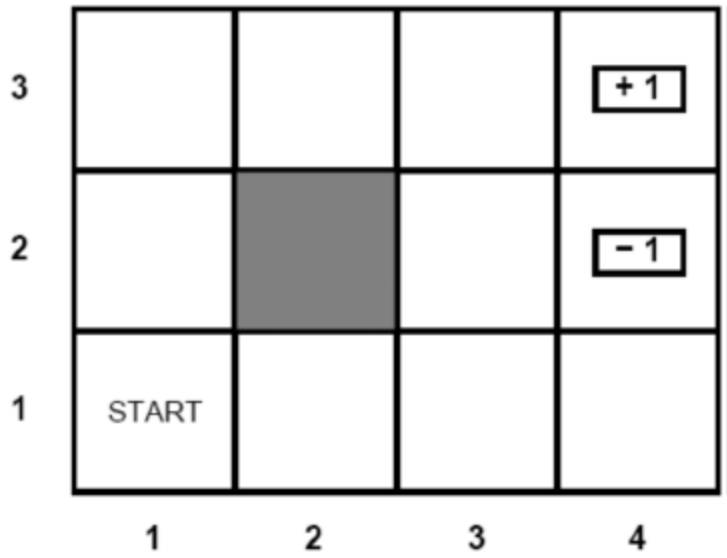
Decision Processes: General Description

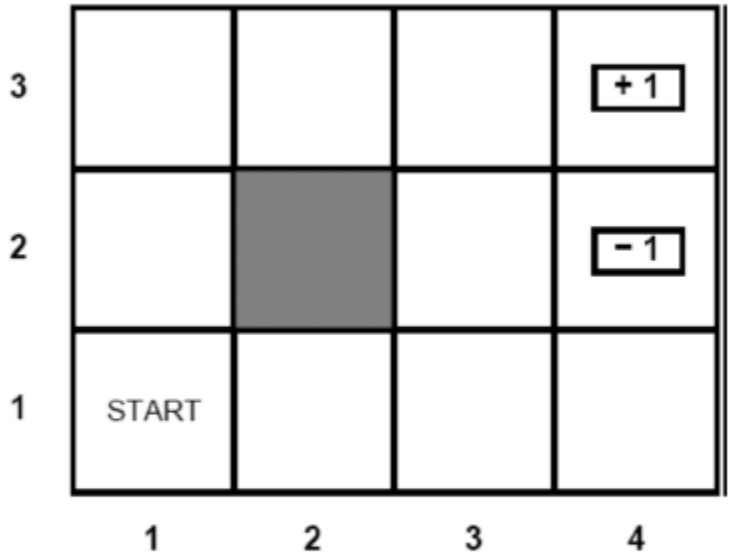
- Decide what action to take next given that your action will affect what happens in the future
- Real world examples:
  - Robot path planning
  - Elevator scheduling
  - Travel route planning
  - Aircraft navigation
  - Manufacturing processes
  - Network switching and routing

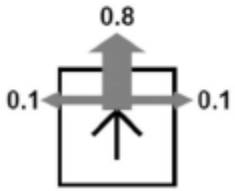
Sequential Decisions



- Assume a fully observable, deterministic environment, like the example shown here
- Each grid cell is a state
- The goal state is marked +1
- At each time step, agent must move Up, Right, Down, or Left
- How do you get from start to the goal state?

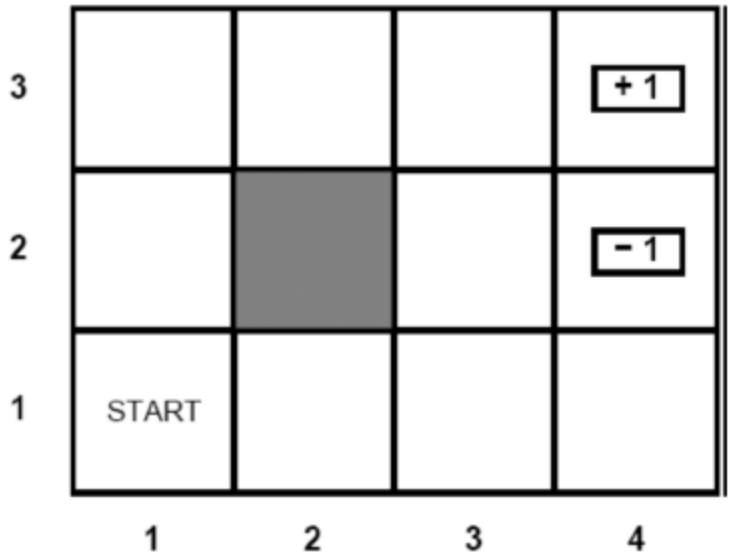
Sequential Decisions





- Suppose the environment is now stochastic
- With 0.8 probability you go in the direction you intend
- With 0.2 probability you move at right angles to the intended direction (0.1 in either direction – if you hit the wall you stay put)
- What is the optimal solution now?

Sequential Decisions



- *Up, Up, Right, Right, Right* reaches the goal state with probability 0.8^5=032768
- But in this stochastic world, going *Up, Up, Right, Right, Right* might end up with you actually going *Right, Right, Up, Up, Right* with probability $(0.1^4)(0.8)$=0.00008
- However, you might end up in the -1 state accidentally

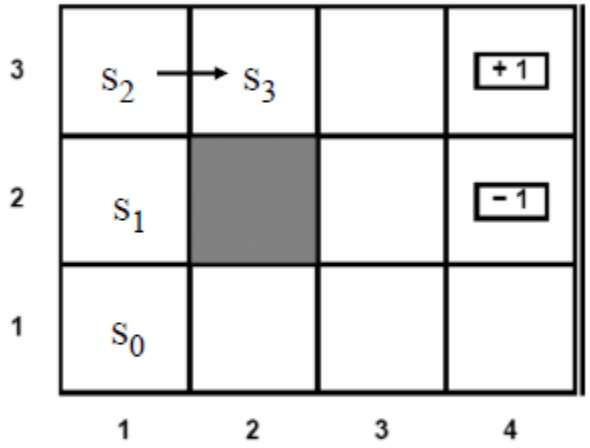
Transition Model

- Transition model: specifies the **outcome probabilities for each action in each possible state**
- T(*s*, *a*, *s'*) = probability of going to state *s'* if you are in state *s* and do action *a*
- The transitions are **Markovian** ie. the probability of reaching state s' from s depends only on s and not on the history of earlier states (aka The Markov Property)
- Mathematically:

  Suppose you visited the following states in chronological order: $s_0, \ldots, s_t$

  $P(s_{t+1} \mid a, s_0, \ldots, s_t) = P(s_{t+1} \mid a, s_t)$

Markov Property Example



Suppose

$s_0 = (1,1)$, $s_1 = (1,2)$, $s_2 = (1,3)$

If I go right from state s_2, the probability of going to s_3 only depends on the fact that I am at state s_2 and not the entire state history $\{s_0, s_1, s_2\}$

The Reward Function

- At each state *s*, the agent receives a reward R(s) which may be positive or negative (but must be bounded)
- For now, we'll define the reward of a sequence of states as the sum of the rewards received

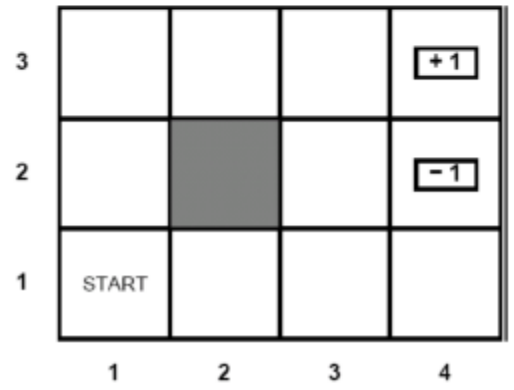
Reward Function Example

R(4,3) = +1 (Agent wants to get here)

R(4,2) = -1 (Agent wants to avoid this)

R(s) = -0.04 (for all other states)

$U(s_1, \ldots, s_n) = R(s_1) + \ldots + R(s_n)$

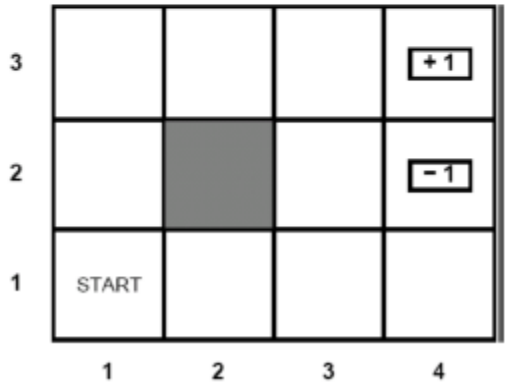


If the states an agent goes through are Up, Up, Right, Right, Right, the utility of this state sequence is:

-0.04-0.04-0.04-0.04-0.04+1

Reward Function Example

If there's no uncertainty, then the agent would find the sequence of actions that maximizes the sum of the rewards of the visited states



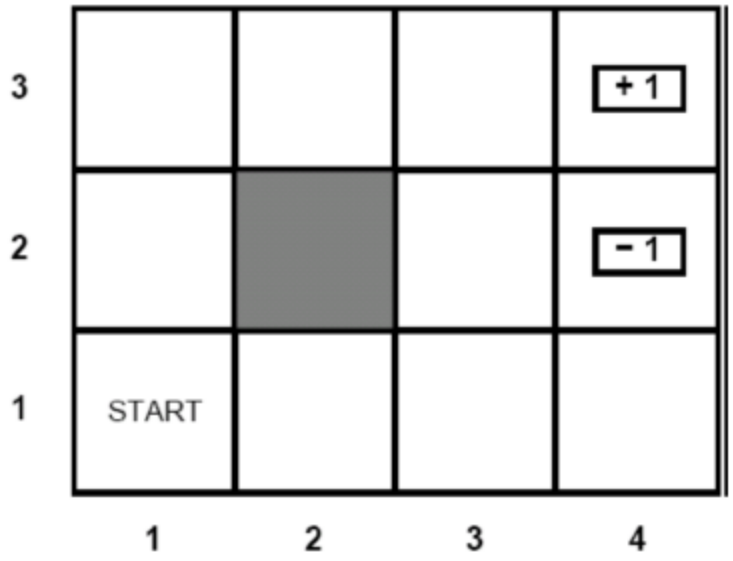
Markov Decision Process

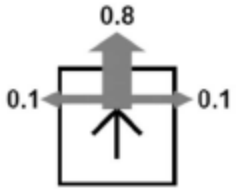
A sequential decision problem with a fully observable environment with a *Markovian transition model* and *additive rewards* is modeled by a Markov Decision Process (MDP)

An MDP has the following components:

1. A (finite) set of states S
2. A (finite) set of actions A
3. Transition Model: $T(s, a, s') = P(s' \mid a, s)$
4. Reward Function: R(s)

Example





- State space:
(1,1), (1,2) ……(4,3)
- Transition probability:

With 0.2 probability you move at right angles to the intended direction (0.1 in either direction – if you hit the wall you stay put)

- Reward:

R(4,3)=1, R(4,2)=-1, everywhere else: -0.04

Optimal Policy

- Many different policies exist for an MDP
- Some are better than others. The "best" one is called the optimal policy π^* (we will define best more precisely in later slides)
- Note: every time we start at the initial state and execute a policy, we get a different state sequence (because the environment is stochastic)
- We get a different utility each time we execute a policy
- Need to measure the expected utility ie. the average of the utilities of the possible state sequences generated by the policy

Utility of a State Sequence

There are 2 ways to assign utilities to sequences

1. Additive rewards: The utility of a state sequence is:

   $U(s_0, s_1, s_2, \ldots) = R(s_0) + R(s_1) + R(s_2) + \ldots$

2. Discounted rewards: The utility of a state sequence is:

   $U(s_0, s_1, s_2, \ldots) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$

   Where $0 \leq \gamma \leq 1$ is the discount factor

The Discount Factor

- Describes preference for current rewards over future rewards
- Compensates for uncertainty in available time (models mortality)
- Eg. Being promised $10000 next year is only worth 90% of being promised $10000 now
- γ near 0 means future rewards don't mean anything
- $\gamma = 1$ makes discounted rewards equivalent to additive rewards

Utilities

We assume infinite horizons. This means that if the agent doesn't get to a terminal state, then environmental histories are infinite, and utilities with additive rewards are infinite. How do we deal with this?

Discounted rewards makes utility finite.

Assuming largest possible reward is R_{max} and $\gamma < 1$,

$$U(s_0, s_1, s_2, \ldots) = \sum_{t=0}^{\infty} \gamma^t R(s_t)$$

$$\leq \sum_{t=0}^{\infty} \gamma^t R_{\max} = \frac{R_{\max}}{(1-\gamma)}$$

Computing Optimal Policies

- A policy π generates a sequence of states
- But the world is stochastic, so a policy π has a range of possible state sequences, each of which has some probability of occurring
- The value of a policy is the *expected sum of discounted rewards* obtained by following this policy

The Optimal Policy

- Given a policy π, we write the expected sum of discounted rewards obtained as:

$$E\left[\sum_{t=0}^{\infty}\gamma^t R(s_t)\,|\,\pi\right]$$

- An optimal policy π* is the policy that maximizes the expected sum above

$$\pi^* = \arg\max_{\pi} E\left[\sum_{t=0}^{\infty}\gamma^t R(s_t)\,|\,\pi\right]$$

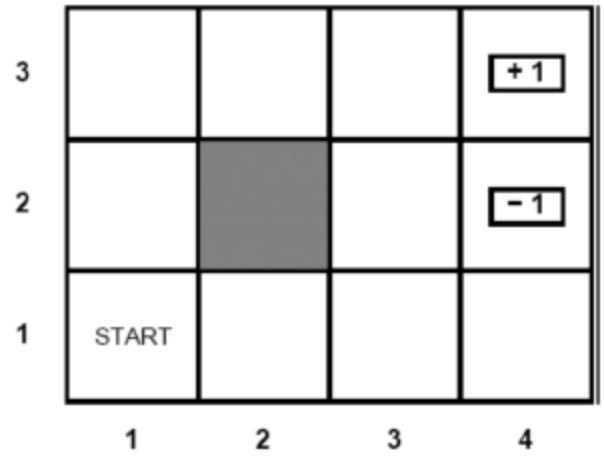
The Optimal Policy

- For every MDP, there exists an optimal policy
- There is no better option (in terms of expected sum of rewards) than to follow this policy
- How do you calculate this optimal policy? Can't evaluate all policies…too many of them
- Instead, we calculate the utility of the states
- Then use the state utilities to select an optimal action in each state

Rewards vs Utilities

- What's the difference between R(s) the reward for a state and U(s) the utility of a state?
  - R(s) – the short term reward for being in s
  - U(s) – The long-term total expected reward for the sequence of states starting at s (not just the reward for state s)

Utilities in the Maze Example

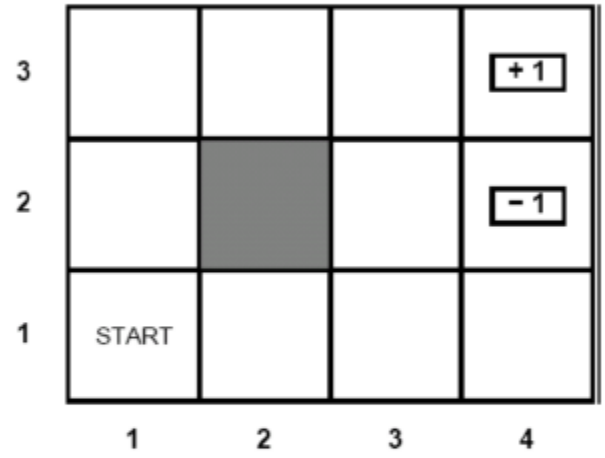


Start at state (1,1). Let's suppose we choose the action Up.

$U(1,1) = R(1,1) + \ldots$

Reward for current state

Utilities in the Maze Example



Start at state (1,1). Let's choose the action Up.

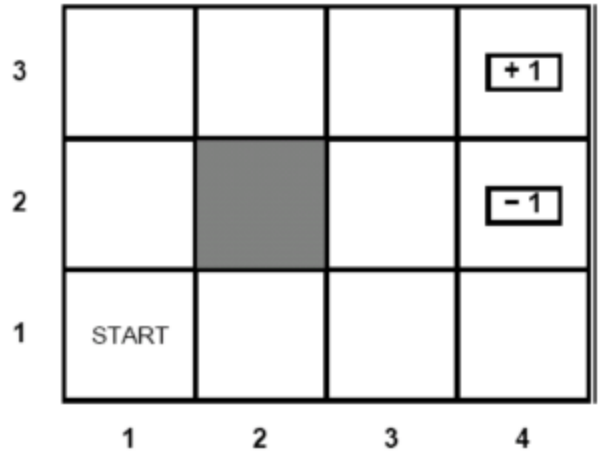
$$U(1,1) = R(1,1) + 0.8*U(1,2) + 0.1*U(2,1) + 0.1*U(1,1)$$

Prob of moving right

Prob of moving up

Prob of moving left (into the wall) and staying put

Utilities in the Maze Example



Now let's throw in the discounting factor

$$U(1,1) = R(1,1) + \gamma*0.8*U(1,2) + \gamma*0.1*U(2,1) + \gamma*0.1*U(1,1)$$

The Utility of a State

If we choose action *a* at state s, expected future rewards (discounted) are:

$$U_a(s) = R(s) + \gamma \sum_{s'} T(s,a,s')U(s')$$

Expected sum of total discounted rewards starting at state s by taking action a

Reward at current state s

Probability of moving from state s to state s' by doing action a

Expected sum of future discounted rewards starting at state s'

The Utility of a State

- In the previous example, we define the utility assuming that action a is taken at state s
- What we want is the utility of a state s assuming that we can choose the optimal action to take at state s.
- We modify the previous formula slightly by adding a max term over actions.

$$U_a(s) = R(s) + \gamma \sum_{s'} T(s,a,s')U(s')$$

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s')U(s')$$

The Utility of a State

The utility of a state is the immediate reward for that state plus the expected discounted utility of the next state, assuming the agent chooses the optimal action

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$$

This is called the Bellman Equation

The Optimal Policy

- Selection of the action π*(s) = a which maximizes the expected utility U(s)
- Intuitively, π* gives us the best action we can take from any state to maximize our future discounted rewards

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')U(s')$$

The Bellman Equation

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s')U(s')$$

- The Bellman equation gives the utility of a state
- If there are n states, there are n Bellman Equations to solve
- This is a system of simultaneous equations
- But the equations are nonlinear because of the max operator

Iterative Solution

Define $U_1(s)$ to be the utility if the agent is at state s and lives for 1 time step

$$U_1(s) = R(s)$$

Calculate this for all states s

Define $U_2(s)$ to be the utility if the agent is at state s and lives for 2 time steps

$$U_2(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s') U_1(s')$$

This has already been calculated above

The Bellman Update

More generally, we have:

$$U_{i+1}(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s')U_i(s')$$

- This is the maximum possible expected sum of discounted rewards (ie. the utility) if the agent is at state *s* and lives for i+1 time steps
- This equation is called the Bellman Update

The Bellman Update

- As the number of iterations goes to infinity, $U_{i+1}(s)$ converges to an equilibrium value $U^*(s)$.
- The final utility values U*(s) are solutions to the Bellman equations. Even better, they are the unique solutions and the corresponding policy is optimal
- This algorithm is called ***Value-Iteration***
- The optimal policy is given by:

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')U^*(s')$$

The Value Iteration Algorithm

$$U_1(s) = R(s)$$

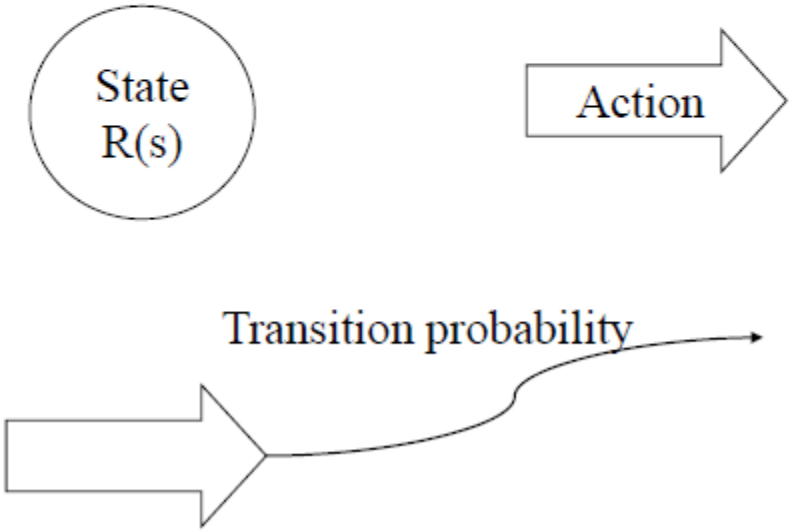
$$U_{i+1}(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s')U_i(s')$$

1. Apply bellman update until it the utility function converges (to $U^*(s)$).
2. The optimal policy is given by:

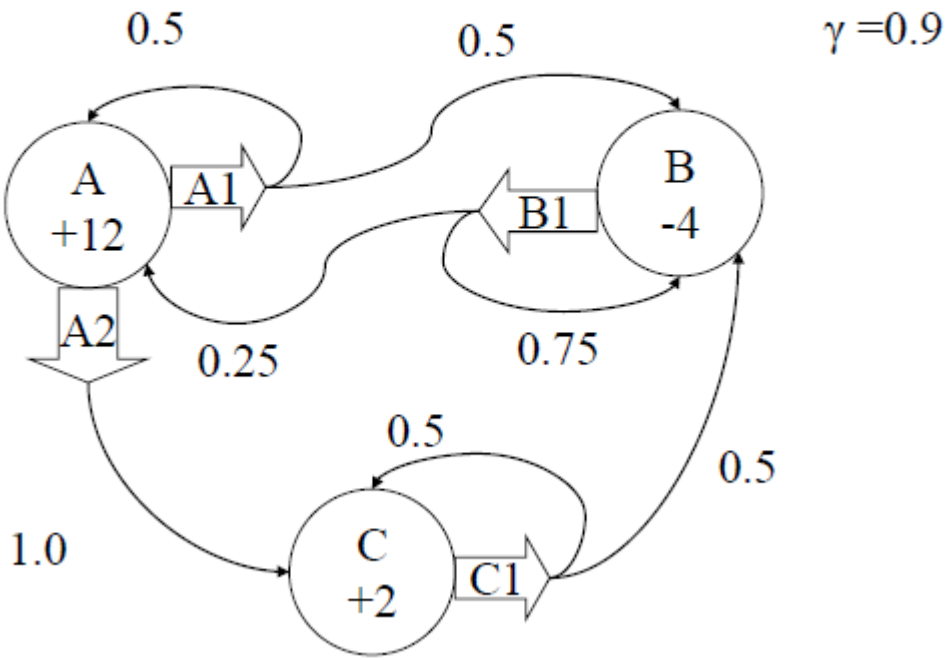
$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')U^*(s')$$

Example

- We will use the following convention when drawing MDPs graphically:



Example



Example

i=1

$U_1(A) = R(A)=12$

$U_1(B) = R(B)=-4$

$U_1(C) = R(C)=2$

Example

$U_1(A)$	$U_1(B)$	$U_1(C)$
12	-4	2

i=2

$U_2(A) = 12 + (0.9) * \max\{(0.5)(12)+(0.5)(-4), (1.0)(2)\}$
$= 12 + (0.9)*\max\{4.0,2.0\} = 12 + 3.6 = 15.6$

$U_2(B) = -4 + (0.9) * \{(0.25)(12)+(0.75)(-4)\} = -4 +$
$(0.9)*0 = -4$

$U_2(C) = 2 + (0.9) * \{(0.5)(2)+(0.5)(-4)\} = 2 + (0.9)*(-1)$
$= 2-0.9 = 1.1$

Example

$U_2(A)$	$U_2(B)$	$U_2(C)$
15.6	-4	1.1

i=3

$U_3(A) = 12 + (0.9) * \max\{(0.5)(15.6)+(0.5)(-4),(1.0)(1.1)\} = 12 + (0.9) * \max\{5.8,1.1\} = 12 + (0.9)(5.8) = 17.22$

$U_3(B) = -4 + (0.9) * \{(0.25)(15.6)+(0.75)(-4)\} = -4 + (0.9)*(3.9-3) = -4 + (0.9)(0.9) = -3.19$

$U_3(C) = 2 + (0.9) * \{(0.5)(1.1)+(0.5)(-4)\} = 2 + (0.9)*\{0.55-2.0\} = 2 + (0.9)(-1.45) = 0.695$

Value-Iteration Termination

When do you stop?

In an iteration over all the states, keep track of the maximum change in utility of any state (call this δ)

When δ is less than some pre-defined threshold, stop

This will give us an approximation to the true utilities, we can act greedily based on the approximated state utilities

Utility of a policy at state s

- $U_\pi(s)$: the utility of policy π at state s
- $U^*(s)$ can be considered as $U_{\pi^*}(s)$ where π^* is an optimal policy
- Given a fixed policy, can compute its utility at state ***s*** as follows:

$$U_\pi(s) = R(s) + \gamma \sum_{s'} T(s, \pi(s), s') \cdot U_\pi(s')$$

Note the difference from:

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$$

Notice that there is no max operator, so the equations are linear, which can be solved directly.

Evaluating a Policy

Once we compute the utilities of the states under policy π, we can easily compute an improved policy π':

$$\pi'(s) = \arg\max_a \sum_{s'} T(s,a,s')U_\pi(s')$$

Iteratively apply this idea, we will eventually converge to the optimal policy

Policy Iteration

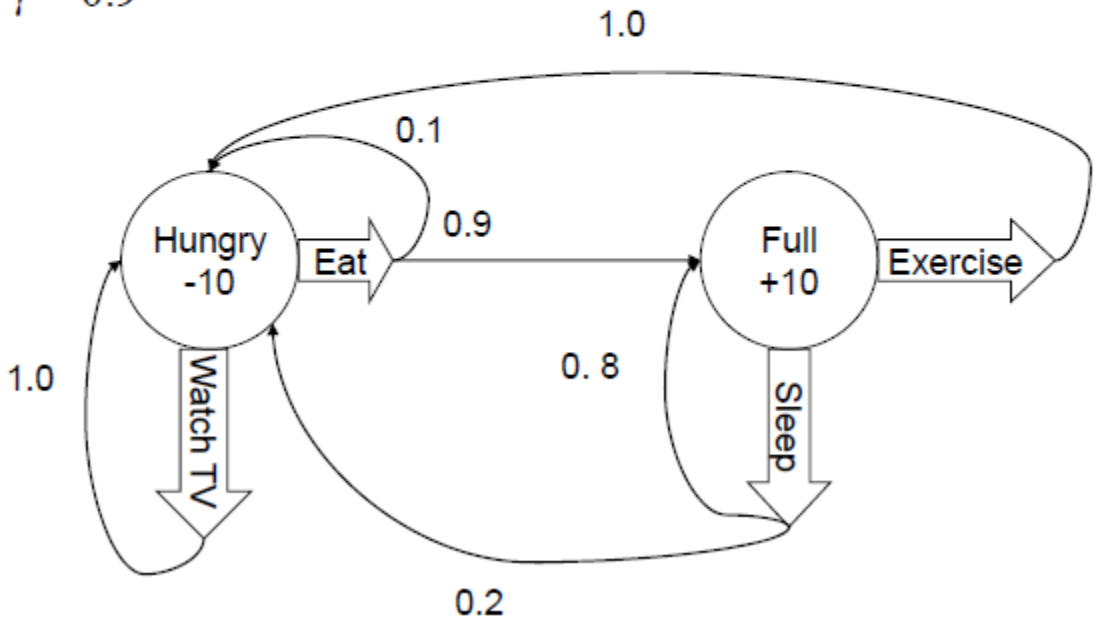
- Start with a randomly chosen initial policy π_0
- Iterate until no change in utilities:
  1. Policy evaluation: given a policy π_i, calculate the utility $U_{\pi i}(s)$ of every state s using policy π_i by solving:
     $$U_{\pi_i}(s) = R(s) + \gamma \sum_{s'} T(s, \pi_i(s), s') U_{\pi_i}(s')$$
  2. Policy improvement: calculate the new policy π_{i+1} based on $U_i(s)$ ie.
     $$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') U_{\pi_i}(s')$$

Policy iteration comments

- In each step of policy iteration:
  - **Policy evaluation** involves solving a set of linear equations
  - **Policy improvement**: straightforward
- Each step of policy iteration is guaranteed to strictly improve the policy at some state when improvement is possible
- Converge to optimal policy
- Gives exact value of optimal policy

Policy Iteration Example

Do one iteration of policy iteration on the MDP below. Assume an initial policy of π_1(Hungry) = Eat and π_1(Full) = Sleep.
Let $\gamma = 0.9$



Policy Iteration Example

Policy Evaluation Phase

Use initial policy for Hungry: $\pi_1(\text{Hungry}) = \text{Eat}$

$U_1(\text{Hungry}) = -10 + (0.9)[(0.1)U_1(\text{Hungry})+(0.9)U_1(\text{Full})]$

$\Rightarrow U_1(\text{Hungry}) = -10 + (0.09)U_1(\text{Hungry})+(0.81)U_1(\text{Full})$

$\Rightarrow (0.91)U_1(\text{Hungry})-(0.81)U_1(\text{Full}) = -10$

Use initial policy for Full: $\pi_1(\text{Full}) = \text{Sleep}$.

$U_1(\text{Full}) = 10 + (0.9)[(0.8)U_1(\text{Full}) + (0.2)U_1(\text{Hungry})]$

$\Rightarrow U_1(\text{Full}) = 10 + (0.72)U_1(\text{Full}) + (0.18)U_1(\text{Hungry})]$

$\Rightarrow (0.28)U_1(\text{Full}) - (0.18)U_1(\text{Hungry}) = 10$

Policy Iteration Example

$(0.91)U_1(\text{Hungry})-(0.81)U_1(\text{Full}) = -10$ ....(Equation 1)

$(0.28)U_1(\text{Full}) - (0.18)U_1(\text{Hungry})=10$ ...(Equation 2)

} Solve for $U_1(\text{Hungry})$ and $U_1(\text{Full})$

From Equation 1:

$(0.91)U_1(\text{Hungry}) = -10+(0.81)U_1(\text{Full})$

$\Rightarrow U_1(\text{Hungry}) = (-10/0.91)+(0.81/0.91)U_1(\text{Full})$

$\Rightarrow U_1(\text{Hungry})=-10.9+(0.89)U_1(\text{Full})$

Policy Iteration Example

$(0.91)U_1(\text{Hungry})-(0.81)U_1(\text{Full}) = -10$ ....(Equation 1)

$(0.28)U_1(\text{Full}) - (0.18)U_1(\text{Hungry})=10$ ...(Equation 2)

} Solve for $U_1(\text{Hungry})$ and $U_1(\text{Full})$

Substitute $U_1(\text{Hungry})=-10.9+(0.89)U_1(\text{Full})$ into Equation 2

$(0.28)U_1(\text{Full}) - (0.18)[-10.9+(0.89)U_1(\text{Full})]=10$

$\Rightarrow (0.28)U_1(\text{Full}) + 1.96-(0.16)U_1(\text{Full})=10$

$\Rightarrow (0.12)U_1(\text{Full})=8.04$

$\Rightarrow U_1(\text{Full})=67$

$\Rightarrow U_1(\text{Hungry})=-10.9+(0.89)(67)=-10.9+59.63=48.7$

Policy Iteration Example

$$\pi_2(\text{Hungry})$$

$$= \underset{\{\text{Eat, WatchTV}\}}{\text{argmax}} \begin{cases} \text{T(Hungry, Eat, Full)U}_1\text{(Full)} + & \\ \quad \text{T(Hungry, Eat, Hungry)U}_1\text{(Hungry)} & \text{[Eat]} \\ \text{T(Hungry, WatchTV, Hungry)U}_1\text{(Hungry)} & \text{[WatchTV]} \end{cases}$$

$$= \underset{\{\text{Eat, WatchTV}\}}{\text{argmax}} \begin{cases} (0.9)\text{U1(Full)} + (0.1)\text{U1(Hungry)} & \text{[Eat]} \\ (1.0)\text{U1(Hungry)} & \text{[WatchTV]} \end{cases}$$

$$= \underset{\{\text{Eat, WatchTV}\}}{\text{argmax}} \begin{cases} (0.9)(67) + (0.1)(48.7) & \text{[Eat]} \\ (1.0)(48.7) & \text{[WatchTV]} \end{cases}$$

$$= \underset{\{\text{Eat, WatchTV}\}}{\text{argmax}} \begin{cases} 65.2 & \text{[Eat]} \\ 48.7 & \text{[Watch]} \end{cases}$$

$$= \text{Eat}$$

Policy Iteration Example

$$\pi_2(\text{Full})$$

$$= \underset{\{\text{Exercise,Sleep}\}}{\text{argmax}} \begin{Bmatrix} \text{T(Full, Exercise, Hungry)U}_1\text{(Hungry)} & \text{[Exercise]} \\ \text{T(Full, Sleep, Full)U}_1\text{(Full)} + & \\ \quad \text{T(Full, Sleep, Hungry)U}_1\text{(Hungry)} & \text{[Sleep]} \end{Bmatrix}$$

$$= \underset{\{\text{Exercise,Sleep}\}}{\text{argmax}} \begin{Bmatrix} (1.0)\text{U}_1\text{(Hungry)} & \text{[Exercise]} \\ (0.8)\text{U}_1\text{(Full)} + (0.2)\text{U}_1\text{(Hungry)} & \text{[Sleep]} \end{Bmatrix}$$

$$= \underset{\{\text{Exercise,Sleep}\}}{\text{argmax}} \begin{Bmatrix} (1.0)(48.7) & \text{[Exercise]} \\ (0.8)(67) + (0.2)(48.7) & \text{[Sleep]} \end{Bmatrix}$$

$$= \underset{\{\text{Exercise,Sleep}\}}{\text{argmax}} \begin{Bmatrix} 48.7 & \text{[Exercise]} \\ 63.34 & \text{[Sleep]} \end{Bmatrix}$$

$$= \text{Sleep}$$

Policy Iteration Example

- π_2(Hungry) =Eat
- π_2(Full) = Sleep

Comparison

- Which would you prefer, policy or value iteration?
- Depends…
  - If you have lots of actions in each state: policy iteration
  - If you have few actions in each state: value iteration