Physics-Based Modeling Framework For Subdivision Surfaces

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Overview

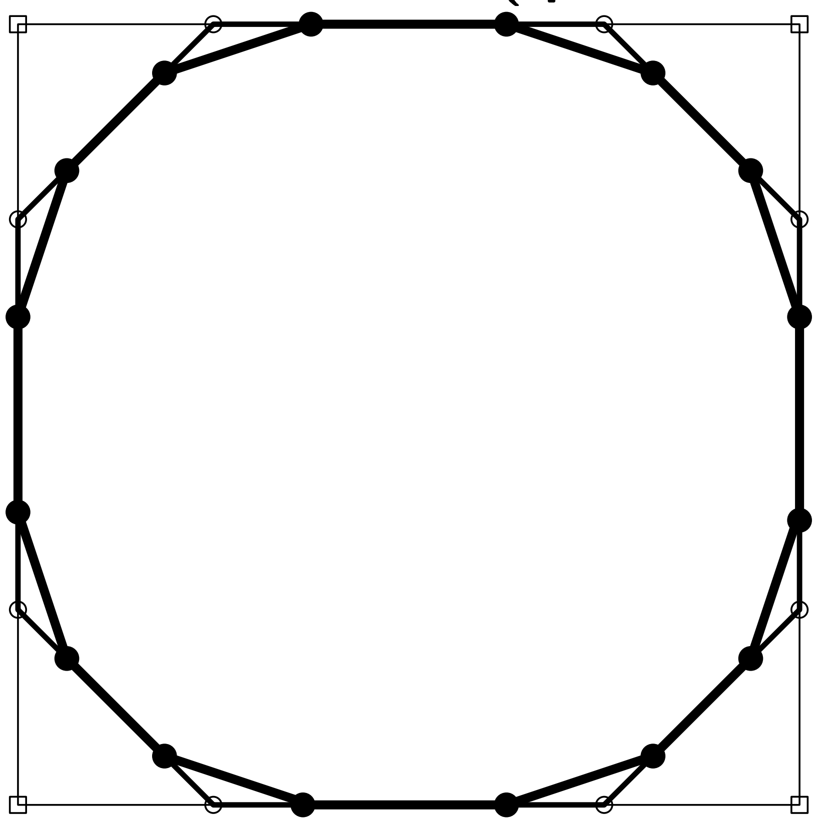
- Recursive subdivision geometry
- Geometric modeling & CAGD based on subdivision
- Physics-based modeling advantages
- Dynamic modeling formulation
- Finite element implementation
- Graphics and modeling applications
- Conclusion & future research

Subdivision Background

- Extremely popular in graphics & CAGD
- Modeling smooth objects of arbitrary topology
- Smooth curve generation
  - Chaikin ('74) algorithm
- Surfaces
  - Doo and Sabin ('78), Catmull and Clark ('78)
  - Loop ('87), Hoppe et al. ('94), Halstead et al. ('93)
  - Taubin ('95), Peters and Reif ('97)
  - Dyn et al. ('90,'93), Zorin ('96), Kobbelt ('95,'96)
  - Kobbelt and Schröder ('97)

Subdivision Curve

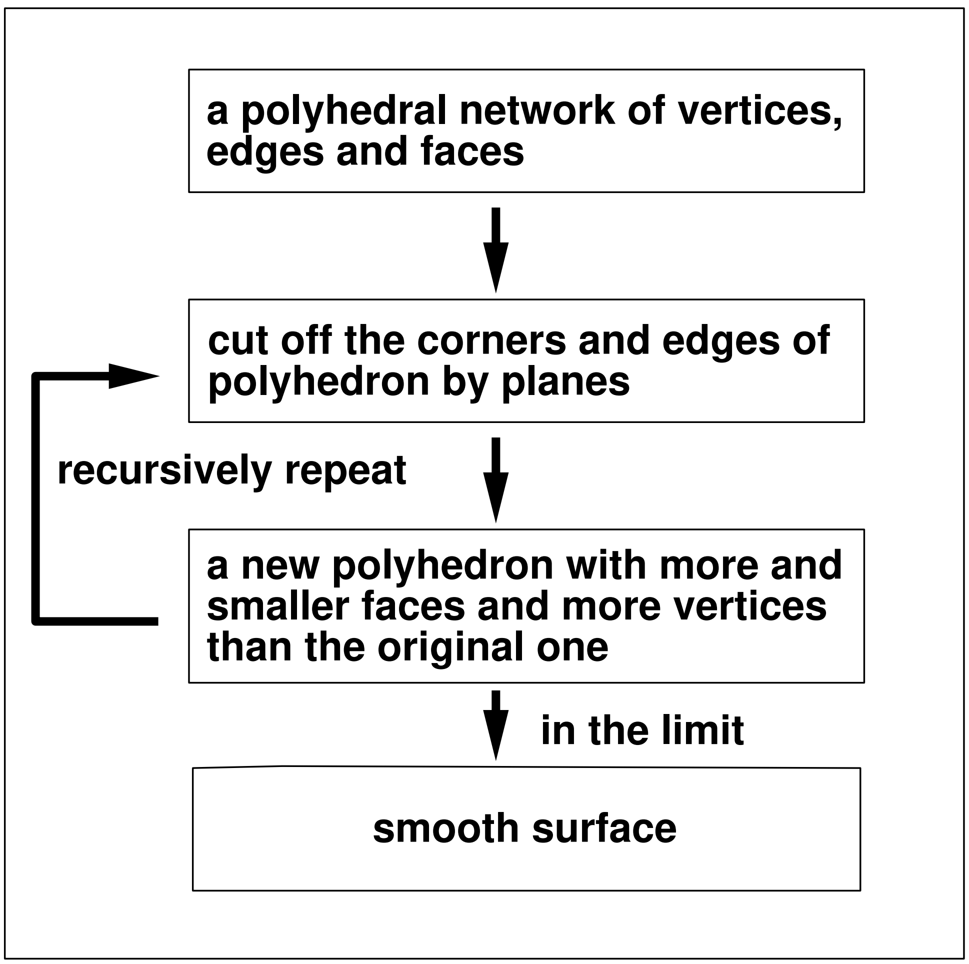
- Chaikin's pioneer work ('74)
- From polygon to smooth curve (quadratic B-spline)



Chaikin Algorithm

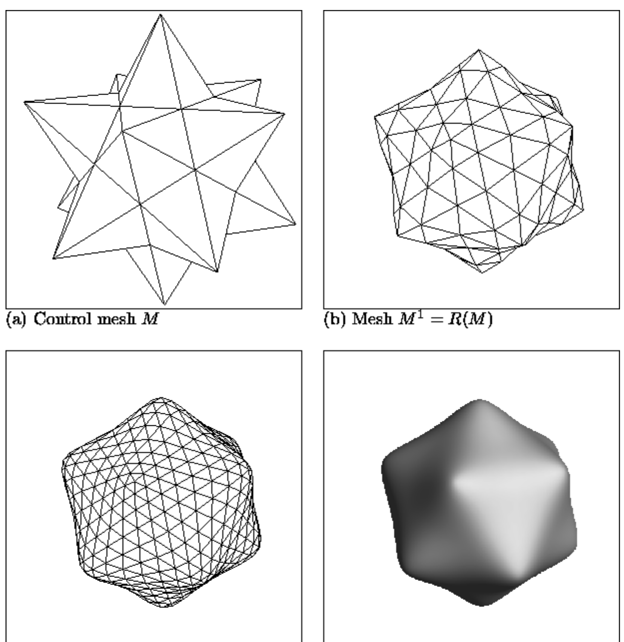
- **Polygon from control vertices:** $\mathbf{p}_0^0, \mathbf{p}_1^0, \ldots, \mathbf{p}_n^0$
- **Subdivision:** $\mathbf{p}_0^k, \mathbf{p}_1^k, \ldots, \mathbf{p}_{2^k n}^k$
- **Rules:** $\mathbf{p}_{2i}^{k+1} = \frac{3}{4}\mathbf{p}_i^k + \frac{1}{4}\mathbf{p}_{i+1}^k, \;\; \mathbf{p}_{2i+1}^{k+1} = \frac{1}{4}\mathbf{p}_i^k + \frac{3}{4}\mathbf{p}_{i+1}^k$
- **Properties:**
  - C^1 **quadratic B-spline curve**
  - **Tangent to each edge at its mid-point**
- **Generalizations**
  - **other rules!**
  - **from polyhedron to smooth surface**

Subdivision Surface



Subdivision Surface

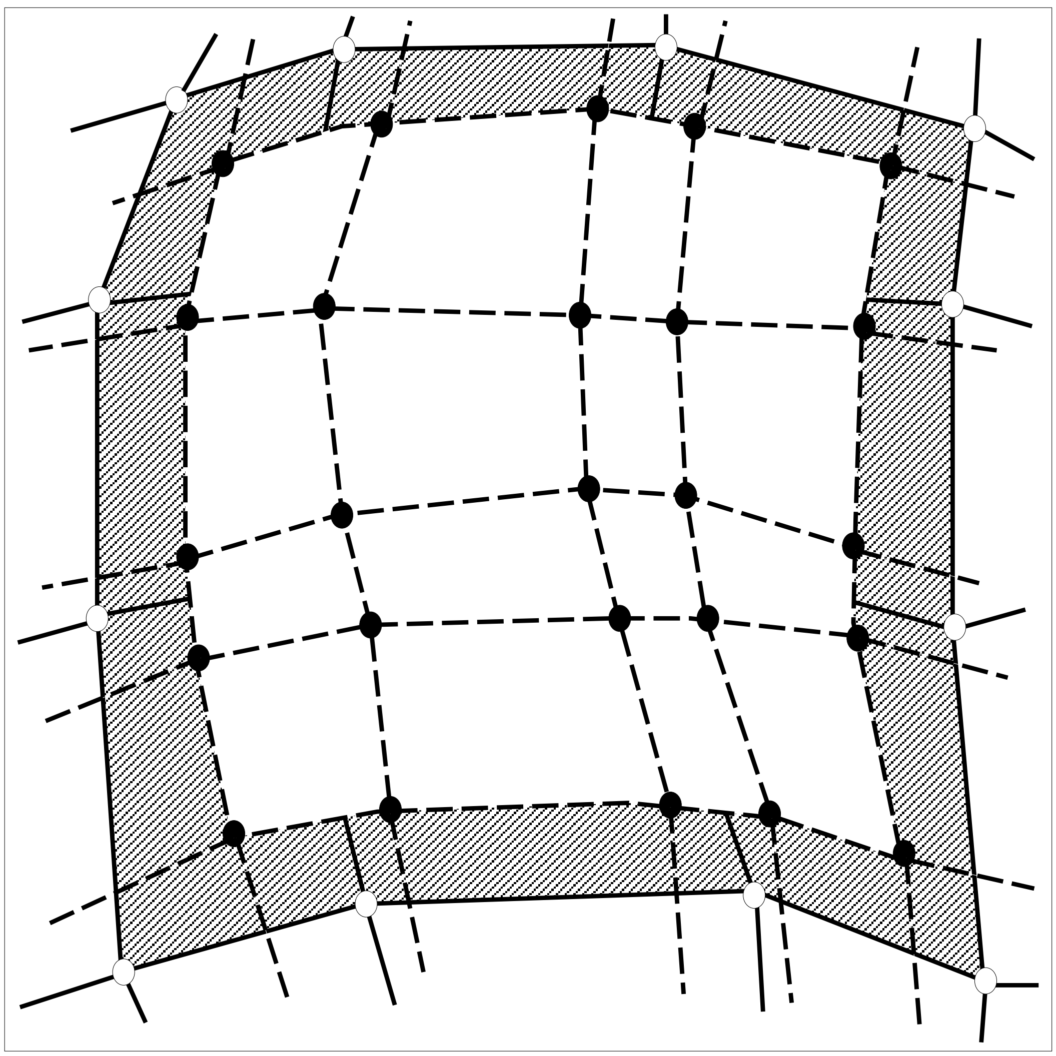
- Models of arbitrary topology
- Initial mesh + recursive subdivision
- Smooth surface as the limit



(a) Control mesh M (b) Mesh $M^1 = R(M)$

(c) Mesh $M^2 = R(R(M))$ (d) Limit surface $S(M) = R^{\infty}(M)$

Subdivision Surface

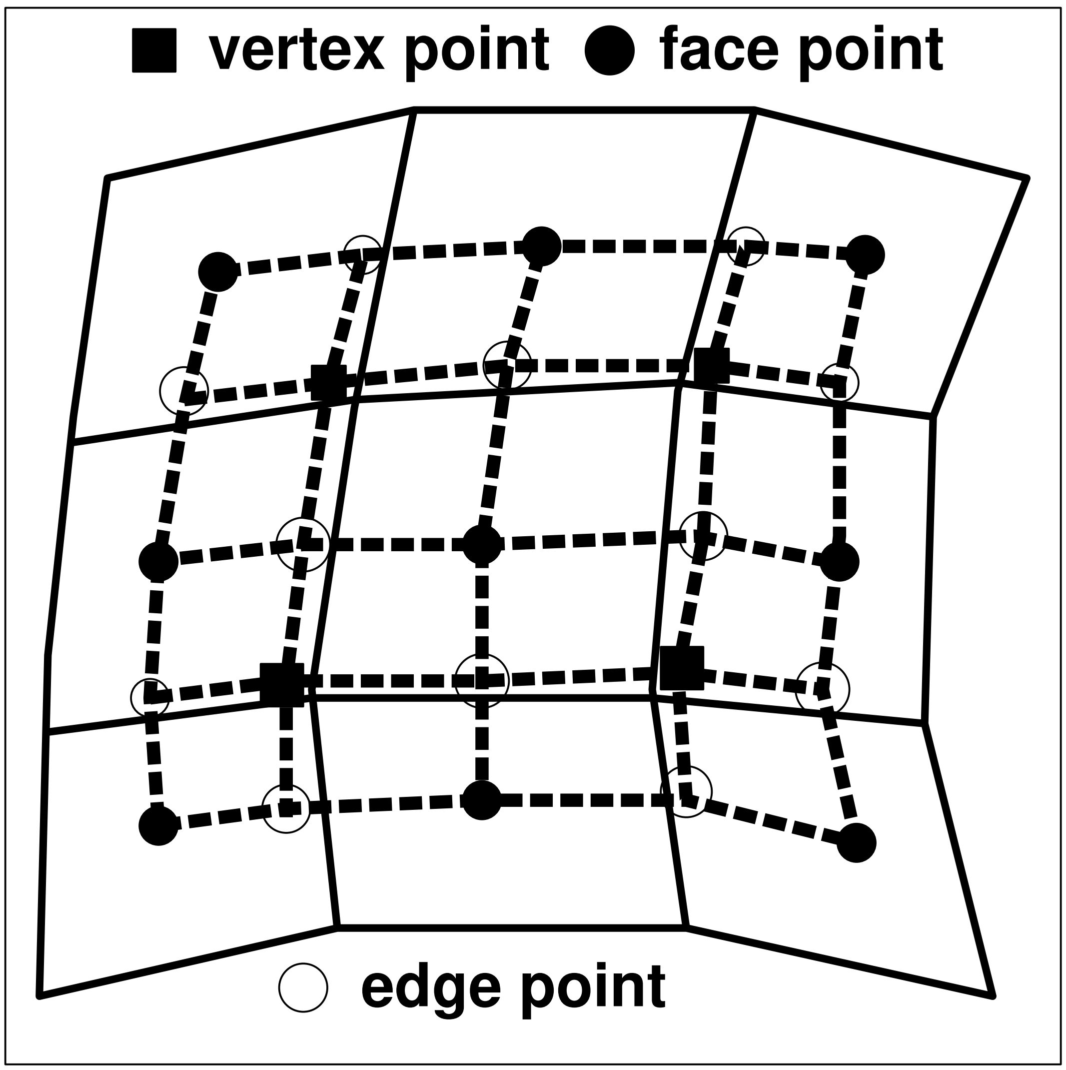


Subdivision Rules

- How NEW vertices are constructed DIFFERS tremendously!

- A large variety of subdivision rules
  - Sabin & Doo approach
  - Butterfly schemes
  - Catmull-Clark surface

Catmull-Clark Surface



Catmull-Clark Rules

- **NEW face point — the average of all OLD points defining the face**
- **NEW edge point — the average of two OLD vertices of the edge & two NEW face points adjacent to the edge**
- **NEW vertex point:** $\frac{F}{n} + \frac{2E}{n} + \frac{(n-3)V}{n}$
  - F: **the average of all NEW face points adjacent to OLD vertex**
  - E: **the average of midpoints of all edges incident on OLD vertex**
  - V: **OLD vertex**
  - n: **vertex degree**

Catmull-Clark Rules

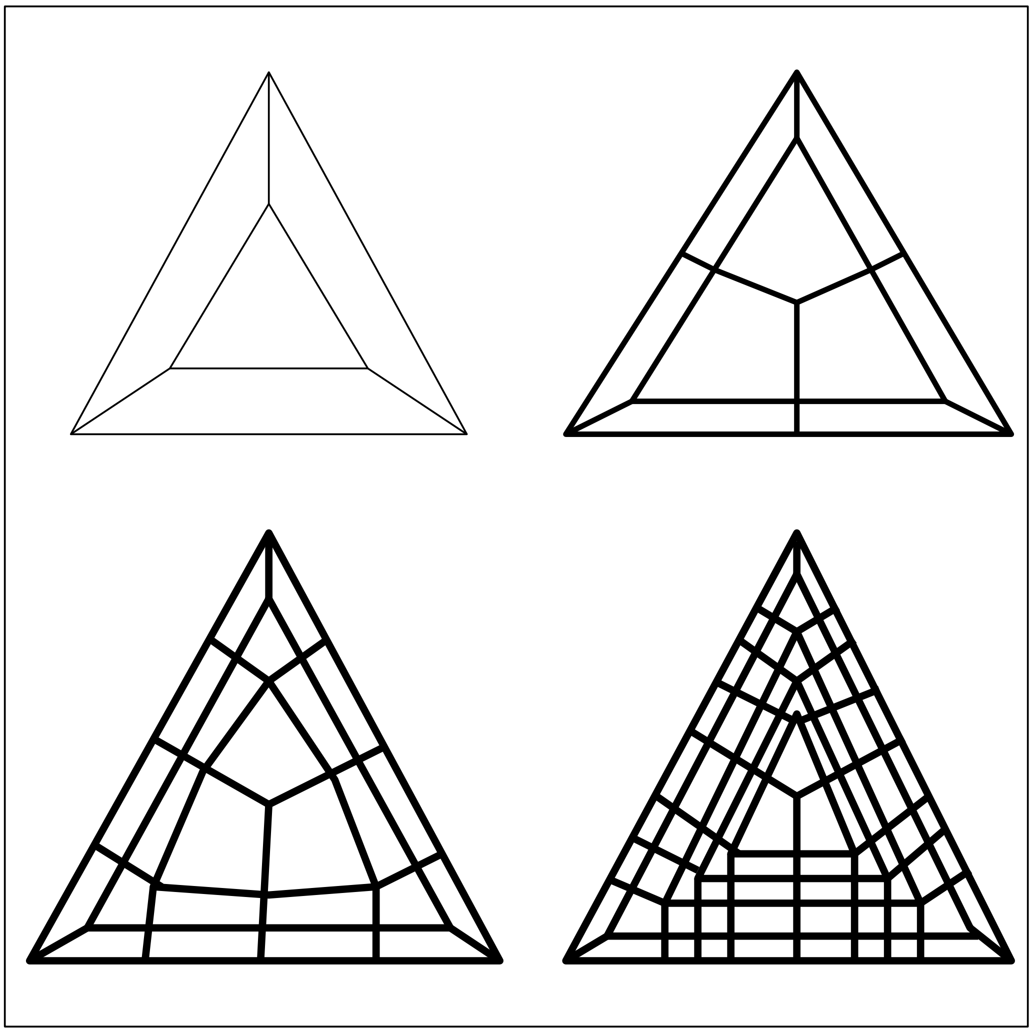
- NEW edge
  - (NEW face point, NEW edge point)
  - (NEW vertex point, NEW edge point)

- NEW face: enclosed by NEW edges

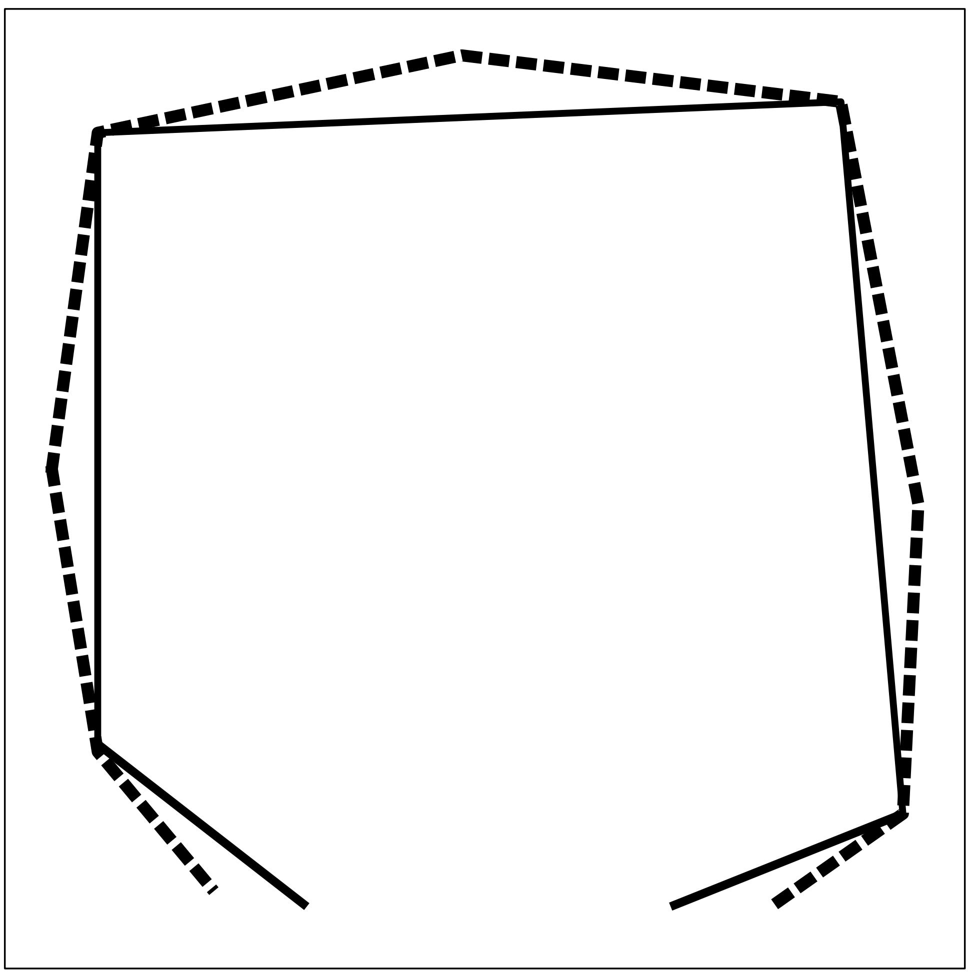
Catmull-Clark Surface

- Geometric objects of arbitrary topology
- Rectangular meshes $\Rightarrow$ B-spline surface
- Non-rectangular meshes $\Rightarrow$ a set of B-spline patches $+$ extraordinary points
- A finite number of extraordinary points
- Curvature-continuous except at extraordinary points
- Tangent-plane-continuous at extraordinary points

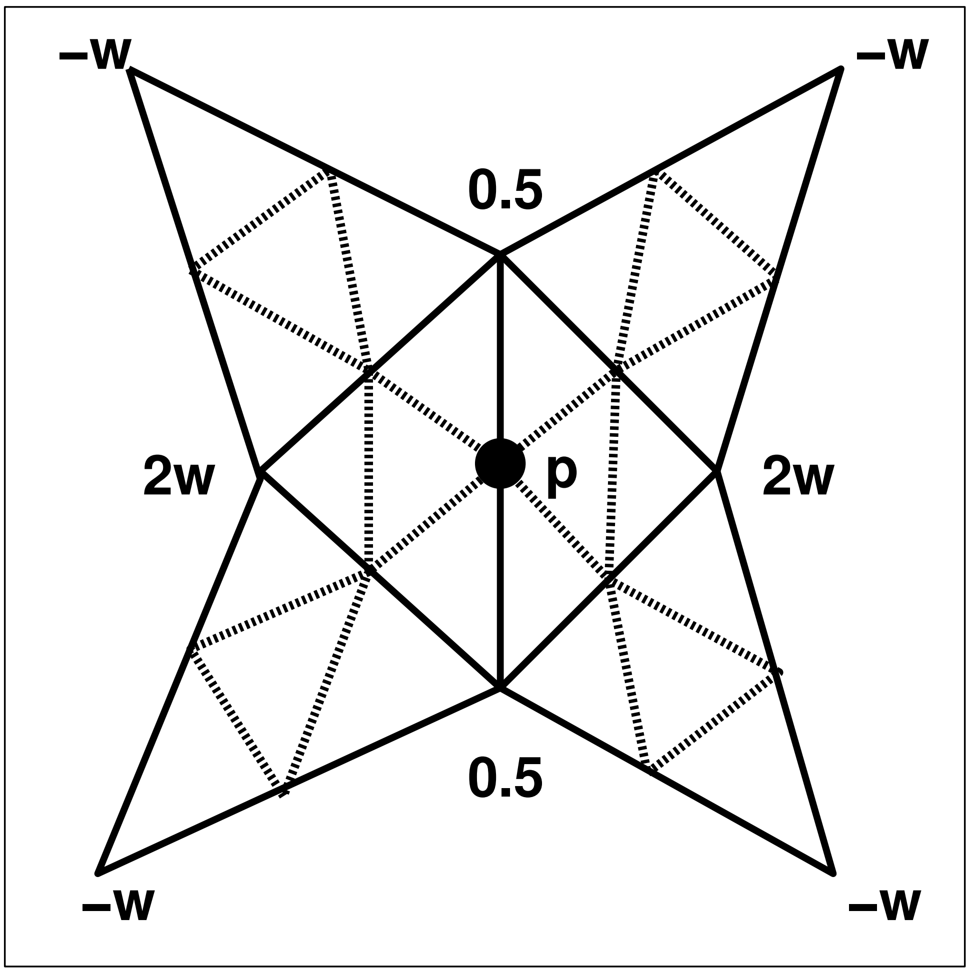
Catmull-Clark Surface



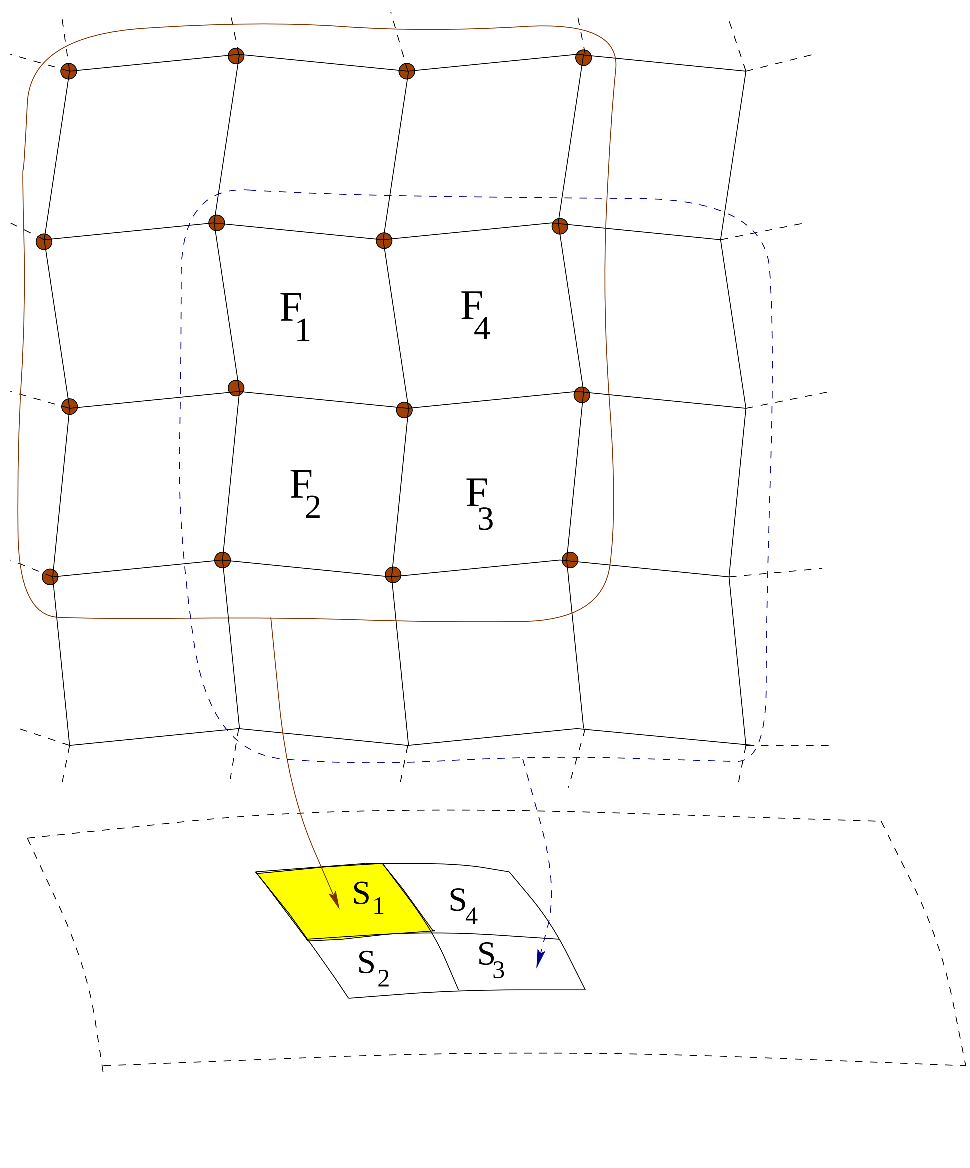
Interpolation Subdivision



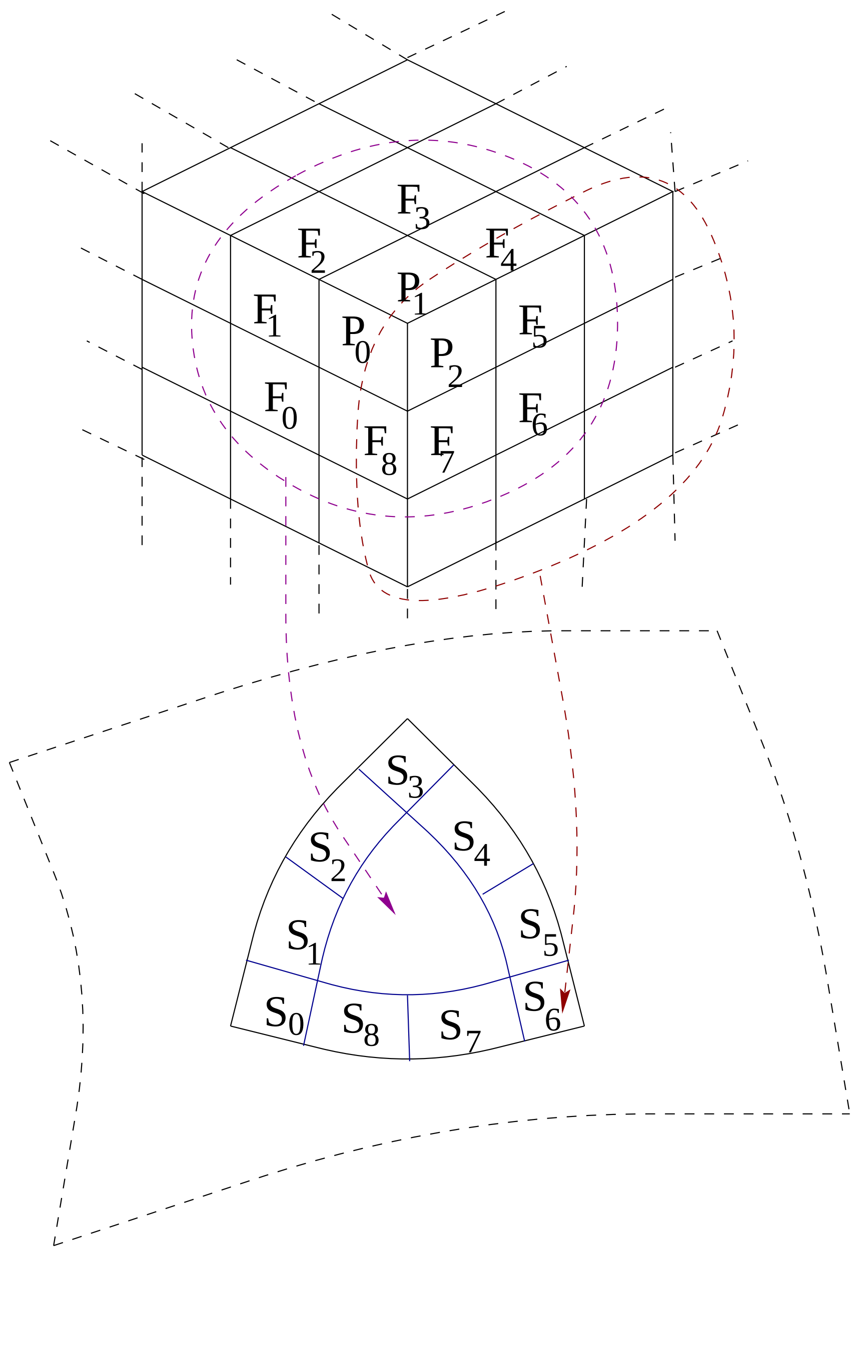
Butterfly Subdivision



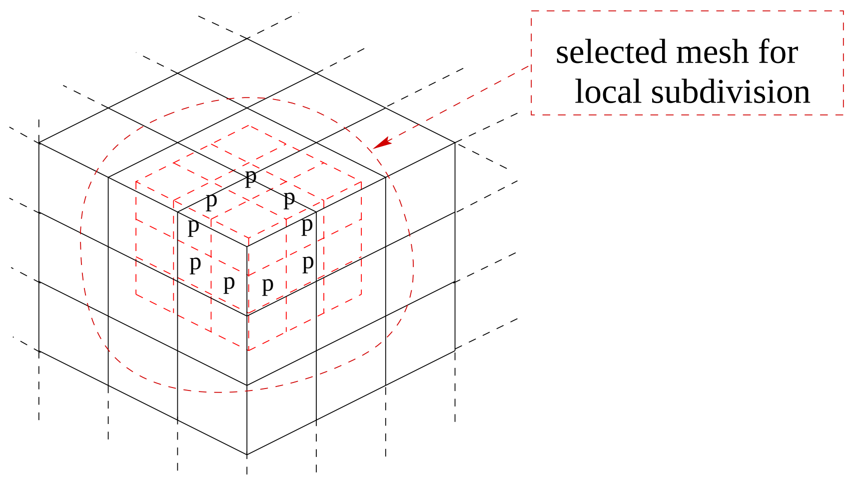
Rectangular Mesh

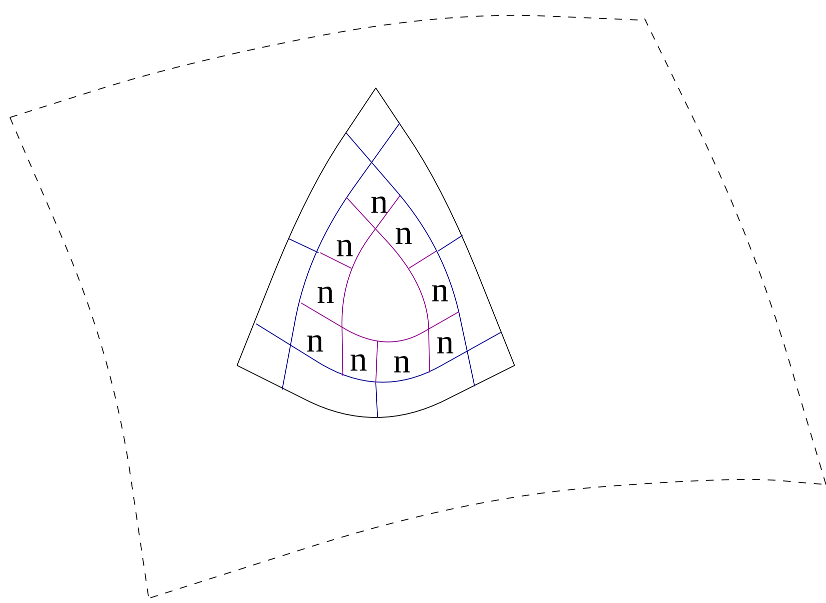


Non-Rectangular Mesh

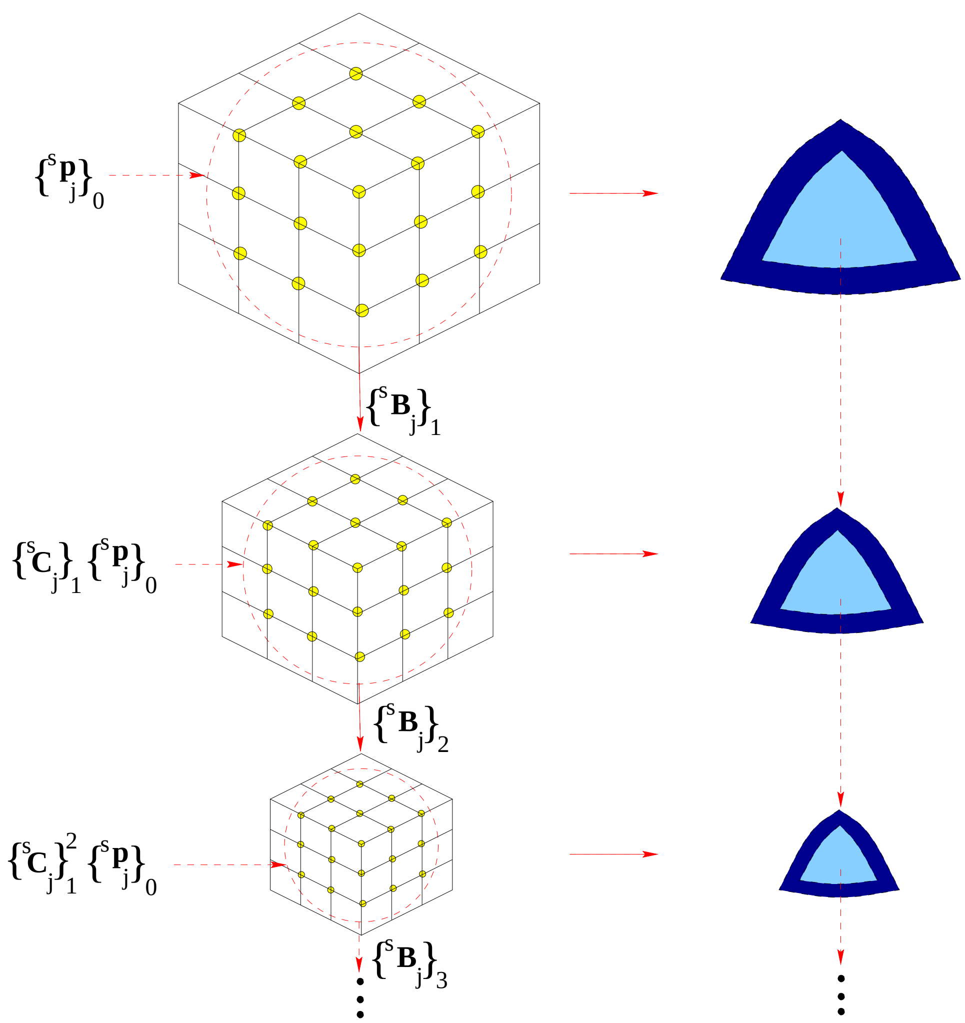


Local Subdivision





Recursive Local Subdivision



Physics-Based Modeling

- Geometric models + physical laws $\Rightarrow$ dynamic models
- Energies express global "fairness" criteria
- Forces support direct manipulation, interactive sculpting
- Constraints permit functional design
- Shape optimization via evolution to equilibrium
- Dynamics allow time-varying conceptual design
- Automatic DOFs selection

Dynamic Subdivision Surface

- Physics-based generalization of geometry
- Lagrangian mechanics formulation
- Control vertices are generalized coordinates
- Mass and damping distributions
- Deformation energy
- Applied forces
- Geometric constraints

B-Spline Formulation

- **Bi-cubic patch**

$$\mathbf{s}_p(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{p}_{i,j} B_{i,4}(u) B_{j,4}(v)$$

- **DOF vector:** $\mathbf{q}_p(t) = \begin{bmatrix} \cdots & \mathbf{p}_{i,j}^{\top} & \cdots \end{bmatrix}^{\top}$

- **Kinematics**

$$\mathbf{s}_p = \mathbf{J}_p \mathbf{q}_p$$

$$\dot{\mathbf{s}}_p = \mathbf{J}_p \dot{\mathbf{q}}_p$$

  - **Jacobian matrix:** $\mathbf{J}_p$

Dynamic Models

- **Finite element decomposition**

$$\mathbf{s} = \sum_{i=1}^{k} \mathbf{n}_i + \sum_{j=1}^{l} \mathbf{s}_j$$

- **Analytic formulation**

$$\mathbf{s} = \mathbf{J}\mathbf{p}$$

- **Kinematics**

$$\dot{\mathbf{s}} = \mathbf{J}\dot{\mathbf{p}}$$

Physics of Dynamic Models

- **Lagrangian equation of motion**

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_p$$

  - M: **mass matrix**
  - D: **damping matrix**
  - K: **stiffness matrix**

- **Generalized force**

$$\mathbf{f}_p = \int\int \mathbf{J}^T \mathbf{f}(u, v, t) du dv$$

  - $\mathrm{f}(u, v, t)$: **applied force distribution**

Global Subdivision

- Increase the number of control points

$$\mathbf{q} = \mathbf{A}\mathbf{p}$$

  - $\mathbf{p}$: initial DOF vector
  - $\mathbf{q}$: new DOF vector
  - $\mathbf{A}$: subdivision matrix

- Limit surface defined by $\mathbf{p}$

$$\mathbf{s} = \mathbf{J}\mathbf{p}$$

$$\mathbf{p} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{q} = \mathbf{B}\mathbf{q}$$

Multilevel Dynamics

- **Limit surface defined by $\mathbf{q}$ after subdivision**

$$\mathbf{s} = (\mathbf{JB})\mathbf{q}$$

- **NEW Lagrangian equation of motion**

$$\mathbf{M}_q\ddot{\mathbf{q}} + \mathbf{D}_q\dot{\mathbf{q}} + \mathbf{K}_q\mathbf{q} = \mathbf{f}_q$$

  - $\mathbf{M}_q$, $\mathbf{D}_q$, $\mathbf{K}_q$: **new matrices**
  - $\mathbf{f}_q$: **new generalized force vector**

Numerical Implementation

- Finite element analysis approach
- New subdivision surface finite element
  - normal elements, special elements
- Gaussian quadrature to assemble element matrices
- Numerical time integration of motion equation
- Efficient parallel algorithm
- Force application
- Hierarchical Model

Applications

- **Interactive sculpting**
- **Shape reconstruction**
- **Scattered data fitting**

Conclusion

- Dynamic framework for subdivision surfaces
- Graphics and visualization applications
- Hierarchical dynamic control

Future Research

- Other subdivision schemes
- Local subdivision
- Dynamic rules
- Unknown topology
- Multi-resolution analysis
- Various applications

Sabin & Doo Approach

- **Subdivision rules:** $\mathrm{v}_i = \sum_{j=1}^{n} \alpha_{ij} \mathrm{w}_j$

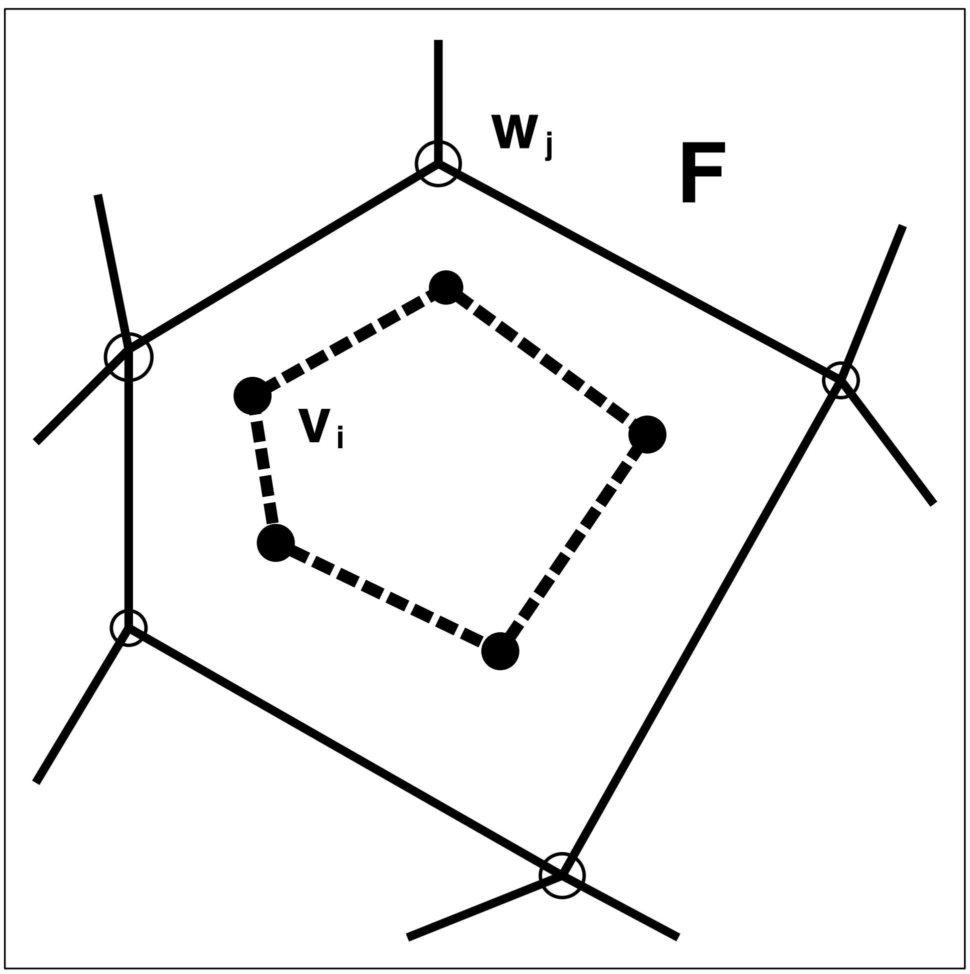
$$\alpha_{ij} = \begin{cases} \frac{n+5}{4n} & i = j \\ \frac{3+2cos(\frac{2(i-j)\pi}{n})}{4n} & otherwise \end{cases}$$

  - **new vertex:** v_i
  - **old vertex of face F:** w_j

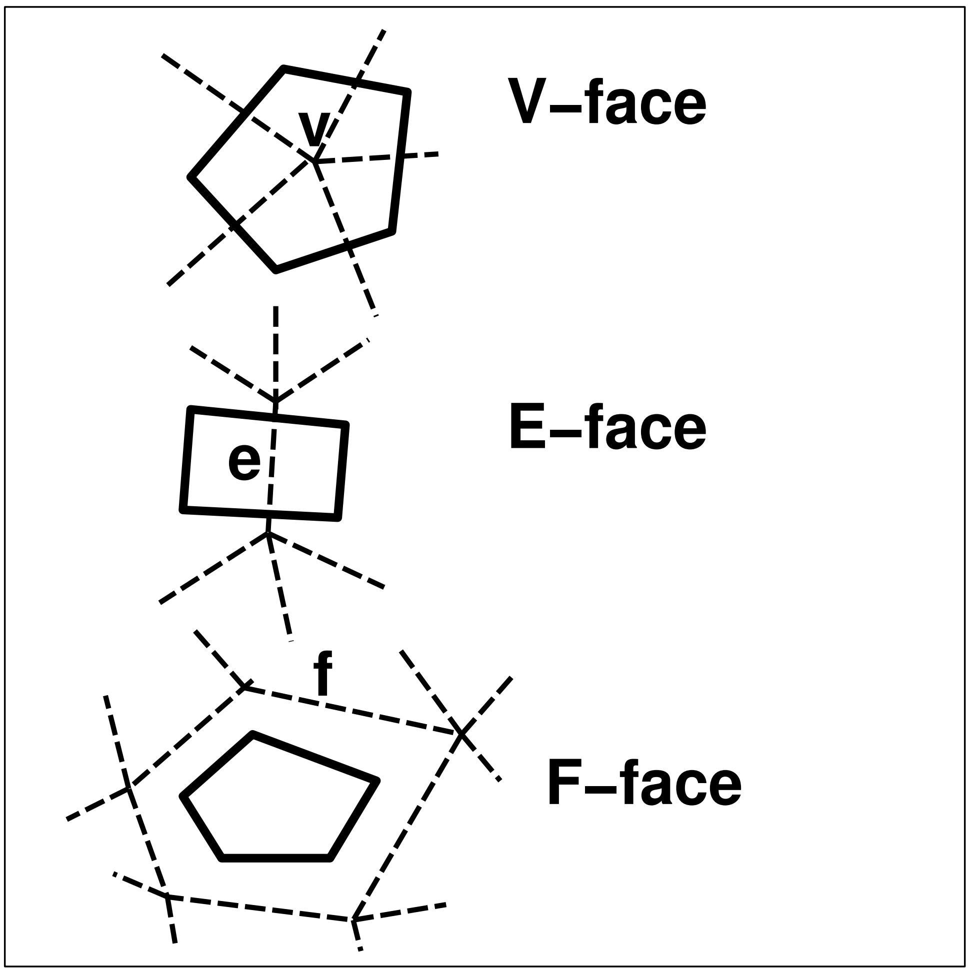
- **New faces after subdivision**
  - **V-face, E-face, F-face**

- **Properties**
  - **the limit surface is C^1**
  - **interpolate centroids of all faces**

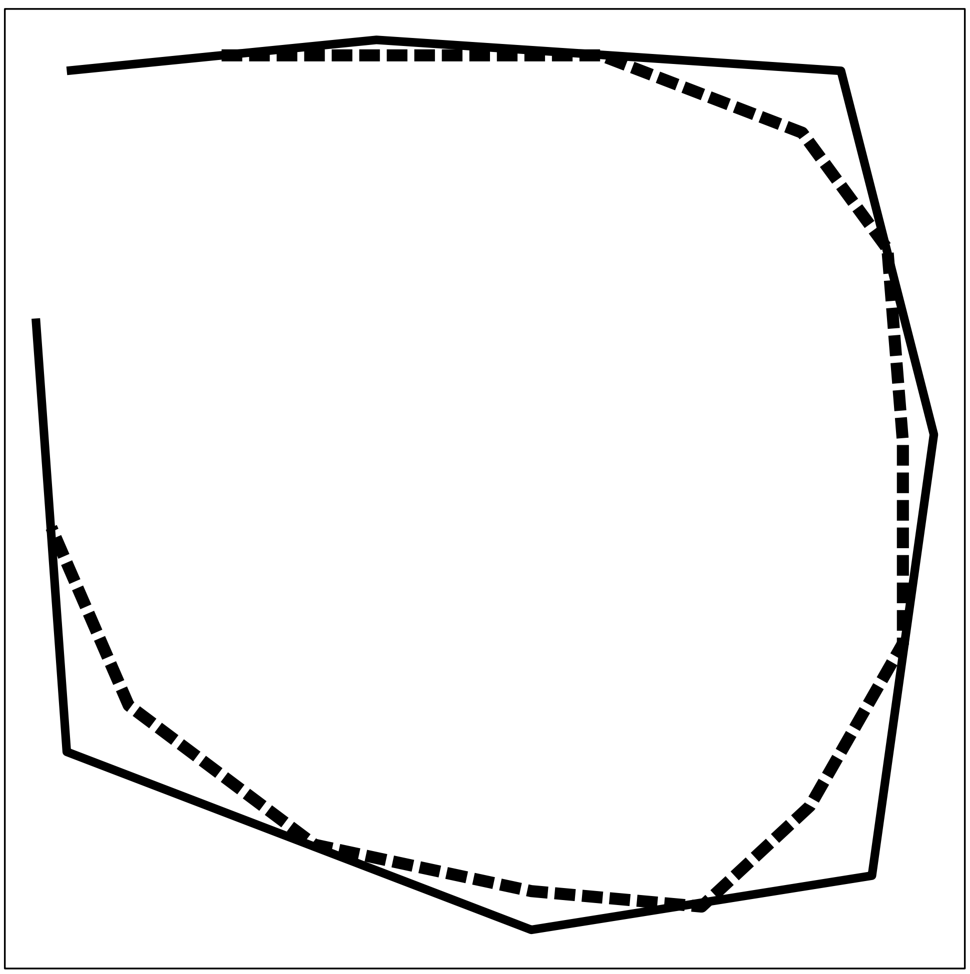
Sabin & Doo Algorithm



Sabin & Doo Algorithm



Cubic Spline



Cubic Spline

- **Subdivision rules:**

$$\mathbf{p}_{2i}^{k+1} = \frac{1}{2}\mathbf{p}_i^k + \frac{1}{2}\mathbf{p}_{i+1}^k$$

$$\mathbf{p}_{2i+1}^{k+1} = \frac{1}{4}(\frac{1}{2}\mathbf{p}_i^k + \frac{1}{2}\mathbf{p}_{i+2}^k) + \frac{3}{4}\mathbf{p}_{i+1}^k$$

- C^2 **cubic B-spline**

- **Corner-chopping**

- **No interpolation**

Interpolation

- **Control points:** $\mathbf{p}^0_{-2}, \mathbf{p}^0_{-1}, \ldots, \mathbf{p}^0_{n+2}$
- **Subdivision rules**

$$\mathbf{p}^{k+1}_{2i} = \mathbf{p}^k_i, -1 <= i <= 2^k n + 1$$

$$\mathbf{p}^{k+1}_{2i+1} = (\frac{1}{2} + w)(\mathbf{p}^k_i + \mathbf{p}^k_{i+1}) - w(\mathbf{p}^k_{i-1} + \mathbf{p}^k_{i+2})$$

$$-1 <= i <= 2^k n$$

- **Interpolate all OLD points**
- **Insert NEW points "in between" the OLD points**
- **Behaviors & properties depend on** w
- **Generalize to SIX-point interpolatory scheme**