Resolution Lemma

Let S be a CNF formula in clause form.

Let R be a resolvent of two clauses C_1 & C_2 in S.

Then $S \equiv S \cup \{R\}$

Proof Sketch

1) $\cancel{S \cup \{R\} \Rightarrow S}$

Show $S \cup \{R\} \models S$

Proof Trivial

2) Show $S \models S \cup \{R\}$

Pick a interpretation that is model for S.

Since R is resolvent of C_1 & C_2

$$R = (C_1 - \{L\}) \cup (C_2 - \{\bar{L}\}).$$

In interpretation I either

L is true or L is false.

Suppose L is true.

Since I is model for S

therefore I is model for $C_2 - \{\bar{L}\}$

Suppose L is false then

I is a model for $C_1 - \{L\}$

Therefore I is a model for

$S \cup \{R\}$.

Proof sketch of Completeness of Resolution for Propositional Logic

$$Res(S) = S \cup \{ R \mid R \text{ is a resolvent of two clauses in } S \}$$

Define: $Res^0(S) = S$

$$Res^{n+1}(S) = Res(Res^n(S))$$

$$\underbrace{Res^*(S)}_{\text{Closure.}} = \bigcup_{n \geq 0} Res^n(C)$$

<u>Theorem</u> If S is unsatisfiable then

$$\Box \in Res^*(S).$$

<u>Proof (Sketch / Outline)</u>

If $n = 0$ then $S = \{\Box\}$. So $\Box \in Res^*(S)$.

Assume theorem holds for all sentences made up of n propositional variables.
(It is assumed that S is in clause form. — i.e. Conjunct of disjuncts.)

Let the n+1 propositional variables be

$P_1 \; P_2 \; - - - \; P_n, \; P_{n+1}$.

Now we show the theorem holds for n+1 vars.

Now create two sentences S_F & S_T.

S_F is created thus:

From any ~~dr~~ disjunct containing

P_{n+1} delete it.

From any disjunct containing $\neg P_{n+1}$

~~elim~~ delete the entire disjunct.

For S_T do the exact dual.

Claim: S_F & S_T are both

unsatisfiable (why?)

$\therefore$ by induction hypothesis

$\square \in Res^*(S_F)$ &

$\square \in Res^*(S_T)$.

So there must be a proof to derive

$\square$ in both S_F & S_T.

Let the proof for unsatisfiability of S_F ~~&~~ S_T be:

$$\begin{array}{ccc} S_F^0 & \qquad & S_T^0 \\ | & & | \\ S_F^1 & & S_T^1 \\ | & & \vdots \\ \vdots & & S_T^{k-1} \\ S_F^{m-1} & & | \\ | & & \square \\ \square & & \end{array}$$

Now put back ~~in~~ ~~di~~ P_{n+1} in proof of S_F in ~~the~~ disjuncts (or clauses) used in the proof from which P_{n+1} was deleted. Analogously for S_T i.e. put back ~~P_{n+1}~~ $\neg P_{n+1}$.

Go through the same steps & you will get the same proof paths for S_F & S_T.

S_F^0	S_T^0
$\vert$	$\vert$
S_F^1	S_T^1
$\vert$	$\vdots$
$S_F^2 \cup \cancel{\{p_{n+1}\}} \leftarrow p_{n+1}$	$S_T^i \leftarrow \neg p_{n+1}$
$\vdots$	$\vdots$
$\{p_{n+1}\}$	$\{\neg p_{n+1}\}$

$$\square$$