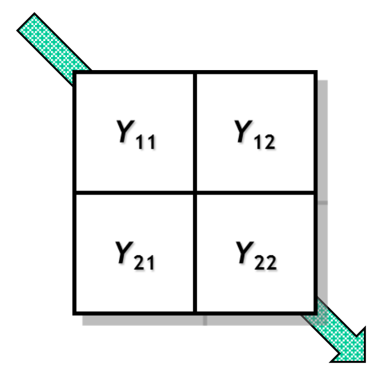

$Y_{r1} = Y_{11} + Y_{12}$

$Y_{r2} = Y_{21} + Y_{22}$




$Y_{c1} = Y_{11} - Y_{21}$

$Y_{c2} = Y_{12} - Y_{22}$

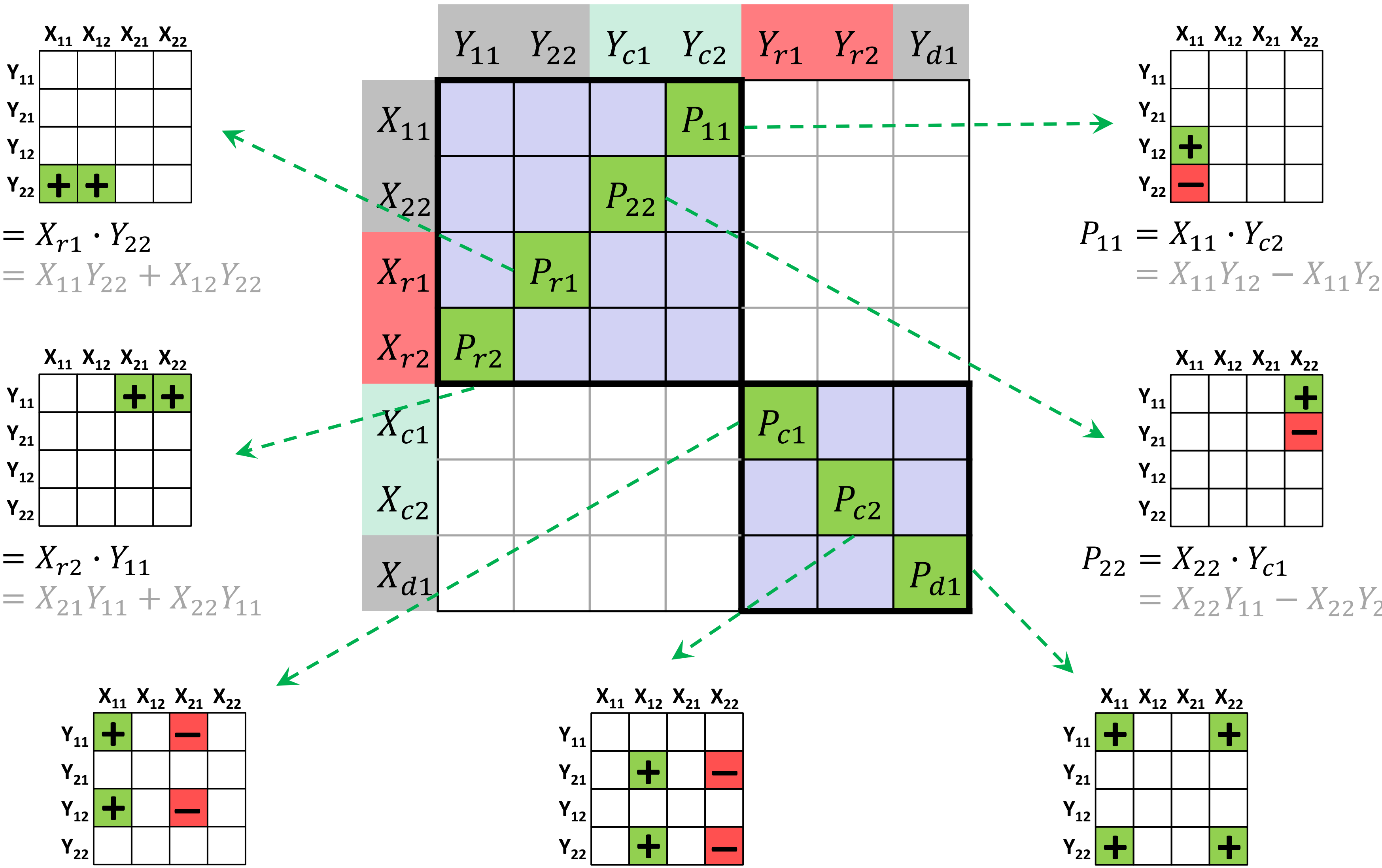



$Y_{d1} = Y_{11} + Y_{22}$

# Strassen's MM: 7 Matrix Products

| | $Y_{11}$ | $Y_{22}$ | $Y_{c1}$ | $Y_{c2}$ | $Y_{r1}$ | $Y_{r2}$ | $Y_{d1}$ |
|---|---|---|---|---|---|---|---|
| $X_{11}$ | | | | $P_{11}$ | | | |
| $X_{22}$ | | | $P_{22}$ | | | | |
| $X_{r1}$ | | $P_{r1}$ | | | | | |
| $X_{r2}$ | $P_{r2}$ | | | | | | |
| $X_{c1}$ | | | | | $P_{c1}$ | | |
| $X_{c2}$ | | | | | | $P_{c2}$ | |
| $X_{d1}$ | | | | | | | $P_{d1}$ |

# Strassen's MM: 7 Matrix Products




$$P_{r1} = X_{r1} \cdot Y_{22} = X_{11}Y_{22} + X_{12}Y_{22}$$

$$P_{11} = X_{11} \cdot Y_{c2} = X_{11}Y_{12} - X_{11}Y_{22}$$

$$P_{r2} = X_{r2} \cdot Y_{11} = X_{21}Y_{11} + X_{22}Y_{11}$$

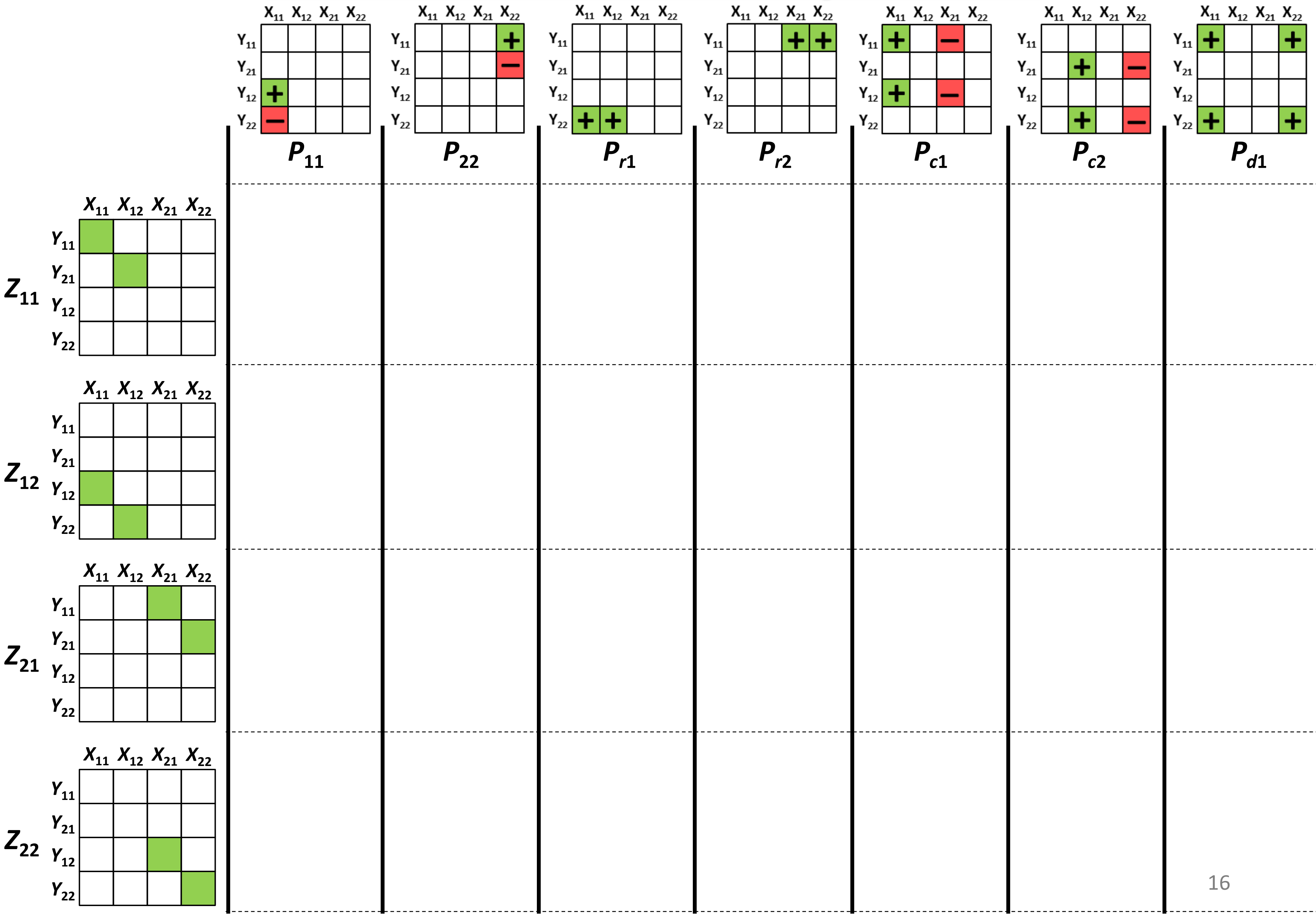
$$P_{22} = X_{22} \cdot Y_{c1} = X_{22}Y_{11} - X_{22}Y_{21}$$

$$P_{c1} = X_{c1} \cdot Y_{r1} = X_{11}Y_{11} + X_{11}Y_{12} - X_{21}Y_{11} - X_{21}Y_{12}$$

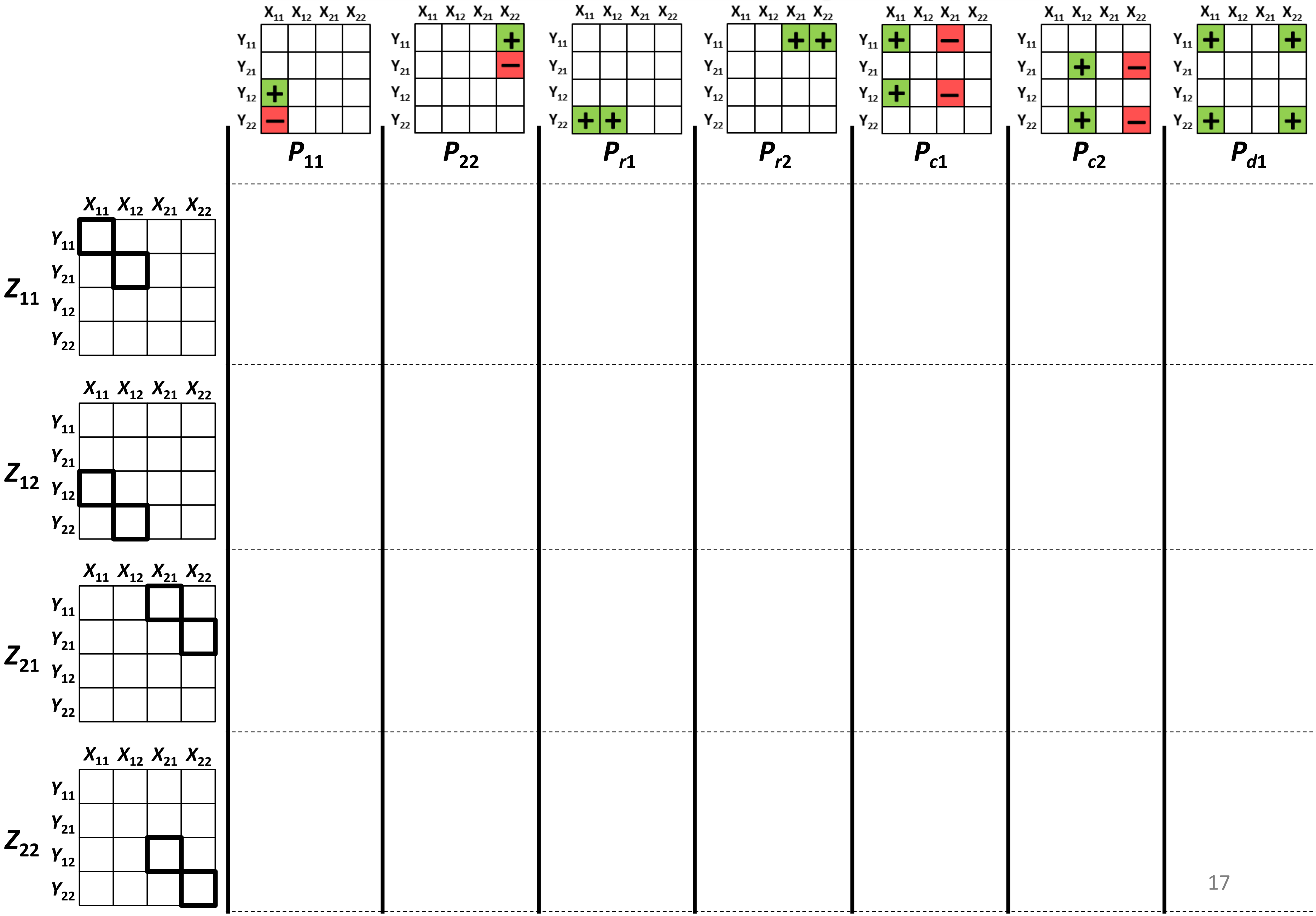
$$P_{c2} = X_{c2} \cdot Y_{r2} = X_{12}Y_{21} + X_{12}Y_{22} - X_{22}Y_{21} - X_{22}Y_{22}$$

$$P_{d1} = X_{d1} \cdot Y_{d1} = X_{11}Y_{11} + X_{11}Y_{22} + X_{22}Y_{11} + X_{22}Y_{22}$$

# Strassen's Matrix Multiplication



# Strassen's Matrix Multiplication



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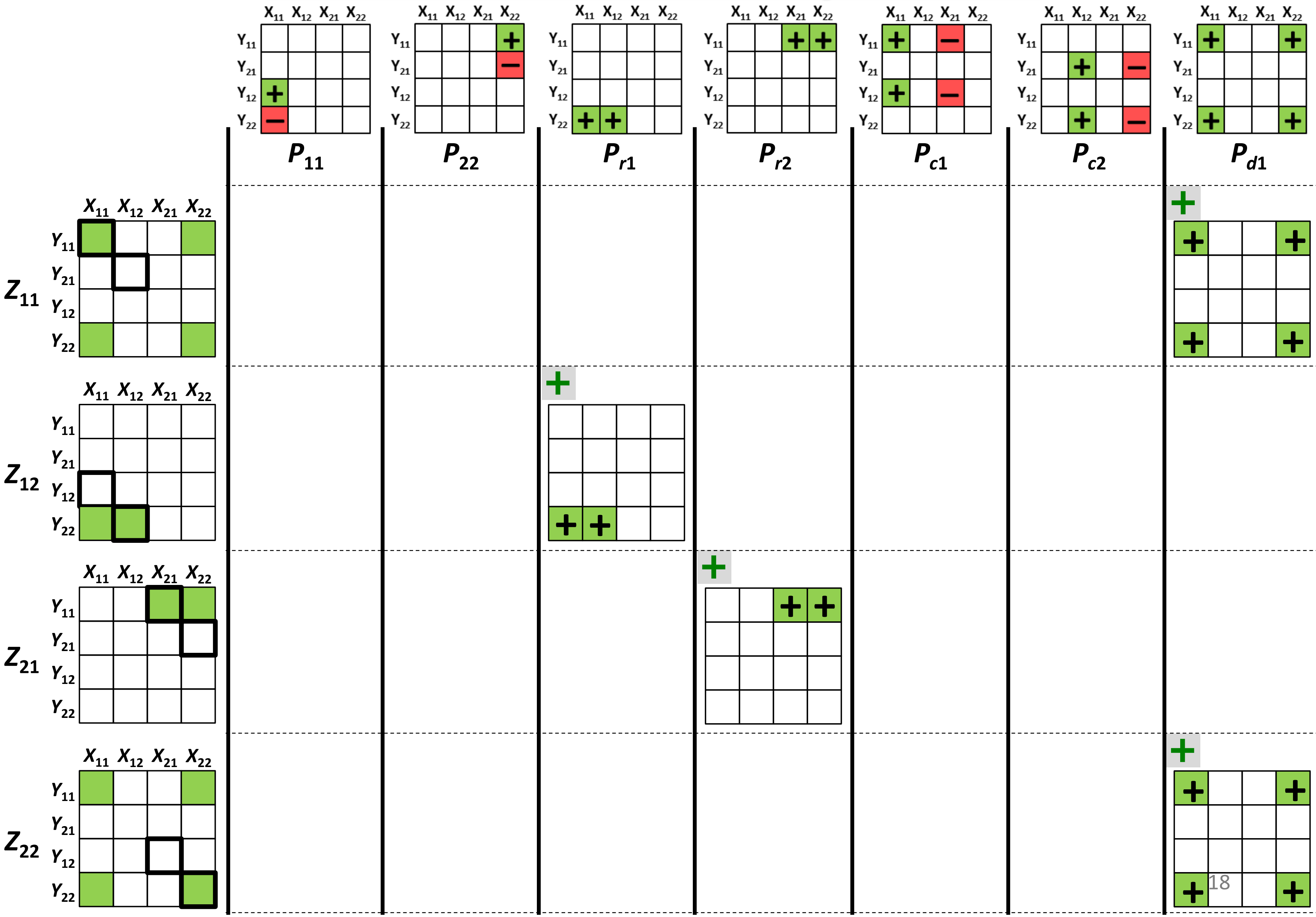

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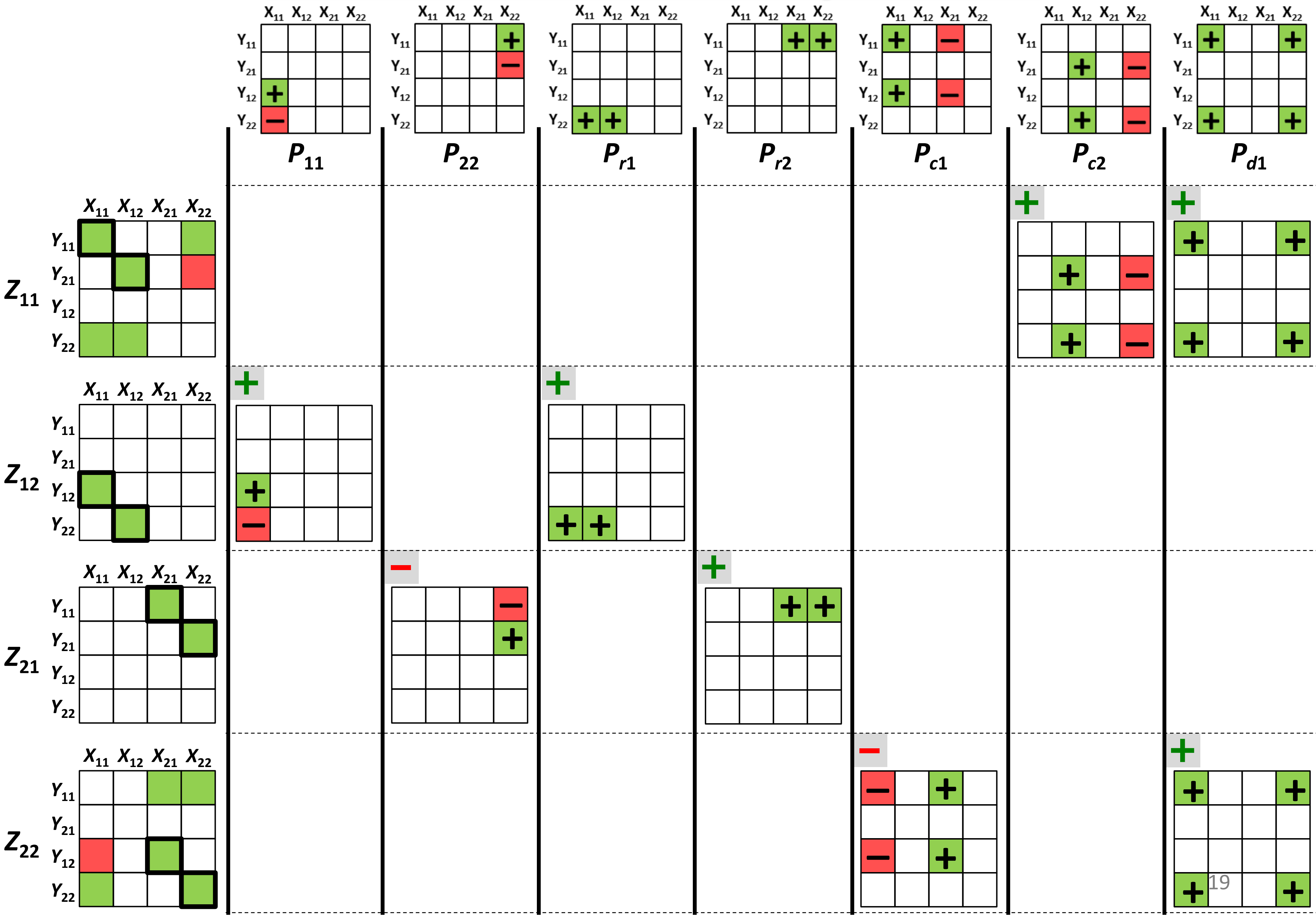

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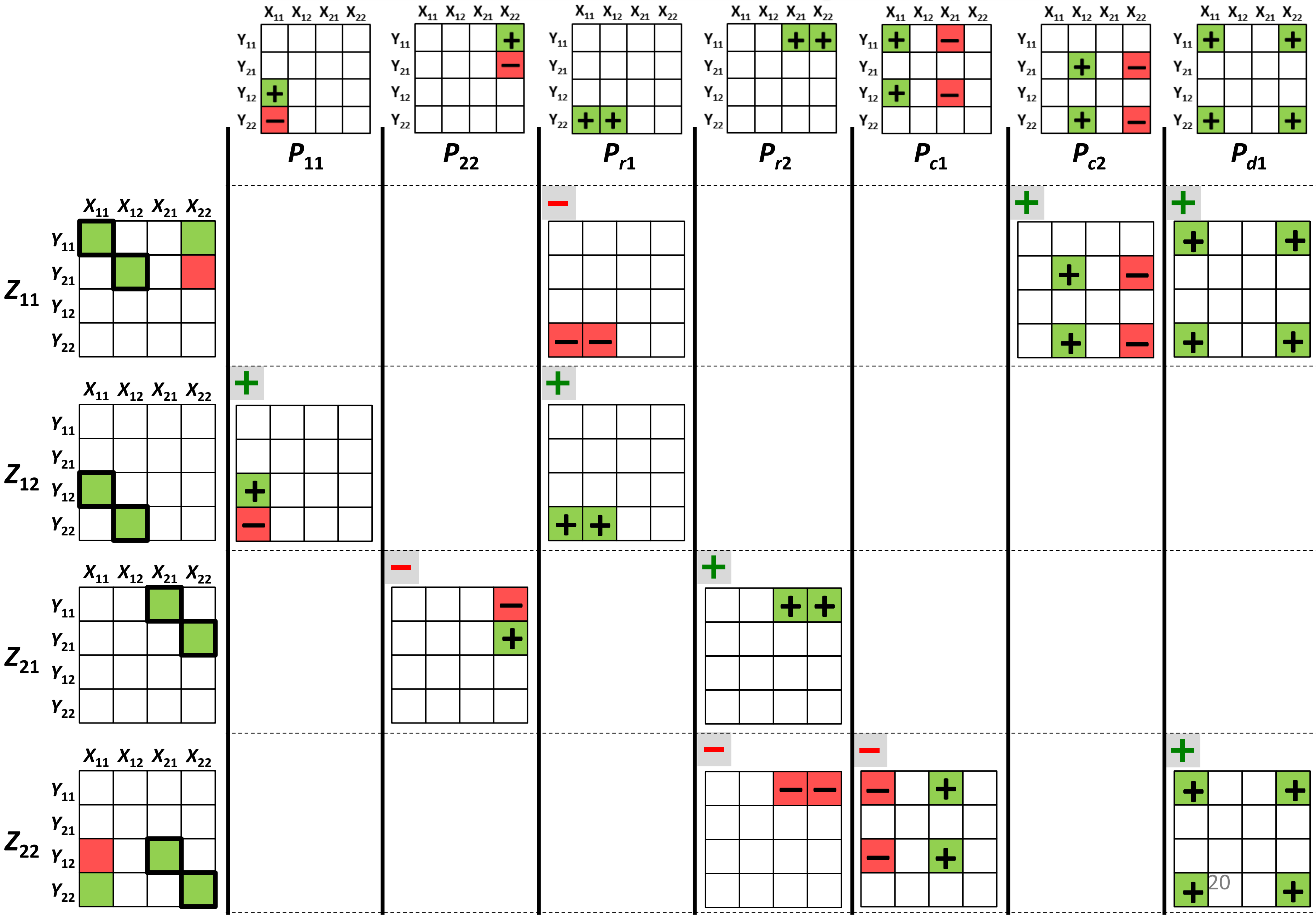
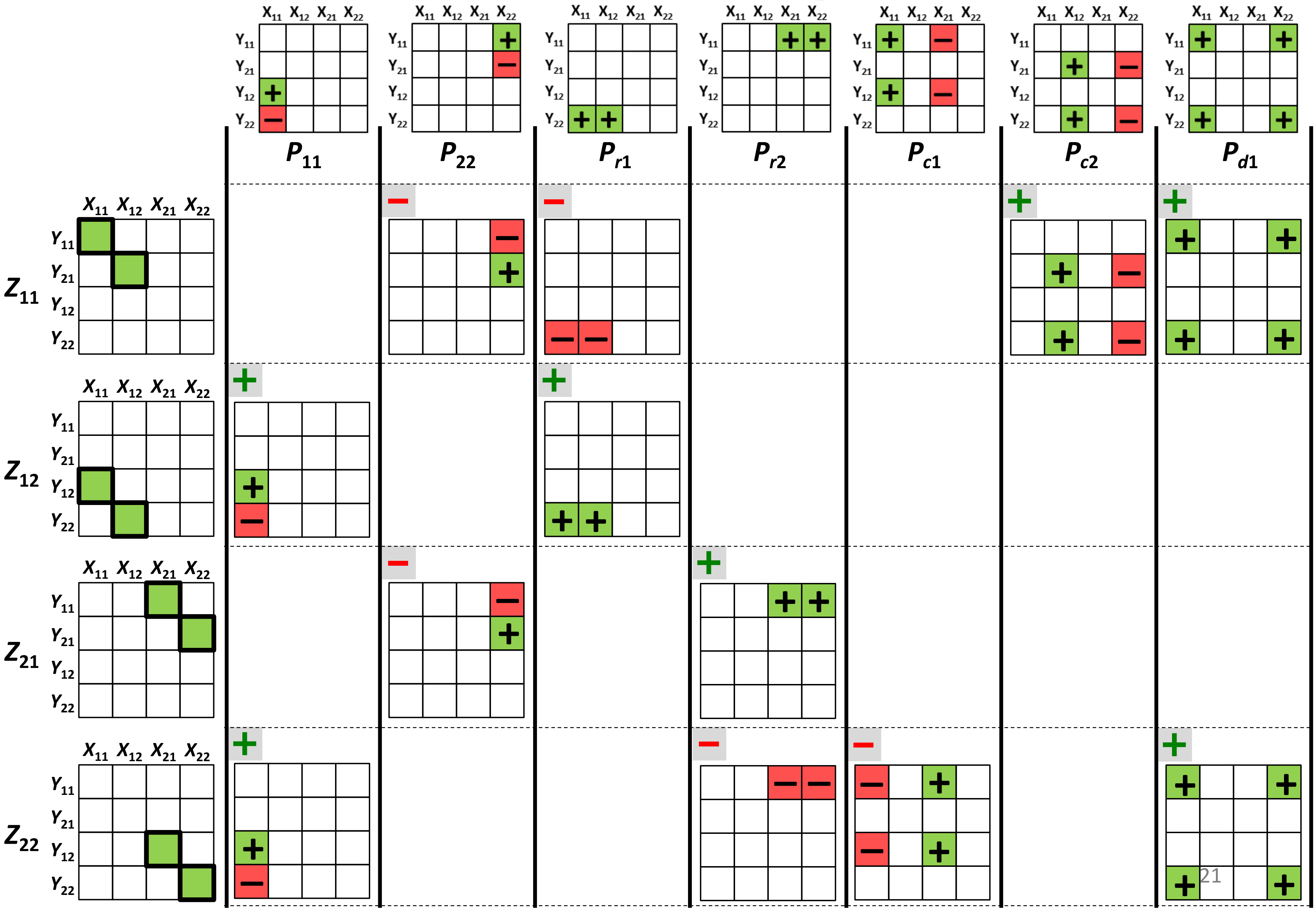


# Strassen's Matrix Multiplication



# Strassen's Matrix Multiplication

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \times \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -P_{r1} - P_{22} + P_{d1} + P_{c2} & +P_{r1} + P_{11} \\ +P_{r2} - P_{22} & -P_{r2} + P_{11} + P_{d1} - P_{c1} \end{bmatrix}$$

## Sums:

$$X_{r1} = X_{11} + X_{12} \qquad Y_{r1} = Y_{11} + Y_{12}$$
$$X_{r2} = X_{21} + X_{22} \qquad Y_{r2} = Y_{21} + Y_{22}$$
$$X_{c1} = X_{11} - X_{21} \qquad Y_{c1} = Y_{11} - Y_{21}$$
$$X_{c2} = X_{12} - X_{22} \qquad Y_{c2} = Y_{12} - Y_{22}$$
$$X_{d1} = X_{11} + X_{22} \qquad Y_{d1} = Y_{11} + Y_{22}$$

## Products:

$$P_{11} = X_{11} \cdot Y_{c2} \qquad P_{c1} = X_{c1} \cdot Y_{r1}$$
$$P_{22} = X_{22} \cdot Y_{c1} \qquad P_{c2} = X_{c2} \cdot Y_{r2}$$
$$P_{r1} = X_{r1} \cdot Y_{22} \qquad P_{d1} = X_{d1} \cdot Y_{d1}$$
$$P_{r2} = X_{r2} \cdot Y_{11}$$

## Running Time:

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 7T\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta\left(n^{\log_2 7}\right) = O\left(n^{2.81}\right)$$

# Deriving Strassen's Algorithm

Use the *Feynman Algorithm*:

**Step 1:** write down the problem

**Step 2:** think real hard

**Step 3:** write down the solution

# Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} e \\ g \\ f \\ h \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} p \\ r \\ q \\ s \end{bmatrix}}_{Z}$$

We will try to minimize the number of multiplications needed to evaluate $Z$ using special matrix products that are easy to compute.

| Type | Product | #Mults |
|---|---|---|
| $(\cdot)$ | $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$ | 4 |
| $(A)$ | $\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ a(e+g) \end{bmatrix}$ | 1 |
| $(B)$ | $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ -a(e+g) \end{bmatrix}$ | 1 |
| $(C)$ | $\begin{bmatrix} a & 0 \\ a-b & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae \\ ae + b(g-e) \end{bmatrix}$ | 2 |
| $(D)$ | $\begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e-g) + bg \\ bg \end{bmatrix}$ | 2 |

# Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} = \underbrace{\begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{\Delta_1}$$

$$\Delta_1 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a-c & b-c \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_2}$$

$$\Delta_2 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ c-b & 0 & 0 & c-b \\ -(c-b) & 0 & 0 & -(c-b) \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ B\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & d-b & 0 & b-c \\ c-b & 0 & a-c & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_3}$$

$$\Delta_3 = \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (a-b)-(a-c) & 0 & a-c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ C\ (\ 2\ Mult\ )} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & d-b & 0 & (d-c)-(d-b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{Type\ D\ (\ 2\ Mult\ )}$$

# Algorithms for Multiplying Two *n*×*n* Matrices

A recursive algorithm based on multiplying two $m \times m$ matrices using $k$ multiplications will yield an $O\left(n^{\log_m k}\right)$ algorithm.

To beat Strassen's algorithm: $\log_m k < \log_2 7 \Rightarrow k < m^{\log_2 7}$.

So, for a $3 \times 3$ matrix, we must have: $k < 3^{\log_2 7} < 22$.

But the best known algorithm uses 23 multiplications!

| Inventor | Year | Complexity |
|---|---|---|
| Classical | — | $\Theta\left(n^3\right)$ |
| Volker Strassen | 1968 | $\Theta\left(n^{2.807}\right)$ |
| Victor Pan<br>( multiply two 70 × 70 matrices using 143,640 multiplications ) | 1978 | $\Theta\left(n^{2.795}\right)$ |
| Don Coppersmith & Shmuel Winograd<br>( arithmetic progressions ) | 1990 | $\Theta\left(n^{2.3737}\right)$ |
| Andrew Stothers | 2010 | $\Theta\left(n^{2.3736}\right)$ |
| Virginia Williams | 2011 | $\Theta\left(n^{2.3727}\right)$ |

Lower bound: $\Omega\left(n^2\right)$ ( why? )