Function Optimization

- Minimization or Maximization
- Consider a single variable function: $f(x)$
- Minimize $f(x)$ (equivalently, maximize $-f(x)$)
- This, in general, leads to a nonlinear equation

$$g(x) = \frac{df(x)}{dx} = 0$$

- Example

$$f(x) = \frac{1}{2}ax^2 - bx + c$$

$$ax - b = 0$$

$$x = \frac{b}{a}$$

- Extend to n-component vector

$$\mathbf{x} = \left[\begin{array}{ccc} x_1 & \cdots & x_n \end{array} \right]^\top$$

- **Minimize f(x)**

$$g_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} = 0$$

- **A set of equations**

- **Solve n-variable non-linear equations**

- **Again, a special case**

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{x}^\top \mathbf{b} + c$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$$

Calculus of Variations

- **Assume $x(u)$, where $u \in [0,1]$**
- **The unknown is a function**
- **The cost function is an integral**
- **Minimize**

$$G(x) = \int_0^1 f(x(u))du$$

$$\frac{\partial G(x)}{\partial x} = 0$$

- **Taylor expansion: $y(u) = \delta x(u)$**

$$\int_0^1 f(x(u) + y(u)) = \int_0^1 f(x)du + \int_0^1 y \frac{\partial f}{\partial x} du + \int_0^1 O(y^2)du$$

- **To minimize the above functional, we need**

$$\frac{\partial G(x)}{\partial x} = 0$$

- **The derivative is the first variation!**

- **Euler equation (strong form)**

$$\frac{\partial f}{\partial x} = 0$$

- **Another example**

$$G(x) = \int_0^1 f(x, x_u)du$$

- **The first variation**

$$\frac{\partial G(x)}{\partial x} = 0$$

- **Taylor expansion**

$$\int_0^1 f(x+y, x_u+y_u)du =$$

$$\int_0^1 f(x, x_u)du + \int_0^1 (y\frac{\partial f}{\partial x} + y_u\frac{\partial f}{\partial x_u})du + \cdots$$

- **Detailed derivation**

$$\int_0^1 (y\frac{\partial f}{\partial x})du + \int_0^1 \frac{\partial f}{\partial x_u}dy$$

$$= \int_0^1 (y\frac{\partial f}{\partial x} - y\frac{d}{du}(\frac{\partial f}{\partial x_u}))du + (\frac{\partial f}{\partial x_u}y(1) - \frac{\partial f}{\partial x_u}y(0))$$

- **For any** y

$$\frac{\partial f}{\partial x} - \frac{d}{du}(\frac{\partial f}{\partial x_u}) = 0$$

- **More complicated examples**

$$G(x) = \int_0^1 f(x, x_u, x_{uu}, \cdots)du$$

- **The first variation**

$$\frac{\partial G(x)}{\partial x} = 0$$

- **The Euler equation is**

$$\frac{\partial f}{\partial x} - \frac{d}{du}(\frac{\partial f}{\partial x_u}) + \frac{d^2}{du^2}(\frac{\partial f}{x_{uu}}) - \frac{d^3}{du^3}(\cdots) + \frac{d^4}{du^4}(\cdots) + \cdots = 0$$

- **Two variable function** $x(u, v)$

$$\int_0^1 \int_0^1 f(x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, \cdots)dudv$$

- **The Euler equation is**

$$\frac{\partial f}{\partial x} - \frac{\partial}{\partial u}(\frac{\partial f}{\partial x_u}) - \frac{\partial}{\partial v}(\frac{\partial f}{\partial x_v})$$

$$+\frac{\partial^2}{\partial u^2}(\frac{\partial f}{\partial x_{uu}})+\frac{\partial^2}{\partial u \partial v}(\frac{\partial f}{\partial x_{uv}})+\frac{\partial^2}{\partial v^2}(\frac{\partial f}{\partial x_{vv}})+\cdots=0$$

Examples

- Quadratic energies $\Rightarrow$ Linear equations
- Non-quadratic energies $\Rightarrow$ Non-linear equations
- The shortest path (arc-length)
- The minimal surface area (soap bubble)
- The minimization of curvature
- x can be a vector, e.g.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- x can be of arbitrary domain
- With constraints
- Differential equations with boundary conditions

- **Differential equations with initial-value conditions (Either u or v is time t)**

- **How we should solve the above equations**
  - **finite element**
  - **finite difference**

Numerical Algorithm

- **Finite difference**

$$\mathbf{c}(u) = \{\mathbf{c}(u_0), \mathbf{c}(u_1), \ldots, \mathbf{c}(u_n)\}$$

$$\mathbf{c}(u) = \mathbf{J} \begin{bmatrix} \vdots \\ \mathbf{c}(u_i) \\ \vdots \end{bmatrix}$$

where

$$\mathbf{J} = \begin{bmatrix} \cdots & B_i(u) & \cdots \end{bmatrix}$$

$B_i(u)$ **is a piecewise constant**

$$\dot{\mathbf{c}}(u) = \{\dot{\mathbf{c}}(u_0), \dot{\mathbf{c}}(u_1), \ldots, \dot{\mathbf{c}}(u_n)\}$$

$$\dot{\mathbf{c}}(u) = \mathbf{J} \begin{bmatrix} \vdots \\ \dot{\mathbf{c}}(u_i) \\ \vdots \end{bmatrix}$$

- **Finite element (B-splines)**

$$\mathbf{c}(u) = \sum_i \mathbf{p}_i B_i(u) = \mathbf{J}\mathbf{p}$$

where J is a collection of B-spline Basis functions

$$\dot{\mathbf{c}}(u) = \sum_i \frac{\partial \mathbf{c}(u)}{\partial \mathbf{p}_i}\dot{\mathbf{p}}_i = \mathbf{J}\dot{\mathbf{p}}$$

- **Finite element (NURBS)**

$$\mathbf{c}(u) = \frac{\sum_i \mathbf{p}_i w_i B_i(u, \mathbf{k})}{\sum_i w_i B_i(u, \mathbf{k})}$$

$$= \mathbf{B}\mathbf{p}$$

where

$$\mathbf{p} = \begin{bmatrix} \vdots \\ \mathbf{p}_i \\ w_i \\ \vdots \\ \vdots \\ k_i \\ \vdots \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \cdots & \frac{w_i B_i(u,\mathbf{k})}{\sum_i w_i B_i(u,\mathbf{k})} & 0 & \cdots & 0 \end{bmatrix}$$

$$\dot{\mathbf{c}}(u) = \sum_i \frac{\partial \mathbf{c}(u)}{\mathbf{p}_i}\dot{\mathbf{p}}_i + \sum_i \frac{\partial \mathbf{c}(u)}{w_i}\dot{w}_i + \sum_i \frac{\partial \mathbf{c}(u)}{k_i}\dot{k}_i$$

$$\dot{\mathbf{c}}(u) = \mathbf{J}\dot{\mathbf{p}}$$

where

$$\mathbf{p} = \begin{bmatrix} \vdots \\ p_i \\ w_i \\ \vdots \\ \vdots \\ k_i \\ \vdots \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \cdots & \frac{w_i B_i(u,\mathbf{k})}{\sum_i w_i B_i(u,\mathbf{k})} & \frac{\partial \mathbf{c}(u)}{\partial w_i} & \cdots & \cdots & \frac{\partial \mathbf{c}(u)}{\partial k_i} & \cdots \end{bmatrix}$$

- Reparameterization
- From NURBS to a set of piecewise rational Bezier curves
- From NURBS control points and weights to a set of new control points and weights

$$\begin{bmatrix} \mathbf{p}_i w_i \\ w_i \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{q}_j(\mathbf{p}) h_j(\mathbf{p}) \\ h_j(\mathbf{p}) \end{bmatrix}$$

$$\mathbf{c}(u) = \frac{\sum_j \mathbf{q}_j(\mathbf{p}) h_j(\mathbf{p}) N_j(v)}{\sum_j h_j(\mathbf{p}) N_j(v)}$$

where all basis functions $N_j(v)$ are defined over $[0, 1]$

- **Jacobian matrix is**

$$\mathbf{J} = \frac{(\sum_j \frac{\partial \mathbf{q}_j h_j}{\partial \mathbf{p}} N_j(v))(\sum_j h_j N_j(v)) - (\sum_j \mathbf{q}_j h_j \sum_j h_j N_j(v))(\sum_j \frac{\partial h_j}{\partial v} N_j(v))}{(\sum_j h_j \mathbf{N}_j(v))^2}$$

$$\dot{\mathbf{c}}(u) = \mathbf{J}_{\mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_{\mathbf{q}} \frac{\partial \mathbf{q}}{\partial \mathbf{p}} \dot{\mathbf{p}}$$

- **Material properties**
  - **mass density, damping density, elasticity, ...**

- **Energies**
  - **kinetic, damping, potential, elastic**

Non-uniform B-Splines

- **Non-uniform knots**

$$\mathbf{c}(u) = \sum_i \mathbf{p}_i B_i(u, \mathbf{k})$$

$$\dot{\mathbf{c}}(u) = \sum_i B_i \dot{\mathbf{p}}_i + \sum_j (\sum_i \mathbf{p}_i \frac{\partial B_i}{\partial k_i}) \dot{k_j}$$

- **Compute** $\frac{\partial c(u)}{\partial k_i}$

- **Reparameterization**

- **Convert B-splines to a set of Bezier splines**

$$\mathbf{c}(u) = \sum_j \mathbf{q}_j N_j(v)$$

where all $N_j(v)$ **are defined over** $[0, 1]$

$$\frac{\partial \mathbf{c}(u)}{\partial k_i} = \sum_j \frac{\partial \mathbf{q}_j}{\partial k_i} N_j(v)$$

- **Assume**

$$\mathbf{J}_k = \left[\begin{array}{ccc} \cdots & \frac{\partial \mathbf{c}(u)}{\partial k_i} & \cdots \end{array} \right]$$

- **The jacobian is**

$$\mathbf{J} = \left[\begin{array}{cccc} \cdots & B_i(u) & \cdots & \mathbf{J}_k \end{array} \right]$$

Toolkits

- Point-based
- Curve-based
- Surface-based
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-
-
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Point-Based Tools

- **Energy** E
- **Virtual work** $dW = \frac{\partial E}{\partial \mathbf{c}(u)} \delta \mathbf{c}(u)$
- **Example — spring**

$$E = \frac{1}{2} k (\mathbf{l} - \mathbf{l}_0)^2$$

$$\mathbf{f} = k(\mathbf{l} - \mathbf{l}_0)$$

- **Manipulate one point on the curve**

$$E = \int \frac{1}{2} k (\mathbf{d}_0 - \mathbf{c}(u,t))^2 \delta(u - u_0) du$$

$$\mathbf{f} = \int k (\mathbf{d}_0 - \mathbf{c}(u,t)) \delta(u - u_0) du$$

where

$$\int \delta(u) du = 1$$

$$\int f(x) \delta(u) du = f(0)$$

- **Manipulate the tangent at one point**

$$E = \int \frac{1}{2} k(\mathbf{d}_0 - \mathbf{t}(\mathbf{c}(u), \mathbf{c}_u(u)))^2 \delta(u - u_0) du$$

$$\mathbf{f} = \int k(\mathbf{d}_0 - \mathbf{t}(\mathbf{c}(u), \mathbf{c}_u(u)))(\frac{\partial \mathbf{t}}{\partial \mathbf{c}(u)}) \delta(u - u_0) du$$

- **If $\mathrm{c}(u)$ is a function of p_i's (control points)**

- **The principle of virtual work**

$$dW = \sum_i \frac{\partial E}{\partial \mathbf{p}_i} \delta \mathbf{p}_i$$

$$\frac{\partial \mathbf{t}}{\partial \mathbf{p}_i} = \frac{\partial \mathbf{t}}{\partial \mathbf{c}(u)} \frac{\partial \mathbf{c}(u)}{\partial \mathbf{p}_i} + \frac{\partial \mathbf{t}}{\partial \mathbf{c}_u(u)} \frac{\partial \mathbf{c}_u(u)}{\partial \mathbf{p}_i}$$

- **Manipulate curvature**

$$E = \int \frac{1}{2} k(\kappa_0 - \kappa(\mathbf{c}(u), \mathbf{c}_u(u), \mathbf{c}_{uu}(u)))^2 \delta(u - u_0) du$$

$$\mathbf{f} = \int k(\kappa_0 - \kappa(\mathbf{c}(u), \mathbf{c}_u(u), \mathbf{c}_{uu}(u))) \frac{\partial \kappa}{\partial \mathbf{c}(u)}$$

- **Manipulate torsion (third-order derivative) at any point**

Curve-Based Tools

- **Manipulate a curve span**

$$E = \int \frac{1}{2} k (d(u) - c(u))^2 du$$

- **The first variation is force**

$$dW = \frac{\partial E}{\partial \mathbf{c}(u)} \delta \mathbf{c}(u)$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{c}(u)}$$

- **Manipulate the tangent function of a curve span**

$$E = \int \frac{1}{2} k (\mathbf{d}(u) - \mathbf{t}(\mathbf{c}(u), \mathbf{c}_u(u)))^2 du$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{c}(u)}$$

- **Manipulate the curvature function of a curve span**

$$E = \int \frac{1}{2} k (d(u) - \kappa(\mathbf{c}(u), \mathbf{c}_{uu}(u)))^2 du$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{c}(u)}$$

- **Manipulate the torsion distribution of a curve span**

- **Manipulate a curve span over the surface**

$$E = \int \frac{1}{2} k(d(u) - \mathbf{s}(c_x(u), c_y(u)))^2 du$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{s}(c_x(u), c_y(u))}$$

Surface-Based Tools

- **Manipulate a surface region**

$$E = \int \frac{1}{2} k(\mathbf{d}(u,v) - \mathbf{s}(u,v))^2 du dv$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{s}(u,v)}$$

- **Modify the normal distribution of a surface region**

$$E = \int \frac{1}{2} k(\mathbf{d}(u,v) - \mathbf{n}(\mathbf{s}(u,v), \mathbf{s}_u(u,v), \mathbf{s}_v(u,v)))^2 du dv$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{s}(u,v)}$$

- **Modify the curvature distribution of a surface region**

$$E = \int \frac{1}{2} k(\kappa(u,v) - \kappa(\mathbf{s}, \mathbf{s}_u, \mathbf{s}_v, \mathbf{s}_{uu}, \mathbf{s}_{uv}, \mathbf{s}_{vv}))^2 du dv$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{s}(u,v)}$$

- **Modify the derivative distribution of a surface region**

$$E = \int \frac{1}{2} k(\mathbf{d}(u, v) - \mathbf{s}_u(u, v))^2 du dv$$

$$\mathbf{f} = \frac{\partial E}{\partial \mathbf{s}(u, v)}$$

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Toolkit Examples

- Boundary curve attractor
- Gravity applied on a surface region
- Pressure on a surface region
- Force from a potential field
- Force from a vector field
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Applications

- **Streamline extraction**

Equations of Motion

- **Lagrangian equations (continuous form)**

$$\mu \frac{\partial^2 \mathbf{s}}{\partial t^2} + \gamma \frac{\partial \mathbf{s}}{\partial t} + \frac{\partial E(\mathbf{s})}{\partial \mathbf{s}} = \mathbf{f}$$

  where $\mathbf{s} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$**,** $\mathbf{s}$ **can be curve, surface, or solid.**

- **Finite difference**

$$\mu_i \frac{d^2 \mathbf{d}_i}{dt^2} + \gamma_i \frac{d\mathbf{d}_i}{dt} + \frac{\partial E(\mathbf{d})}{\partial \mathbf{d}_i} = \mathbf{f}_i$$

  where $\mathbf{d} = \begin{bmatrix} \dots & \mathbf{d}_i^\top & \dots \end{bmatrix}^\top$

- **Mass-spring discretization**

$$\mu_i \frac{d^2 \mathbf{d}_i}{dt^2} + \gamma_i \frac{d\mathbf{d}_i}{dt} + \mathbf{K}(\mathbf{d})\mathbf{d}_i = \mathbf{f}_i$$

- **Finite element method**

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}} = \mathbf{f}_\mathbf{p}$$

  where $\mathbf{s} = \sum_i \mathbf{p}_i B_i$**,** $\mathbf{p} = \begin{bmatrix} \dots & \mathbf{p}_i^\top & \dots \end{bmatrix}$

Curve Functionals

- **Simple forms (derivatives of various orders)**
  $\mathrm{c}(u)$ **is a parametric curve**

$$\int_a^b (\mathbf{c}^{(2)}(u))^2 du$$

$$\int_a^b (\mathbf{c}^{(3)}(u))^2 du$$

$$\int_a^b (\mathbf{c}^{(i)}(u))^2 du$$

- **Curvature** $\kappa(\mathbf{c}(u))$

$$\int_a^b \kappa^2(\mathbf{c}(u)) du$$

- **Arc-length**

$$\int_a^b l(\mathbf{c}(u)) du$$

- **Variation of curvature**

$$\int_a^b (\frac{d\kappa}{du})^2(\mathbf{c}(u)) du$$

- **Variation of arc-length**

$$\int_a^b (\frac{dl}{du})^2(\mathbf{c}(u))du$$

- **Least motion for curvature and arc-length**
  Preserving the curvature and arc-length
  $\mathbf{c}_n(u)$ **denotes the new curve after the manipulation**
  $\mathbf{c}_o(u)$ **denotes the old curve before the manipulation**

$$\int_a^b (\kappa(\mathbf{c}_n(u)) - \kappa(\mathbf{c}_o(u)))^2 du$$

$$\int_a^b (l(\mathbf{c}_n(u)) - l(\mathbf{c}_o(u)))^2 du$$

- **Torsion related functional**

- **Combination of all of the above**

- **Generic forms**
  Minimize $f(x)$ **(where** $x \in R^n$**)**
  subject to

$$\begin{cases} c_i(x) = 0, & i = 1, \ldots, k \\ c_i(x) >= 0, & i = k+1, \ldots, m \end{cases}$$

Surface Functionals

- **Simple forms**

$$\int_u \int_v (\mathrm{s}_u^2(u,v) + \mathrm{s}_v^2(u,v))dudv$$

$$\int_u \int_v (\mathrm{s}_{uu}^2(u,v) + \mathrm{s}_{uv}^2(u,v) + \mathrm{s}_{vv}^2(u,v))dudv$$

- **Principal curvatures of the surface κ_1 and κ_2**

$$\int_{\mathrm{s}} (\kappa_1^2(\mathrm{s}(u,v)) + \kappa_2^2(\mathrm{s}(u,v)))d\mathrm{s}$$

- **Area $A(\mathrm{s}(u,v))$**

$$\int \int A(\mathrm{s}(u,v))dudv$$

- **Variation of curvature**
  κ_1 and κ_2 are two principle curvatures
  u_1 and u_2 represent u and v, respectively

$$\int \int (\sum_{i,j} (\frac{\partial \kappa_i}{\partial u_j})^2 (\mathrm{s}(u,v)))dudv$$

- **Variation of area**

$$\int\int(\sum_i(\frac{\partial A}{\partial u_i})^2(\mathbf{s}(u,v)))dudv$$

- **Least motion for curvature and area**

$$\int\int(\sum_i(\kappa_i(\mathbf{s}_n(u,v))-\kappa_i(\mathbf{s}_o(u,v)))^2)dudv$$

$$\int\int(\sum_i(A(\mathbf{s}_n(u,v))-A(\mathbf{s}_o(u,v)))^2)dudv$$

- **Combination of all of the above**

- **Area preserving, convex-preserving, etc.**

- **For surface cases, generic forms are the same**
  Minimize $f(x)$ (where $x \in R^n$)
  subject to

$$\begin{cases} c_i(x) = 0, & i = 1, \ldots, k \\ c_i(x) >= 0, & i = k+1, \ldots, m \end{cases}$$