

# **CSE 548: Analysis of Algorithms**

## **Lecture 6**

### **( Divide-and-Conquer Algorithms: Akra-Bazzi Recurrences )**

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# Deterministic Select

**Input:** An array  $A[ q : r ]$  of distinct elements, and integer  $k \in [1, r - q + 1]$ .

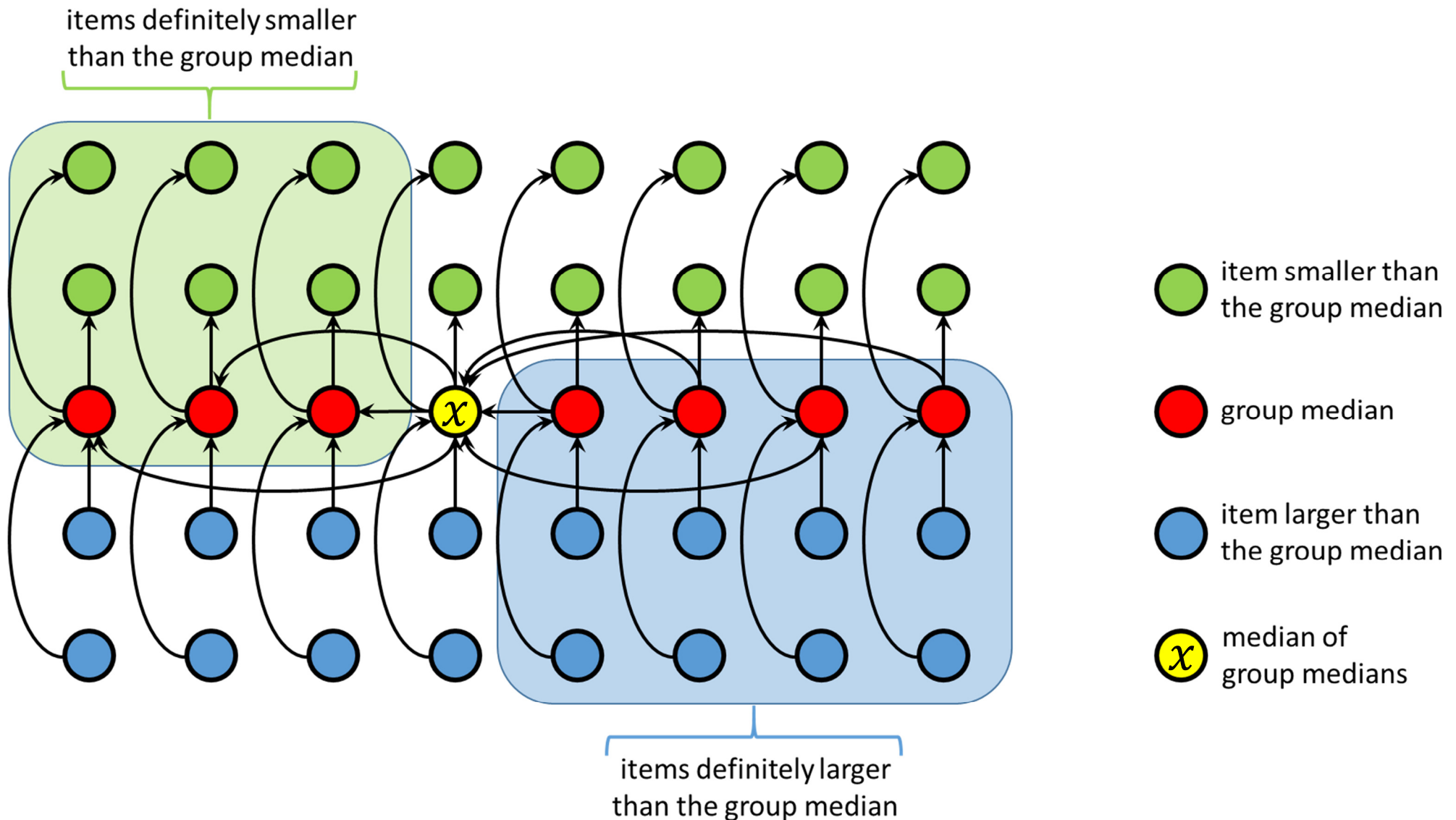
**Output:** An element  $x$  of  $A[ q : r ]$  such that  $rank(x, A[ q : r ]) = k$ .

*Select* (  $A[ q : r ]$ ,  $k$  )

1.  $n \leftarrow r - q + 1$
2. *if*  $n \leq 140$  *then*
3.     sort  $A[ q : r ]$  and *return*  $A[ q + k - 1 ]$
4. *else*
5.     divide  $A[ q : r ]$  into blocks  $B_i$ 's each containing 5 consecutive elements  
      ( last block may contain fewer than 5 elements )
6.     *for*  $i \leftarrow 1$  *to*  $\lceil n / 5 \rceil$  *do*
7.          $M[ i ] \leftarrow$  median of  $B_i$  using sorting
8.      $x \leftarrow$  *Select* (  $M[ 1 : \lceil n / 5 \rceil ]$ ,  $\lfloor (\lceil n / 5 \rceil + 1) / 2 \rfloor$  ) { median of medians }
9.      $t \leftarrow$  *Partition* (  $A[ q : r ]$ ,  $x$  )     { partition around  $x$  which ends up at  $A[ t ]$  }
10.     *if*  $k = t - q + 1$  *then return*  $A[ t ]$
11.     *else if*  $k < t - q + 1$  *then return* *Select* (  $A[ q : t - 1 ]$ ,  $k$  )
12.     *else return* *Select* (  $A[ t + 1 : r ]$ ,  $k - t + q - 1$  )

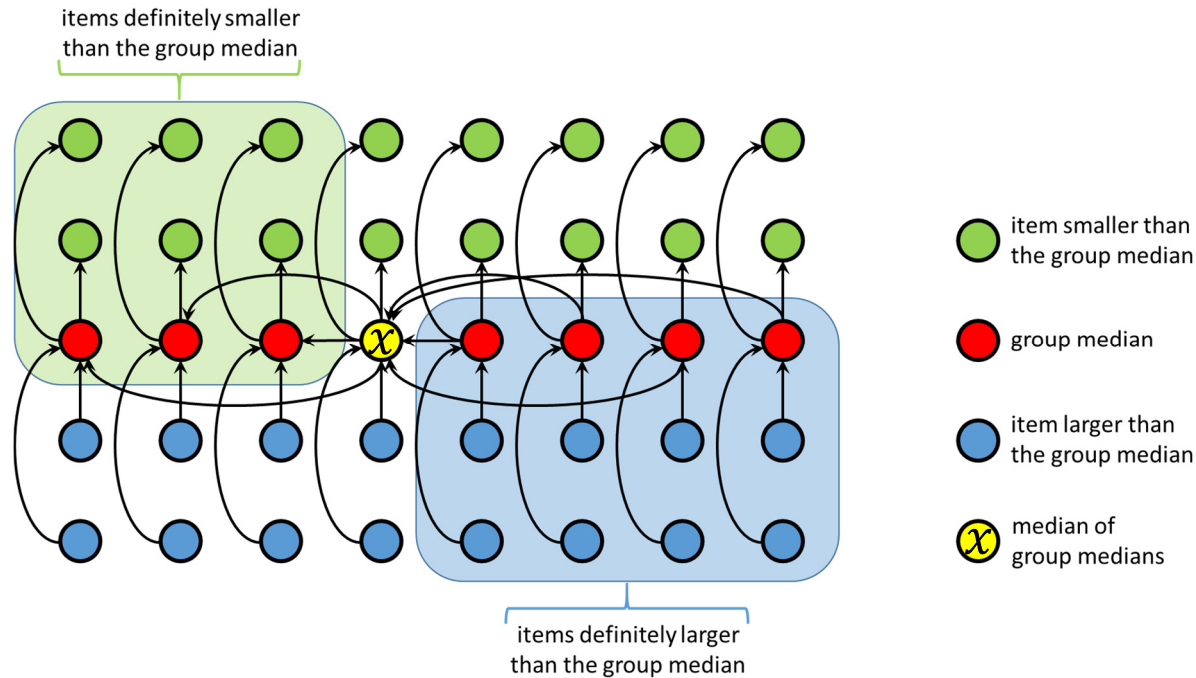
# Deterministic Select

SELECT( $A, k$ ): Given an unsorted set  $A$  of  $n$  ( $= |A|$ ) items, find the  $k^{\text{th}}$  smallest item in the set



# Deterministic Select

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$$\# \text{items definitely smaller than } x \text{ is} \quad \geq 3 \left( \left\lfloor \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1 \right) \geq \frac{3n}{10} - 6$$

$$\# \text{items definitely larger than } x \text{ is} \quad \geq 3 \left( \left\lfloor \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1 \right) \geq \frac{3n}{10} - 6$$

$$\# \text{items in any recursive call (lines 11/12)} \leq n - \left( \frac{3n}{10} - 6 \right) = \frac{7n}{10} + 6$$

## Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm ( given in Section 9.3 of CLRS ):

$$T(n) \leq \begin{cases} \Theta(1), & \text{if } n < 140, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

Dropping the ceiling for simplicity, and observing that  $\frac{7n}{10} + 6 \leq \frac{8n}{10}$  when  $n \geq 60$ , we obtain the following upper bound on  $T(n)$ .

$$T'(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

How do you solve for  $T'(n)$ ?

## Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm ( given in Section 9.3 of CLRS ):

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Dropping the ceiling for simplicity, and observing that  $\frac{7n}{10} + 6 \leq \frac{7.5n}{10}$  when  $n \geq 120$ , we obtain the following upper bound on  $T(n)$ .

$$T''(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

How do you solve for  $T''(n)$ ?

# Akra-Bazzi Recurrences

Consider the following recurrence:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0; \end{cases}$$

where,

1.  $k \geq 1$  is an integer constant
2.  $a_i > 0$  is a constant for  $1 \leq i \leq k$
3.  $b_i \in (0,1)$  is a constant for  $1 \leq i \leq k$
4.  $x \geq 1$  is a real number
5.  $x_0 \geq \max \left\{ \frac{1}{b_i}, \frac{1}{1-b_i} \right\}$  is a constant for  $1 \leq i \leq k$
6.  $g(x)$  is a nonnegative function that satisfies a *polynomial-growth condition* ( to be specified soon )

# Polynomial-Growth Condition

We say that  $g(x)$  satisfies the *polynomial-growth condition* if there exist positive constants  $c_1$  and  $c_2$  such that for all  $x \geq 1$ , for all  $1 \leq i \leq k$ , and for all  $u \in [b_i x, x]$ ,

$$c_1 g(x) \leq g(u) \leq c_2 g(x),$$

where  $x$ ,  $k$ ,  $b_i$  and  $g(x)$  are as defined in the previous slide.



# The Akra-Bazzi Solution

Consider the recurrence given in the previous two slides under the conditions specified there:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0. \end{cases}$$

Let  $p$  be the unique real number for which  $\sum_{i=1}^k a_i b_i^p = 1$ . Then

$$T(x) = \Theta \left( x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right) \right)$$

# Deterministic Select

$$T'(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

From  $\left(\frac{1}{5}\right)^p + \left(\frac{4}{5}\right)^p = 1$  we get  $p = 1$ .

$$\begin{aligned} \text{Hence, } T'(n) &= \Theta\left(n^p \left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right) \\ &= \Theta\left(n \left(1 + \int_1^n \frac{du}{u}\right)\right) \\ &= \Theta(n \ln n) \end{aligned}$$

# Deterministic Select

$$T''(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

From  $\left(\frac{1}{5}\right)^p + \left(\frac{3}{4}\right)^p = 1$  we get  $p < 1$ .

$$\begin{aligned} \text{Hence, } T''(n) &= \Theta\left(n^p \left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right) \\ &= \Theta\left(n^p \left(1 + \int_1^n \frac{du}{u^p}\right)\right) \\ &= \Theta\left(\left(\frac{1}{1-p}\right)n - \left(\frac{p}{1-p}\right)n^p\right) \\ &= \Theta(n) \end{aligned}$$

# Examples of Akra-Bazzi Recurrences

**Example 1:**  $T(x) = 2T\left(\frac{x}{4}\right) + 3T\left(\frac{x}{6}\right) + \Theta(x \log x)$

Then  $p = 1$  and  $T(x) = \Theta\left(x\left(1 + \int_1^x \frac{u \log u}{u^2} du\right)\right) = \Theta(x \log^2 x)$

**Example 2:**  $T(x) = 2T\left(\frac{x}{2}\right) + \frac{8}{9}T\left(\frac{3x}{4}\right) + \Theta\left(\frac{x^2}{\log x}\right)$

Then  $p = 2$  and  $T(x) = \Theta\left(x^2\left(1 + \int_1^x \frac{u^2 / \log u}{u^3} du\right)\right) = \Theta(x^2 \log \log x)$

**Example 3:**  $T(x) = T\left(\frac{x}{2}\right) + \Theta(\log x)$

Then  $p = 0$  and  $T(x) = \Theta\left(1 + \int_1^x \frac{\log u}{u} du\right) = \Theta(\log^2 x)$

**Example 4:**  $T(x) = \frac{1}{2}T\left(\frac{x}{2}\right) + \Theta\left(\frac{1}{x}\right)$

Then  $p = -1$  and  $T(x) = \Theta\left(\frac{1}{x}\left(1 + \int_1^x \frac{1}{u} du\right)\right) = \Theta\left(\frac{\log x}{x}\right)$

# A Helping Lemma

**Lemma:** If  $g(x)$  is a nonnegative function that satisfies the polynomial-growth condition, then there exist positive constants  $c_3$  and  $c_4$  such that for  $1 \leq i \leq k$  and all  $x \geq 1$ ,

$$c_3 g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq c_4 g(x).$$

**Proof:**

$$b_i x \leq u \leq x$$

$$\Rightarrow \frac{1}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \leq \frac{1}{u^{p+1}} \leq \frac{1}{\min\{(b_i x)^{p+1}, x^{p+1}\}}$$

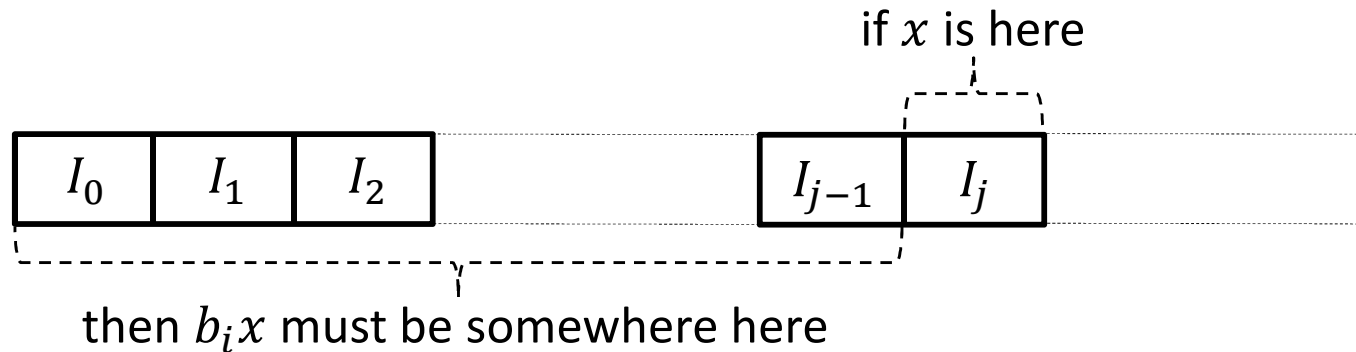
$$\Rightarrow \frac{x^p c_1 g(x)}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^x du \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq \frac{x^p c_2 g(x)}{\min\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^x du$$

$$\Rightarrow \frac{(1 - b_i) c_1}{\max\{1, b_i^{p+1}\}} g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq \frac{(1 - b_i) c_2}{\min\{1, b_i^{p+1}\}} g(x)$$

$$\Rightarrow c_3 g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq c_4 g(x)$$

# Partitioning the Domain of $x$

Let  $I_0 = [1, x_0]$  and  $I_j = (x_0 + j - 1, x_0 + j]$  for  $j \geq 1$ .

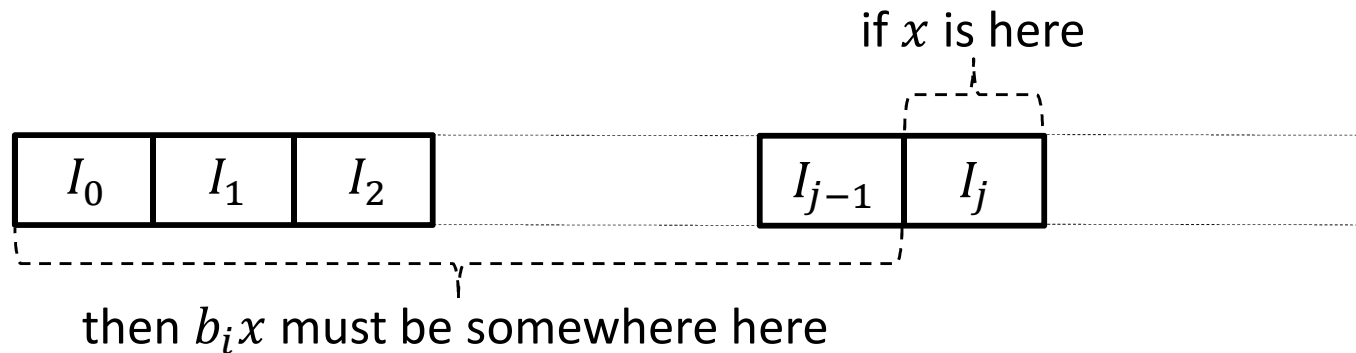


That allows us to use induction in the proof of:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0. \end{cases}$$

# Partitioning the Domain of $x$

Let  $I_0 = [1, x_0]$  and  $I_j = (x_0 + j - 1, x_0 + j]$  for  $j \geq 1$ .



**Proof:**

$$x_0 + j - 1 < x \leq x_0 + j$$

$$\Rightarrow b_i(x_0 + j - 1) < b_i x \leq b_i(x_0 + j)$$

$$\Rightarrow b_i x_0 < b_i x \leq b_i x_0 + j$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - (1 - b_i)x_0$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - 1$$

# Derivation of the Akra-Bazzi Solution

**Lower Bound:** There exists a constant  $c_5 > 0$  such that for all  $x > x_0$ ,

$$T(x) \geq c_5 x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right).$$

**Proof:** By induction on the interval  $I_j$  containing  $x$ .

Base case ( $j = 0$ ) follows since  $T(x) = \Theta(1)$  when  $x \in I_0 = [1, x_0]$ .

Induction: 
$$T(x) = \sum_{i=1}^k a_i T(b_i x) + g(x) \geq \sum_{i=1}^k a_i c_5 (b_i x)^p \left( 1 + \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du \right) + g(x)$$

$$= c_5 x^p \sum_{i=1}^k a_i b_i^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \right) + g(x)$$

$$\geq c_5 x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \frac{c_4}{x^p} g(x) \right) \sum_{i=1}^k a_i b_i^p + g(x)$$

$$= c_5 x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right) + (1 - c_4 c_5) g(x) \geq c_5 x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right)$$

( assuming  $c_4 c_5 \leq 1$  )



# Derivation of the Akra-Bazzi Solution

**Upper Bound:** There exists a constant  $c_6 > 0$  such that for all  $x > x_0$ ,

$$T(x) \leq c_6 x^p \left( 1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right).$$

**Proof:** Similar to the lower bound proof.

