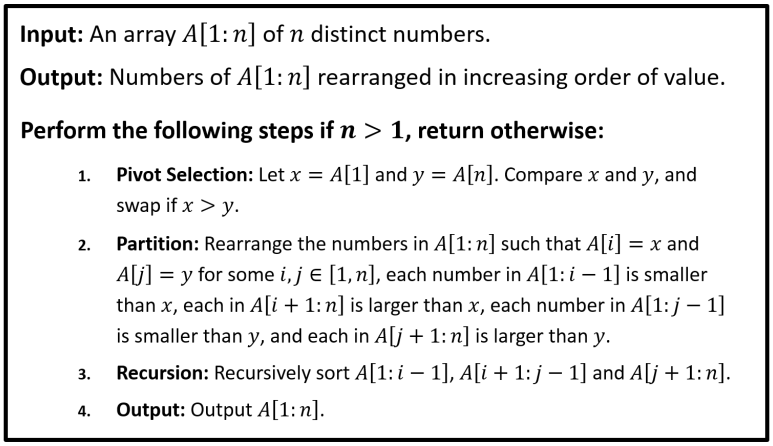
CSE548, AMS542: Analysis of Algorithms, Fall 2016 **Date:** Oct 17

Homework #2

(Due: Nov 1)

Task 1. [**80 Points**] **Average Case Analysis of 3-way Quicksort**

Consider the 3-way quicksort algorithm given in Figure 1.



Input: An array $A[1:n]$ of n distinct numbers.

Output: Numbers of $A[1:n]$ rearranged in increasing order of value.

Perform the following steps if $n > 1$, return otherwise:

1. **Pivot Selection:** Let $x = A[1]$ and $y = A[n]$. Compare x and y, and swap if $x > y$.
2. **Partition:** Rearrange the numbers in $A[1:n]$ such that $A[i] = x$ and $A[j] = y$ for some $i, j \in [1, n]$, each number in $A[1:i-1]$ is smaller than x, each in $A[i+1:n]$ is larger than x, each number in $A[1:j-1]$ is smaller than y, and each in $A[j+1:n]$ is larger than y.
3. **Recursion:** Recursively sort $A[1:i-1]$, $A[i+1:j-1]$ and $A[j+1:n]$.
4. **Output:** Output $A[1:n]$.

Figure 1: The 3-way quicksort algorithm.

Given an input of size n, in this task we will analyze the average number of element comparisons (i.e., comparisons between two numbers of the input array) performed by this algorithm over all $n!$ possible permutations of the input numbers. We will assume that the partionting algorithm is *stable*, i.e., if two numbers p and q end up in the same partition and p appears before q in the input, then p must also appear before q in the resulting partition.

(*a*) [**8 Points**] Show how to implement steps 1 and 2 of Figure 1 to get a stable partitioning of $A[1:n]$ using only $2n - \text{rank}(\min(x, y)) - 2$ element comparisons, where $\text{rank}(u)$ gives you the number of entries in $A[1:n]$ with value not larger than u.

(*b*) [**12 Points**] Let t_n be the average number of element comparisons performed by the algorithm given in Figure 1 to sort $A[1:n]$, where $n \geq 1$ and the average is taken over all $n!$ possible permutations of the numbers in A. Show that

$$t_n = \begin{cases} 0 & \text{if } n < 2, \\ \frac{5n-7}{3} + \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(t_{i-1} + t_{j-i-1} + t_{n-j} \right) & \text{otherwise.} \end{cases}$$

(*c*) [**20 Points**] Give an algorithm that can compute t_n in $\mathcal{O}(n)$ time and $\mathcal{O}(1)$ space, where $n \geq 1$ is an integer.

(*d*) [**20 Points**] Let $T(z)$ be a generating function for t_n:

$$T(z) = t_0 + t_1 z + t_2 z^2 + \ldots + t_n z^n + \ldots \quad \ldots$$

Show that $T''(z) = \frac{6}{(1-z)^2}\, T(z) + \frac{2}{(1-z)^3} + \frac{10z}{(1-z)^4}$.

(*e*) [**20 Points**] Solve the differential equation from part (*d*) to express the precise value of t_n in terms of n.

Task 2. [100 Points] Amortized Analysis of a Priority Queue

In this task we will consider a priority queue $\mathcal{Q}$ that supports *Insert* and *Extract-Min* operations. An *Insert*$(\mathcal{Q}, x)$ operation inserts an item x into $\mathcal{Q}$, and *Extract-Min*$(\mathcal{Q})$ deletes and returns the item with the smallest key from $\mathcal{Q}$. We will assume for simplicity that all values stored in $\mathcal{Q}$ are distinct.

The structure of $\mathcal{Q}$ at any given time is determined by three constants: $\mu_{max} > 1$, $\mu \in (1, \mu_{max})$ and $\alpha \in (1, 2]$, where μ_{max} and α remain fixed throughout the lifetime of $\mathcal{Q}$, but μ may change when the data structure is reconstructed periodically.

The data structure consists of $L+1$ levels, where $L = \log_\alpha \log_\mu (N_0)$ and $\frac{N_0}{2}$ is the number of items in $\mathcal{Q}$ at the time of its last reconstruction. The data structure is reconstructed periodically in order to ensure that $\frac{1}{4}N_0 \leq N \leq \frac{3}{4}N_0$ always holds, where N is the number of items currently in $\mathcal{Q}$.

Let $n_i = \left\lfloor \mu^{\alpha^i} \right\rfloor$ for $0 \leq i \leq L$. Level 0 consists of two buffers F_0 and S_0, where $|F_0|, |S_0| \leq n_0$. For

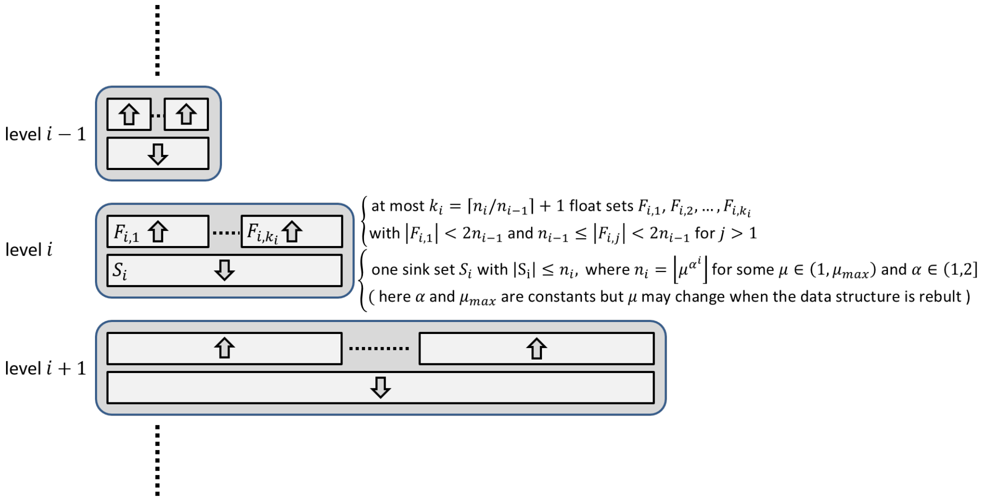


Figure 2: Structure of the priority queue in task 2. Intuitively, float sets store items that are on their way up (i.e., floating) and sink sets store items that are on their way down (i.e., sinking). The arrow inside each box shows this general direction of movements.

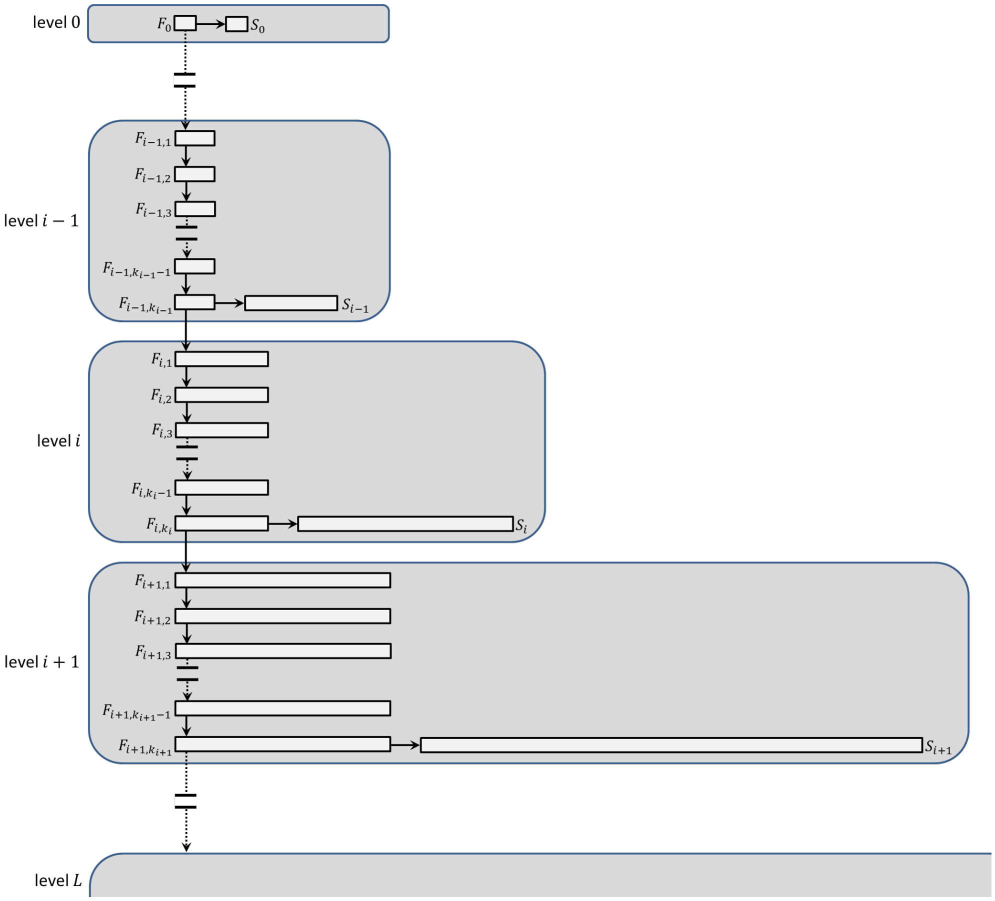


Figure 3: Pictorial depiction of Invariants 1–4. An arrow directed from box B_1 to box B_2 means that the key of every item in box B_1 is smaller than the key of every item in box B_2.

$1 \le i \le L$, level i consists of a *sink set* S_i with $|S_i| \le n_i$, and at most $k_i = \left\lceil \frac{n_i}{n_{i-1}} \right\rceil + 1$ *float sets* $F_{i,1}, F_{i,2}, \ldots, F_{i,k_i}$, where $|F_{i,1}| < 2n_{i-1}$ and $n_{i-1} \le |F_{i,j}| < 2n_{i-1}$ for $2 \le j \le k_i$. See Figure 2.

The entire priority queue is stored in a linear array $\mathcal{A}$ as follows. The first n_0 locations of $\mathcal{A}$ are reserved for S_0 and the next n_0 locations for F_0. Then space is reserved for the sink set and float sets of level 1, followed by those of level 2, then level 3, and so on. For any level $i \in [1, L+1]$, first n_i locations are reserved for S_i followed by $2n_{i-1}$ locations for each of the k_i possible float sets of level i. The float sets are not necessarily stored in sorted order[1], but they are always linked together to form an ordered linked list.

The following four invariants are always maintained (see Figure 3 for a pictorial depiction).

Invariant 1 *For $1 \le i \le L$ and $1 \le j < k_i$, keys in $F_{i,j}$ are smaller than those in $F_{i,j+1}$.*

Invariant 2 *For $1 \le i \le L$ and $1 \le j \le k_i$, keys in $F_{i,j}$ are smaller than those in S_i.*

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[1] e.g., $F_{i,j+1}$ may be necessarily be stored next to $F_{i,j}$ and it may appear before or after $F_{i,j}$

Invariant 3 *For $1 \leq i < L$ and $1 \leq j \leq k_i$, keys in $F_{i,j}$ are smaller than those in $F_{i+1,1}$.*

Invariant 4 *Keys in F_0 are smaller than those in S_0 and $F_{1,1}$.*

The item with the largest key in $F_{i,j}$ will be called a *splitter*, and will be denoted by $f_{i,j}$.

The *Insert* and *Extract-Min* operations internally use the following two recursive operations.

- Lift(i, V): Extracts the n_i items with the smallest keys from level $i+1$ (and below), and returns them in V.
- Sink(i, V): Inserts the n_i items stored in V (each with a key larger than all keys in the float sets of level i) into level $i+1$.

The Lift and Sink operations work as follows.

• **Sink**(i, V)**:** First we sort the items in V. Then we distribute those items among the float sets of level $i+1$ by simultaneously scanning the sorted items in V and visiting the float sets $F_{i+1,1}, F_{i+1,2}, \ldots, F_{i+1,k_{i+1}}$ in that linked list order. When visiting $F_{i+1,j}$ for some $j < k_{i+1}$ we keep copying items from V (in sorted order) to the end of $F_{i+1,j}$ until we encounter an item with key larger than the key of the splitter $f_{i+1,j}$, and at that point we move to $F_{i+1,j+1}$ and continue copying. Items with keys larger than the largest key in the last float set are inserted into S_{i+1}. If at any point a float set overflows (i.e., contains $2n_i$ items), it is split into two float sets of size n_i each. If the number of float sets after the split does not exceed k_{i+1}, the new float set is stored in any empty float set slot for that level and the linked list is updated accordingly. Otherwise, we move the at most $2n_i - 1$ items of the last float set to S_{i+1}, store the new float set in the newly freed space and update the linked list. If S_{i+1} itself overflows because of the insertion, i.e., it contains more than n_{i+1} items, we move the n_{i+1} items with the largest n_{i+1} keys from S_{i+1} to a temporary array V' leaving the remaining at most n_{i+1} items in S_{i+1}. We then call Sink$(i+1, V')$ to push the items in V' into level $i+2$.

• **Lift**(i, V)**:** If $F_{i+1,1}$ has at least n_i items, we sort those items, copy the n_i items with the smallest keys to V, and leave the remaining items (if any) in $F_{i+1,1}$.

If $F_{i+1,1}$ has $m < n_i$ items but $F_{i+1,2}$ is nonempty (i.e., has at least n_i items), we first remove $F_{i+1,1}$ by moving all its items to V. Then $F_{i+1,2}$ takes the role of $F_{i+1,1}$. The new $F_{i+1,1}$ has between n_i and $2n_i - 1$ items. We sort those items, move the $n_i - m$ items with the smallest keys to V and leave the remaining items in this new $F_{i+1,1}$.

If $F_{i+1,1}$ is the only nonempty float set in level $i+1$, and it has $m < n_i$ items, we move those m items to V leaving $F_{i+1,1}$ empty, and recursivly extract n_{i+1} items with the smallest keys from level $i+2$ by calling Lift$(i+1, V')$ where V' is a temporary array into which the extracted items are copied. Let m' $(\leq n_{i+1})$ be the number of items in S_{i+1}. We now move all items of S_{i+1} to V', sort the items in V', move the $n_i - m$ items with the smallest keys from V' to V, and move the m' items with largest keys to S_{i+1}. The remaining $n_{i+1} - n_i + m < n_{i+1} \leq k_{i+1} n_i$ items are distributed among the float sets of level $i+1$ with $F_{i+1,1}$ getting at most n_i items and at most $k_{i+1} - 1$ other float sets getting n_i items each.

Now the *Insert* and *Extract-Min* operations are performed as follows.

Insert($\mathcal{Q}$, x)**:** We compare the key k of item x with the largest key in F_0. If k is larger, we simply insert x into S_0, otherwise we insert it into F_0 and move the item with the largest key in F_0 to S_0. If S_0 now contains n_0 items we empty it by calling $\textsc{Sink}(0, S_0)$.

Extract-Min($\mathcal{Q}$)**:** If F_0 is empty, we call $\textsc{Lift}(0, F_0)$ to fill F_0 with n_0 items, and keep in F_0 the n_0 items with the smallest keys among F_0 and S_0 and leave the remaining items in S_0. We then extract and return the item with the smallest key from F_0.

We reconstruct $\mathcal{Q}$ periodically in order to make sure that $N_0 = \Theta(N)$ always holds, where $\frac{N_0}{2}$ is the number of items in the data structure at the time of the latest rebuild and N is the number of items currently in $\mathcal{Q}$. The structure is rebuilt right after the $\frac{N_0}{4}$-th *Insert*/*Extract-Min* operation is performed on it. First we find the largest value $\mu < \mu_{max}$ such that $\mu^{\alpha^l} = 2N$ holds for some integer l, and this value of l is our new L meaning that the rebuilt priority queue will have $l + 1 = L + 1$ levels. We first sort the N items currently in $\mathcal{Q}$. Then we rebuild $\mathcal{Q}$ top-down starting from level 0 and going up to level L while maintaining Invariants 1–4 as follows. We leave F_0 and S_0 empty. Then for each level $i \in [1, L)$, the sink set S_i and $F_{i,1}$ will be left completely empty while exactly n_{i-1} items will be stored in each $F_{i,j}$ for $2 \leq j \leq k_i$. The remaining items are stored in level L such that sink set S_L is empty while $F_{L,1}$ contains at most n_{L-1} item and at most k_L of the remaining float sets of level L contains n_{L-1} items each.

Now answer the following questions.

(a) [**4 Points**] Argue that $\mathcal{Q}$ uses only $\Theta(N)$ space.

(b) [**14 Points**] Show that both $\textsc{Lift}(\ i,\ V\ )$ and $\textsc{Sink}(\ i,\ V\ )$ maintain invariants 1–3.

(c) [**6 Points**] Use your results from part (b) to argue that both *Insert*($\mathcal{Q}$, x) and *Extract-Min*($\mathcal{Q}$) operations maintain invariants 1–4.

(d) [**14 Points**] What is the worst-case running time of $\textsc{Lift}(\ i,\ V\ )$ without considering any recursive use of this operation? What about $\textsc{Sink}(\ i,\ V\ )$?

(e) [**6 Points**] Use your results from part (d) to find the worst-case time needed for performing the *Insert*($\mathcal{Q}$, x) and *Extract-Min*($\mathcal{Q}$) operations.

(f) [**36 Points**] Argue that the amortized running time of $\textsc{Lift}(\ i,\ V\ )$ is only $\mathcal{O}(n_i \log n_i)$ without considering any recursive use of this operation. Show the same for $\textsc{Sink}(\ i,\ V\ )$.

(g) [**20 Points**] Use your results from part (f) to show that Argue that the amortized running time of and *Insert*($\mathcal{Q}$, x) operation is only $\mathcal{O}(\log N)$. Show that same for *Extract-Min*($\mathcal{Q}$).