CSE 373: Analysis of Algorithms

Lectures 21 & 22
(Iterated Log, Inverse Ackermann, and the Union-Find Data Structure)

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Iterated Functions

$$f^*(n) = \begin{cases} 0 & if\ n \le 1 \\ 1 + f^*\big(f(n)\big) & if\ n > 1 \end{cases}$$

$$= \min\left\{i \ge 0: \underbrace{f\left(f\big(f(\ldots f(n)\ldots)\big)\right)}_{i\ times} \le 1\right\}$$

$$= \min\{i \ge 0: f^{(i)}(n) \le 1\},$$

where $f^{(i)}(n) = \begin{cases} n & if\ i = 0 \\ f\left(f^{(i-1)}(n)\right) & if\ i > 0 \end{cases}$

Example: If $f = \log$, we have:

$\log^{(0)}(65536) = 65536$

$\log^{(1)}(65536) = 16$

$\log^{(2)}(65536) = 4$

$\log^{(3)}(65536) = 2$

$\log^{(4)}(65536) = 1$

$\therefore \log^*(65536) = 4$

Iterated Functions

$f(n)$	$f^*(n)$
$n-1$	$n-1$
$n-2$	$\frac{n}{2}$
$n-c$	$\frac{n}{c}$
$\frac{n}{2}$	$\log_2 n$
$\frac{n}{c}$	$\log_c n$
$\sqrt{n}$	$\log \log n$
$\log n$	$\log^* n$

The Inverse Ackermann Function: $\boldsymbol{\alpha(n)}$

$$
k = \alpha(n) \text{ rows} \left\{
\begin{array}{lll}
f(n) & f^*(n) & \\
\hline
\log n & \log^* n & > 3 \\
\log^* n & \log^{**} n & > 3 \\
\log^{**} n & \log^{***} n & > 3 \\
\cdots & \cdots & \\
\cdots & \cdots & \\
\cdots & \cdots & \\
\log^{\overbrace{*\cdots*}^{k-2}} n & \log^{\overbrace{*\cdots*}^{k-1}} n & > 3 \\
\log^{\overbrace{*\cdots*}^{k-1}} n & \log^{\overbrace{*\cdots*}^{k}} n & \leq 3
\end{array}
\right.
$$

$$
\alpha(n) = \min\left\{k \geq 1 : \log^{\overbrace{*\cdots*}^{k}} n \leq 3\right\}
$$

Union-Find:
A Disjoint-Set Data Structure

Disjoint Set Operations

A *disjoint-set data structure* maintains a collection of disjoint dynamic sets. Each set is identified by a *representative* which must be a member of the set.

The collection is maintained under the following operations:

MAKE-SET($\boldsymbol{x}$): create a new set $\{x\}$ containing only element x. Element x becomes the representative of the set.

FIND($\boldsymbol{x}$): returns a pointer to the representative of the set containing x

UNION($\boldsymbol{x}, \boldsymbol{y}$): replace the dynamic sets S_x and S_y containing x and y, respectively, with the set $S_x \cup S_y$

Union-Find Data Structure with *Union by Rank* and *Find with Path Compression*

MAKE-SET (x)

$\pi(x) \leftarrow x$

$rank(x) \leftarrow 0$

LINK (x, y)

if $rank(x) > rank(y)$ *then* $\pi(y) \leftarrow x$

else $\pi(x) \leftarrow y$

    if $rank(x) = rank(y)$ *then* $rank(y) \leftarrow rank(y) + 1$

UNION (x, y)

LINK (FIND (x), FIND (y))

FIND (x)

if $x \neq \pi(x)$ *then* $\pi(x) \leftarrow$ FIND ($\pi(x)$)

return $\pi(x)$

Some Useful Properties of Rank

- If x is not a root then $rank(x) < rank\big(\pi(x)\big)$
- Node ranks strictly increase along any simple path towards a root
- Once a node becomes a non-root its rank never changes
- If $\pi(x)$ changes from y to z then $rank(z) > rank(y)$
- If the root of x's tree changes from y to z then $rank(z) > rank(y)$
- If x is the root of a tree then $size(x) \geq 2^{rank(x)}$
- If there are only n nodes the highest possible rank is $\lfloor \log_2 n \rfloor$
- There are at most $\frac{n}{2^r}$ nodes with rank $r \geq 0$

Some Useful Properties of Rank

- We will analyze the total running time of m' MAKE-SET, UNION and FIND operations of which exactly n $(\leq m')$ are MAKE-SET
- But each UNION can be replaced with two FIND and one LINK
- Hence, we can simply analyze the total running time of m MAKE-SET, LINK and FIND operations of which exactly n $(\leq m)$ are MAKE-SET and where $m' \leq m \leq 3m'$

Compress

```
COMPRESS ( x, y )            { y is an ancestor of x }
1.   if x ≠ y then π(x) ← COMPRESS ( π(x), y )
2.   return π(x)
```

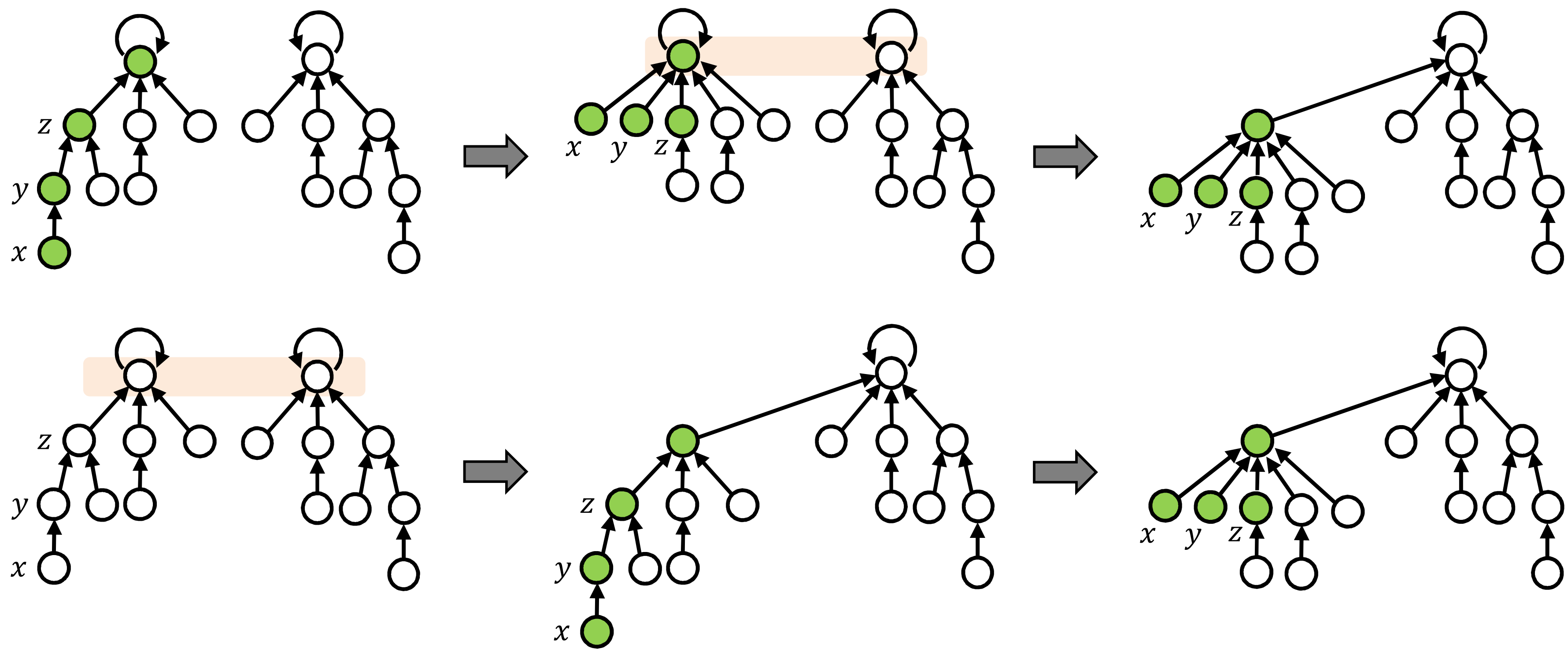
- We will analyze the total running time of m MAKE-SET, LINK and FIND operations of which exactly n $(\leq m)$ are MAKE-SET
- But FIND(x) is nothing but COMPRESS(x, y), where y is the root of the tree containing x
- Hence, we can analyze the total running time of m MAKE-SET, LINK and COMPRESS operations of which exactly n $(\leq m)$ are MAKE-SET

Compress

COMPRESS (x, y) { y is an ancestor of x }

1. *if* $x \neq y$ *then* $\pi(x) \leftarrow$ COMPRESS ($\pi(x), y$)
2. *return* $\pi(x)$

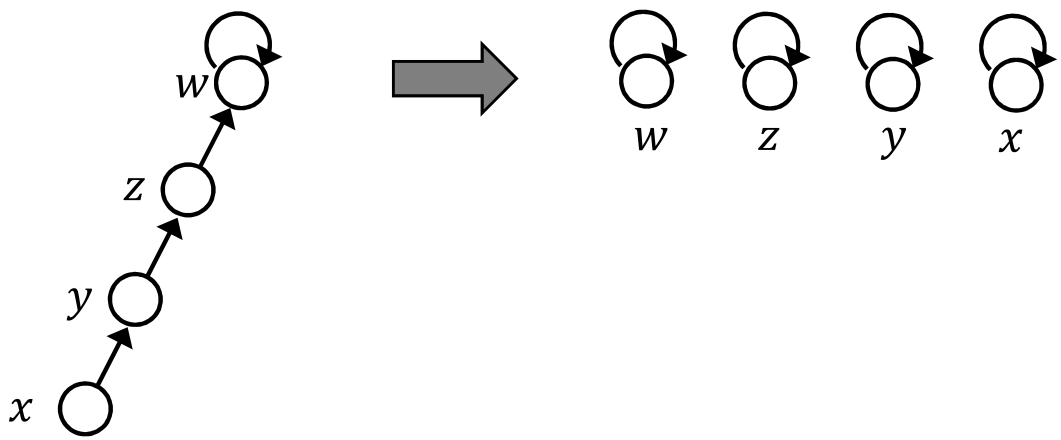
We can reorder the sequence of LINK and COMPRESS operations so that all LINK's are performed before all COMPRESS operations without changing the number of parent pointer reassignments!



Shatter

SHATTER (x)

1. *if* $x \neq \pi(x)$ *then* SHATTER ($\pi(x)$)
2. $\pi(x) \leftarrow x$



Bound 0

Let $T(m, n, r) =$ worst-case number of parent pointer assignments

- during any sequence of at most m COMPRESS operations
- on a forest of n nodes
- with maximum rank r

Bound 0: $T(m, n, r) \leq nr.$

Proof: Since there are at most r distinct ranks, and each new parent of a node has a higher rank than its previous parent, any node can change parents fewer than r times.

Bound 1

Bound 1: $T(m, n, r) \leq m + 2n \log^* r$.

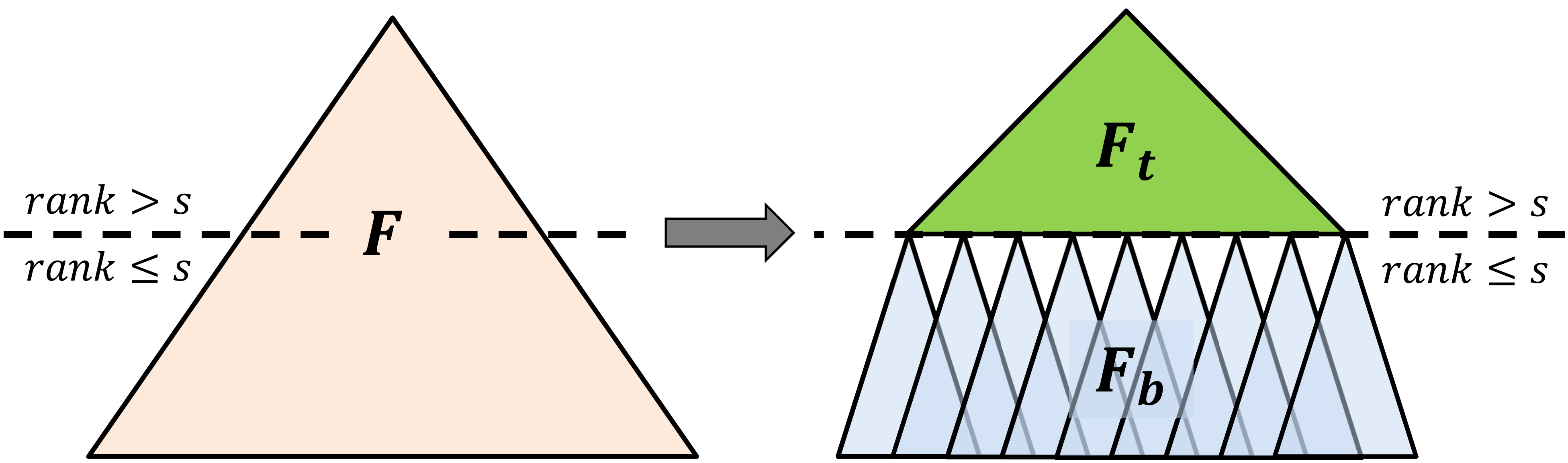
Proof: Let F be the forest, and C be the sequence of COMPRESS operations performed on F.

Let $T(F, C)$ be the number of parent pointer assignments by C in F.

Let s be an arbitrary rank. We partition F into two subforests:

F_b containing all nodes with rank $\leq s$, and

F_t containing all nodes with rank $> s$.



Bound 1

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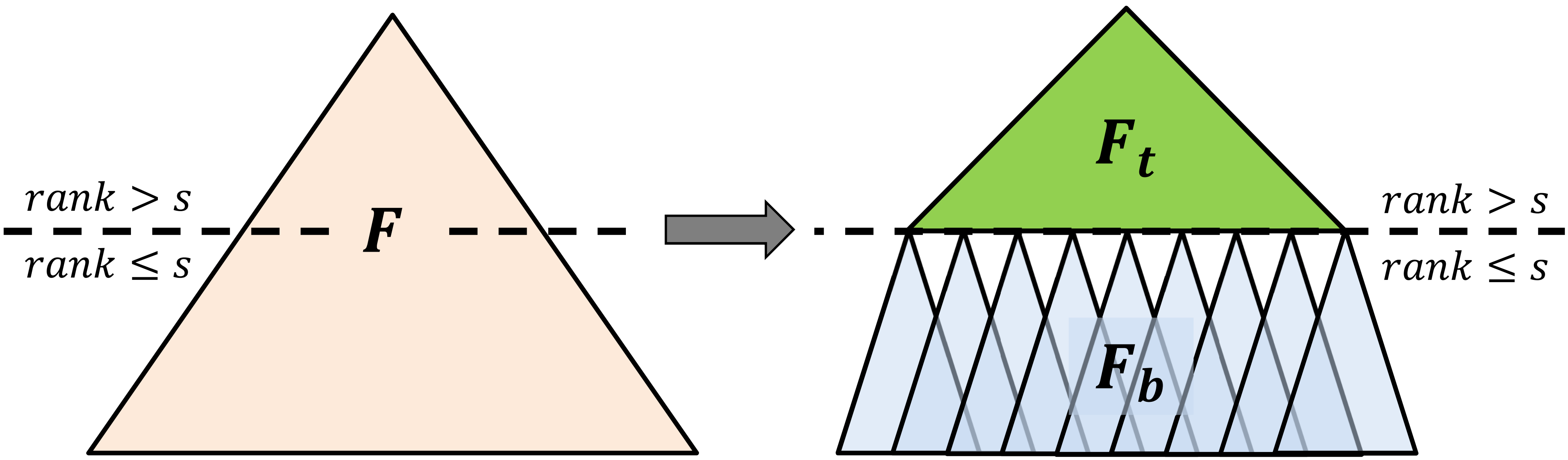


Let n_t = #nodes in F_t, and n_b = #nodes in F_b

Let m_t = #COMPRESS operations with at least one node in F_t, and

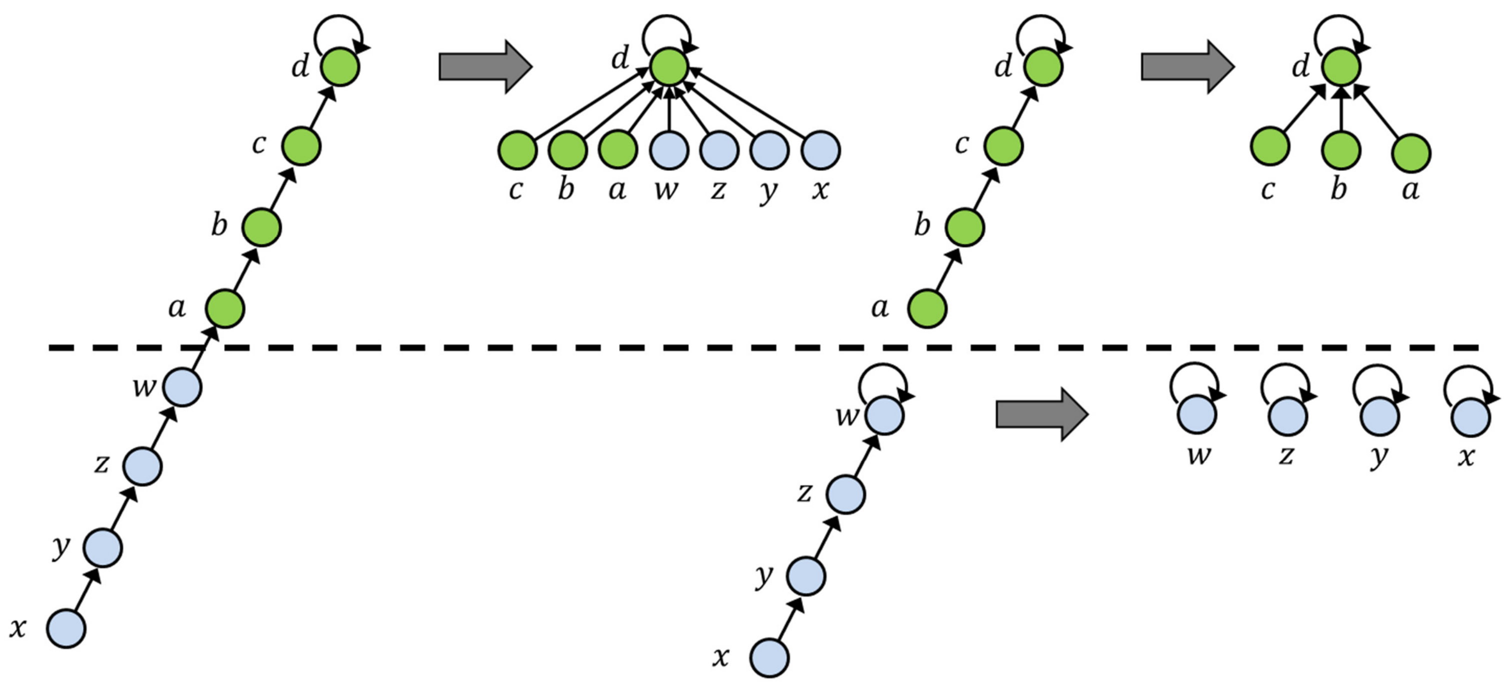
$m_b = m - m_t$

Bound 1

Bound 1: $T(m, n, r) \le m + 2n \log^* r$.

Proof: The sequence C on F can be decomposed into

- a sequence of COMPRESS operations in F_t, and
- a sequence of COMPRESS and SHATTER operations in F_b



Suppose, this decomposition partitions C into two subsequences

- C_t in F_t, and
- C_b in F_b

Bound 1

Bound 1: $T(m, n, r) \leq m + 2n \log^* r.$

Proof: We get the following recurrence:

$$T(F, C) \leq T(F_t, C_t) + T(F_b, C_b) + m_t + n_b$$

Cost on Left Side	Corresponding Cost on Right Side
node $\in F_t$ gets new parent $\in F_t$	$T(F_t, C_t)$
node $\in F_b$ gets new parent $\in F_b$	$T(F_b, C_b)$
node $\in F_b$ gets new parent $\in F_t$ (for the first time)	n_b
node $\in F_b$ gets new parent $\in F_t$ (again)	m_t

Bound 1

Bound 1: $T(m, n, r) \leq m + 2n \log^* r.$

Proof: We get the following recurrence:

$$T(F, C) \leq T(F_t, C_t) + T(F_b, C_b) + m_t + n_b$$

Now $n_t \leq \sum_{i>s} \frac{n}{2^i} = \frac{n}{2^s}$, and $r_t = r - s < r$.

Hence, using bound 0: $T(F_t, C_t) \leq n_t r_t < \frac{nr}{2^s}$

Let $s = \log r$. Then $T(F_t, C_t) < n$.

Hence, $\quad T(F, C) \leq T(F_b, C_b) + m_t + 2n$

$$\Rightarrow T(F, C) - m \leq T(F_b, C_b) - m_b + 2n$$

Bound 1

Bound 1: $T(m,n,r) \leq m + 2n\log^* r.$

Proof:

We got $T(F,C) - m \leq T(F_b, C_b) - m_b + 2n$

Let $T_1(m,n,r) = T(m,n,r) - m$

Then $T_1(m,n,r) \leq T_1(m_b, n_b, r_b) + 2n$

$\Rightarrow T_1(m,n,r) \leq T_1(m,n,\log r) + 2n$

Solving, $T_1(m,n,r) \leq 2n\log^* r$

Hence, $T(m,n,r) \leq m + 2n\log^* r$

Bound 2

Bound 2: $T(m, n, r) \leq 2m + 3n \log^{**} r$.

Proof: Similar to the proof of bound 1.

But we solve $T(F_t, C_t)$ using bound 1, instead of bound 0!

We fix $s = \log^* r$ (instead of $\log r$ for bound 1)

Then using bound 1: $T(F_t, C_t) \leq m_t + 2n_t \log^* r_t$

$$\leq m_t + 2 \frac{n}{2^{\log^* r}} \log^* r$$

$$\leq m_t + 2n$$

Then from $T(F, C) \leq T(F_t, C_t) + T(F_b, C_b) + m_t + n_b$, we get

$$T(F, C) \leq T(F_b, C_b) + 2m_t + 3n_b$$

Bound 2

Bound 2: $T(m,n,r) \leq 2m + 3n\log^{**} r$.

Proof: Our recurrence:

$$T(F,C) \leq T(F_b, C_b) + 2m_t + 3n_b$$

$$\Rightarrow T(F,C) - 2m \leq T(F_b, C_b) - 2m_b + 3n_b$$

Let $T_2(m,n,r) = T(m,n,r) - 2m$

Then $T_2(m,n,r) \leq T_2(m_b, n_b, r_b) + 3n$

$\Rightarrow T_2(m,n,r) \leq T_2(m,n,\log^* r) + 3n$

Solving, $T_2(m,n,r) \leq 3n\log^{**} r$

Hence, $T(m,n,r) \leq 2m + 3n\log^{**} r$

Bound k

Bound k: $T(m,n,r) \leq km + (k+1)n \log^{\overbrace{*\cdots*}^{k}} r$.

Observation: As we increase k:

- the dependency on m increases
- the dependency on r decreases

When $k = \alpha(r)$, we have $\log^{\overbrace{*\cdots*}^{k}} r \leq 3$!

Bound α: $T(m,n,r) \leq m\alpha(r) + 3(\alpha(r)+1)n$.

The α Bound

Bound $\boldsymbol{\alpha}$: $T(m,n,r) \leq m\alpha(r) + 3(\alpha(r) + 1)n$.

Observing that $r < n$, we have:

Bound $\boldsymbol{\alpha}$: $T(m,n,r) \leq (m + 3n)\alpha(n) + 3n = \mathrm{O}\big((m + n)\alpha(n)\big)$.

Assuming $m \geq n$, we have:

Bound $\boldsymbol{\alpha}$: $T(m,n,r) = \mathrm{O}\big(m\alpha(n)\big)$.

So, amortized complexity of each operation is only $\mathrm{O}\big(\alpha(n)\big)$!