# Physics-Based Design

- Long-term objective
  - new interactive design environment for CAD/CAM
  - variety of new force-based design tools
- New approach
  - physics-based geometric modeling and design
- Rationales
  - difficulties with conventional approaches
  - integration of geometry with physics
    - improve interactive design
    - support intuitive interaction via forces
- D-NURBS theory & practice
- Future research topics

# Engineering Design

- CAD/CAM
  - conceptual design, analysis, evaluation, prototyping, manufacturing, assembly, production, etc.
- Iterative & innovative procedure
- Critical for other downstream CAD/CAM activities
  - design decisions affect final products in terms of quality, feasibility, cost, time, etc.
- Primary objective: define product geometry
- Techniques & tools
  - advanced graphics interface
  - efficient algorithm & software
  - specialized CAD hardware system

# Geometric Modeling

- Shape representation
  - parametric polynomial
  - piecewise rational spline
  - recursive subdivision form
  - implicit function
- Design paradigms
  - interpolation
  - approximation
  - optimization
  - cross section design
  - blend and offset
  - solid modeling

# Geometric NURBS

- Non-Uniform Rational B-Splines
- CAGD industry standard — useful properties
- Degrees of freedom
  - control points
  - weights

# Geometric Design with NURBS

- Control polygon manipulation
- Weight adjustment
- Knot vector modification
- Data/curve interpolation & approximation
- Geometric design can be difficult
  - indirect geometric operation
  - DOF-oriented shape specification
  - restricted functional requirements
  - limited fairness control

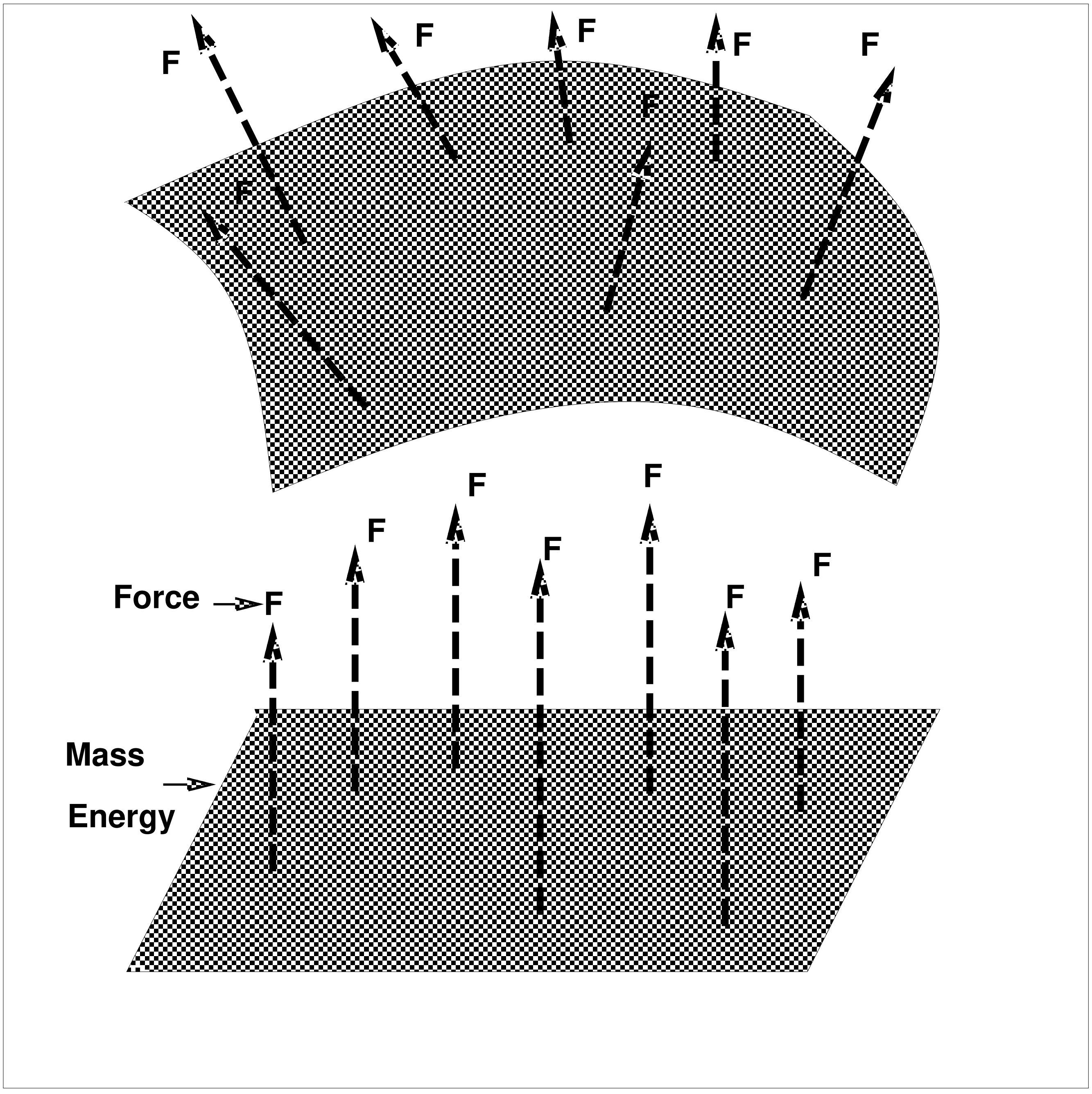
# Geometric Design Tools

- Interactive DOF (degree of freedom) manipulation
- Interpolation/Approximation
- Cross-sectional design: curve network creation & manipulation
- Free-form surface fitting
- Reverse engineering from clay models or CAD data
- Constraint-based iterative optimization
- Conventional approaches can be difficult
- New design techniques & tools are necessary

# Limitations of Geometric Design

- Indirect & kinematic geometric operation
- DOF-oriented shape specification
- Restricted functional requirements
- Limited fairness control
- Confined time-varying design
- Lack of intuitive interaction

# Physics-Based Modeling



# Physics-Based Design

- Geometric models + physical laws $\Rightarrow$ dynamic models
- Integration of static geometry with dynamic behavior
- Energies express global "fairness" criteria
- Forces support direct manipulation, interactive sculpting, & intuitive interaction
- Constraints permit functional design
- Shape optimization via evolution to equilibrium
- Dynamics allow time-varying shape design & control
- Automatic DOF selection

# Optimization $\Rightarrow$ Dynamics

- Generalization of variational design
- Smooth & natural motions for shape control
- Intuitive interaction for real-time shape variation
- Interactive manipulation with force-based tools
- Unified formulation for design & manufacturing

# Industrial Benefits

- Improve product quality
  - supply intuitive & effective CAD tools
  - facilitate the design of high-quality products
- Shorten product development life cycle
  - incorporate manufacturing constraints in design process
  - unify design, analysis, assembly, & manufacturing
- Reduce product cost
- Enhance the effectiveness of design engineers
- Stimulate future technologies for concurrent engineering

# D-NURBS Overview

- D-NURBS: Dynamic NURBS
- From geometric to physics-based modeling
- Tensor product D-NURBS
- Swung D-NURBS
- Triangular D-NURBS
- Graphics, CAGD, and visualization applications

# Dynamic NURBS

- D-NURBS
  Physics-based generalization of geometric NURBS
- Lagrangian mechanics formulation
- NURBS DOFs (control points and weights) are D-NURBS generalized coordinates
- Mass and damping distributions
- Deformation energy
- Applied forces
- Geometric constraints

# NURBS Curve

- **B-spline curve:**

$$c(u) = \sum_{i=0}^{n} \mathbf{p}_i B_{i,k}(u)$$

- **B-spline basis functions:**

$$B_{i,1}(u) = \begin{cases} 1 & \textbf{for } u_i \leq u < u_{i+1} \\ 0 & \textbf{otherwise} \end{cases}$$

**with**

$$B_{i,k}(u) = \frac{u - u_i}{u_{i+k-1} - u_i} B_{i,k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} B_{i+1,k-1}(u)$$

- **NURBS curve:**

$$\mathbf{c}(u) = \frac{\sum_{i=0}^{n} \mathbf{p}_i w_i B_{i,k}(u)}{\sum_{i=0}^{n} w_i B_{i,k}(u)}$$

- **Geometric components:**
  - **control point, weights**
  - **parametric domain**
  - **knots**

# NURBS Surface

- **B-spline surface:**

$$\mathrm{s}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathrm{p}_{i,j} B_{i,k}(u) B_{j,l}(v)$$

- **NURBS surface**

$$\mathrm{s}(u,v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} \mathrm{p}_{i,j} w_{i,j} B_{i,k}(u) B_{j,l}(v)}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{i,k}(u) B_{j,l}(v)}$$

- **Parametric variables:** $u$ **and** $v$

- **Control points** $\mathrm{p}_{i,j}$ **and weights** $w_{i,j}$**:** $(m+1)(n+1)$

- **Degree of basis functions:** $k-1$ **and** $l-1$

- **Knot sequence:**

$$u_0 \leq u_1 \leq \ldots \leq u_{m+k}$$

$$v_0 \leq v_1 \leq \ldots \leq v_{n+l}$$

- **Parametric domain:**

$t_{k-1} \leq u \leq t_{m+1}$ **and** $s_{l-1} \leq v \leq s_{n+1}$

# D-NURBS Curve

- **Curve:**

$$\mathbf{c}(u,t) = \frac{\sum_{i=0}^{n} \mathbf{p}_i(t) w_i(t) B_{i,k}(u)}{\sum_{i=0}^{n} w_i(t) B_{i,k}(u)}$$

- **Control points** $\mathbf{p}_i(t)$ **and weights** $w_i(t)$

- **DOF vector:**

$$\mathbf{p}(t) = \left[ \begin{array}{cccccc} \mathbf{p}_0^\top & w_0 & \cdots & \mathbf{p}_n^\top & w_n \end{array} \right]^\top$$

- **Kinematics:** $\dot{\mathbf{c}}(u,\mathbf{p}) = \mathbf{J}(u,\mathbf{p})\dot{\mathbf{p}}$

- **Jacobian matrix** $\mathbf{J}(u,\mathbf{p})$**:**

$$\mathbf{J}(u,\mathbf{p}) = \left[ \begin{array}{cccc} \cdots & \frac{\partial \mathbf{c}}{\partial \mathbf{p}_i} & \frac{\partial \mathbf{c}}{\partial w_i} & \cdots \end{array} \right]$$

- $\mathbf{J}$ **is a** $3 \times 4(n+1)$ **matrix**

$$\mathbf{J}(u,\mathbf{p}) = \left[ \begin{array}{cccccc} \mathbf{B}_0 & \mathbf{w}_0 & \cdots & \mathbf{B}_n & \mathbf{w}_n \end{array} \right]$$

  **where**

$$\mathbf{B}_i(u,\mathbf{p}) = \left[ \begin{array}{ccc} N_i(u,\mathbf{p}) & 0 & 0 \\ 0 & N_i(u,\mathbf{p}) & 0 \\ 0 & 0 & N_i(u,\mathbf{p}) \end{array} \right]$$

$$N_i(u,\mathbf{p}) = \frac{\partial \mathbf{c}}{\partial \mathbf{p}_i} = \frac{w_i B_{i,k}}{\sum_{j=0}^{n} w_j B_{j,k}}$$

$$\mathbf{w}_i(u,\mathbf{p}) = \frac{\partial \mathbf{c}}{\partial w_i} = \frac{\sum_{j=0}^{n}(\mathbf{p}_i - \mathbf{p}_j) w_j B_{i,k} B_{j,k}}{(\sum_{j=0}^{n} w_j B_{j,k})^2}$$

- $$\mathbf{c}(u,\mathbf{p}) = \mathbf{J}\,\mathbf{p}$$

- **Why?**

# Jacobian Matrix

- **We define:**

$$\mathbf{p}_b(t) = \left[\begin{array}{ccc} \mathbf{p}_0^\top & \cdots & \mathbf{p}_n^\top \end{array}\right]^\top$$

$$\mathbf{p}_w(t) = \left[\begin{array}{ccc} w_0 & \cdots & w_n \end{array}\right]^\top$$

$$\mathbf{B}(u, \mathbf{p}) = \left[\begin{array}{ccc} \mathbf{B}_0 & \cdots & \mathbf{B}_n \end{array}\right]$$

$$\mathbf{W}(u, \mathbf{p}) = \left[\begin{array}{ccc} \mathbf{w}_0 & \cdots & \mathbf{w}_n \end{array}\right]$$

- **We have**

$$\mathbf{J}\mathbf{p} = \mathbf{B}\mathbf{p}_b + \mathbf{W}\mathbf{p}_w$$

$$\mathbf{c} = \mathbf{B}\mathbf{p}_b$$

$$\mathbf{W}\mathbf{p}_w = \frac{\sum_{i=0}^{n}(\sum_{j=0}^{n}(\mathbf{p}_i - \mathbf{p}_j) w_j B_{i,k}(u) B_{j,k}(u)) w_i}{(\sum_{j=0}^{n} w_j B_{j,k}(u))^2}$$

$$= -\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}(\mathbf{p}_j - \mathbf{p}_i) w_i w_j B_{i,k}(u) B_{j,k}(u)}{(\sum_{j=0}^{n} w_j B_{j,k}(u))^2}$$

$$= -\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}(\mathbf{p}_i - \mathbf{p}_j)w_i w_j B_{i,k}(u)B_{j,k}(u)}{(\sum_{j=0}^{n} w_j B_{j,k}(u))^2} = -\mathbf{W}\mathbf{p}_w$$

$$\mathbf{W}\mathbf{p}_w = \mathbf{0}$$

$$\dot{\mathbf{c}}(u, \mathbf{p}) = \mathbf{J}\dot{\mathbf{p}} + \dot{\mathbf{J}}\mathbf{p}$$

$$\dot{\mathbf{J}}\mathbf{p} = \mathbf{0}$$

# Tensor Product D-NURBS

- **Surface:** $\mathbf{s}(u,v,t) = \dfrac{\sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{i,j}(t) w_{i,j}(t) B_{i,k}(u) B_{j,l}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}(t) B_{i,k}(u) B_{j,l}(v)}$

- **Control points** $\mathbf{p}_{i,j}(t)$ **and weights** $w_{i,j}(t)$

- **DOF vector:**

$$\mathbf{p}(t) = \left[ \begin{array}{cccccc} \mathbf{p}_{0,0}^\top & w_{0,0} & \cdots & \mathbf{p}_{m,n}^\top & w_{m,n} \end{array} \right]^\top$$

- **Kinematics:** $\dot{\mathbf{s}}(u,v,\mathbf{p}) = \mathbf{J}(u,v,\mathbf{p})\,\dot{\mathbf{p}}$

- **Jacobian matrix: J**

$$\mathbf{J}(u,v,\mathbf{p}) = \left[ \begin{array}{cccc} \cdots & \frac{\partial \mathbf{s}}{\partial \mathbf{p}_{i,j}} & \frac{\partial \mathbf{s}}{\partial w_{i,j}} & \cdots \end{array} \right]$$

- **J is a** $3 \times 4(m+1)(n+1)$ **matrix**

$$\mathbf{J}(u,v,\mathbf{p}) = \left[ \begin{array}{cccccc} \mathbf{B}_{0,0} & \mathbf{w}_{0,0} & \cdots & \mathbf{B}_{m,n} & \mathbf{w}_{m,n} \end{array} \right]$$

**where**

$$\mathbf{B}_{i,j}(u,v,\mathbf{p}) = \left[ \begin{array}{ccc} N_{i,j}(u,v,\mathbf{p}) & 0 & 0 \\ 0 & N_{i,j}(u,v,\mathbf{p}) & 0 \\ 0 & 0 & N_{i,j}(u,v,\mathbf{p}) \end{array} \right]$$

$$N_{i,j}(u,v,\mathbf{p}) = \frac{\partial \mathbf{s}}{\partial \mathbf{p}_{i,j}} =$$

$$\frac{w_{i,j}B_{i,k}(u)B_{j,l}(v)}{\sum_{c=0}^{m}\sum_{d=0}^{n} w_{c,d}B_{c,k}(u)B_{d,l}(v)}$$

$$\mathbf{w}_{i,j}(u,v,\mathbf{p}) = \frac{\partial \mathbf{s}}{\partial w_{i,j}} =$$

$$\frac{\sum_{c=0}^{m}\sum_{d=0}^{n}(\mathbf{p}_{i,j} - \mathbf{p}_{c,d})w_{c,d}B_{c,k}(u)B_{d,l}(v)B_{i,k}(u)B_{j,l}(v)}{(\sum_{c=0}^{m}\sum_{d=0}^{n} w_{c,d}B_{c,k}(u)B_{d,l}(v))^2}$$

- $$\mathbf{s}(\mathbf{p},u,v) = \mathbf{J}\,\mathbf{p}$$

- **Why?**

# Lagrangian Mechanics

- Equation of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_i} - \frac{\partial T}{\partial p_i} + \frac{\partial F}{\partial \dot{p}_i} + \frac{\partial U}{\partial p_i} = f_i$$

  - $T$: kinetic energy
  - $F$: damping energy
  - $U$: potential energy
  - $p_i$: generalized coordinate
  - $f_i$: external generalized force on $p_i$

# Kinetic and Damping Energies

- **Kinetic energy**

$$T = \frac{1}{2}\iint \mu\, \dot{\mathbf{s}}^\top \dot{\mathbf{s}}\, du\, dv = \frac{1}{2}\dot{\mathbf{p}}^\top \mathbf{M}\dot{\mathbf{p}}$$

  - $\mu(u,v)$: **mass density**
  - $\mathrm{M}$: **mass matrix**

- **Damping energy**

$$F = \frac{1}{2}\iint \gamma\, \dot{\mathbf{s}}^\top \dot{\mathbf{s}}\, du\, dv = \frac{1}{2}\dot{\mathbf{p}}^\top \mathbf{D}\dot{\mathbf{p}}$$

  - $\gamma(u,v)$: **damping density**
  - $\mathrm{D}$: **damping matrix**

# Potential Energy

- **Thin-plate under tension energy**

$$U=\frac{1}{2}\iint(\alpha_{1,1}\mathrm{s}_u^2+\alpha_{2,2}\mathrm{s}_v^2+\beta_{1,1}\mathrm{s}_{uu}^2+\beta_{1,2}\mathrm{s}_{uv}^2+\beta_{2,2}\mathrm{s}_{vv}^2)\,du\,dv$$

  - $\alpha_{i,j}(u,v)$: **tension control functions**
  - $\beta_{i,j}(u,v)$: **rigidity control functions**

# D-NURBS Matrices

- **Mass matrix**

$$\mathbf{M}(\mathbf{p}) = \iint \mu \, \mathbf{J}^\top \mathbf{J} \, du \, dv$$

- **Damping matrix**

$$\mathbf{D}(\mathbf{p}) = \iint \gamma \, \mathbf{J}^\top \mathbf{J} \, du \, dv$$

- **Stiffness matrix** $\mathbf{K}(\mathbf{p})$

$$\iint \alpha_{1,1} \mathbf{J}_u^\top \mathbf{J}_u + \alpha_{2,2} \mathbf{J}_v^\top \mathbf{J}_v +$$

$$\beta_{1,1} \mathbf{J}_{uu}^\top \mathbf{J}_{uu} + \beta_{1,2} \mathbf{J}_{uv}^\top \mathbf{J}_{uv} + \beta_{2,2} \mathbf{J}_{vv}^\top \mathbf{J}_{vv} \, du \, dv$$

# D-NURBS Dynamics

- **Motion equation**

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_p - \mathbf{I}\dot{\mathbf{p}}$$

  - **generalized forces**

$$\mathbf{f}_p = \iint \mathbf{J}^\top \mathbf{f}\, du\, dv$$

    **where** $\mathbf{f}(u, v, t)$ **is applied force distribution**
  - **inertia matrix**

$$\mathbf{I} = \iint \mu\, \mathbf{J}^\top \dot{\mathbf{J}}\, du\, dv$$

# Forces and Constraints

- **Spring**

$$\mathbf{f}(u,v,t) = \iint k(\mathbf{d}_0 - \mathbf{s}(u,v,t))\delta(u-u_0, v-v_0)\,du\,dv$$

- **Contraining the weights!**
  - **negative weights**
  - **penalty method:** $c(\mathbf{p}_w - \mathbf{p}_w^0)^\top(\mathbf{p}_w - \mathbf{p}_w^0)$

# D-NURBS with Geometric Constraints

- **Linear constraint:** $\mathbf{C}(\mathbf{p}) = \mathbf{A}\mathbf{p} + \mathbf{b} = \mathbf{0}$

$$\mathbf{p} = \mathbf{G}\mathbf{q} + \mathbf{q}_0$$

  - $\mathbf{q}$: **new DOF vector with dimensionality** $M < N$
  - $\mathbf{G}$: $N \times M$ **transition matrix**
  - $\mathbf{q}_0$: **constant vector**

- **Kinematics:**

$$\mathbf{s}(u, v, \mathbf{q}) = \mathbf{J}(\mathbf{G}\mathbf{q} + \mathbf{q}_0) = \mathbf{L}\mathbf{q} + \mathbf{J}\mathbf{q}_0$$

$$\dot{\mathbf{s}}(u, v, \mathbf{q}) = \mathbf{J}(\mathbf{G}\dot{\mathbf{q}} + \dot{\mathbf{p}}_0) = \mathbf{J}\mathbf{G}\dot{\mathbf{q}} = \mathbf{L}\dot{\mathbf{q}}$$

  **where**

$$\mathbf{L} = \mathbf{J}\mathbf{G}$$

- **Energies:**

$$T = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{G}^\top \mathbf{M}\mathbf{G}\dot{\mathbf{q}}$$

$$F = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{G}^\top \mathbf{D}\mathbf{G}\dot{\mathbf{q}}$$

$$U = \frac{1}{2}(\mathbf{q}^\top \mathbf{G}^\top + \mathbf{q}_0)\mathbf{K}(\mathbf{G}\mathbf{q} + \mathbf{q}_0)$$

- **Dynamics:** $\mathbf{M}_q\ddot{\mathbf{q}} + \mathbf{D}_q\dot{\mathbf{q}} + \mathbf{K}_q\mathbf{q} = \mathbf{f}_q + \mathbf{g}_q$
  - **matrices**

$$\mathbf{M}_q = \mathbf{G}^\top \mathbf{M}\mathbf{G}; \quad \mathbf{D}_q = \mathbf{G}^\top \mathbf{D}\mathbf{G}; \quad \mathbf{K}_q = \mathbf{G}^\top \mathbf{K}\mathbf{G}$$

  - **generalized forces**

$$\mathbf{f}_q = \mathbf{G}^\top \mathbf{f}_p; \quad \mathbf{g}_q = -\mathbf{G}^\top(\mathbf{K}\mathbf{q}_0 + \mathbf{I}\mathbf{G}\dot{\mathbf{q}})$$

# D-NURBS with Geometric Constraints

- **Nonlinear constraint:** $\mathbf{C}(\mathbf{p}) = \mathbf{0}$

- **Equation of motion:** $\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_p - \mathbf{I}\dot{\mathbf{p}} - \mathbf{C}_{\mathbf{p}}^{\top}\boldsymbol{\lambda}$

  - $\boldsymbol{\lambda}$: **Lagrange multiplier vector**
  - $\mathbf{C}_{\mathbf{p}}$: **constraint Jacobian matrix**
  - $\mathbf{C}_{\mathbf{p}}^{\top}\boldsymbol{\lambda}$: **generalized constraint forces**

- **Dynamics:**

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{p}}^{\top} \\ \mathbf{C}_{\mathbf{p}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}\dot{\mathbf{p}} - \mathbf{K}\mathbf{p} - \mathbf{I}\dot{\mathbf{p}} + \mathbf{f}_p \\ \boldsymbol{\sigma} - 2a\mathbf{C}_{\mathbf{p}}\dot{\mathbf{p}} - b^2\mathbf{C} \end{bmatrix}$$

# Numerical Implementation

- Finite element method approach
- New D-NURBS finite element
  - curve arc
  - surface patch
- Gaussian quadrature to assemble element matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$
- Numerical time integration of motion equation
- Efficient parallel implementation

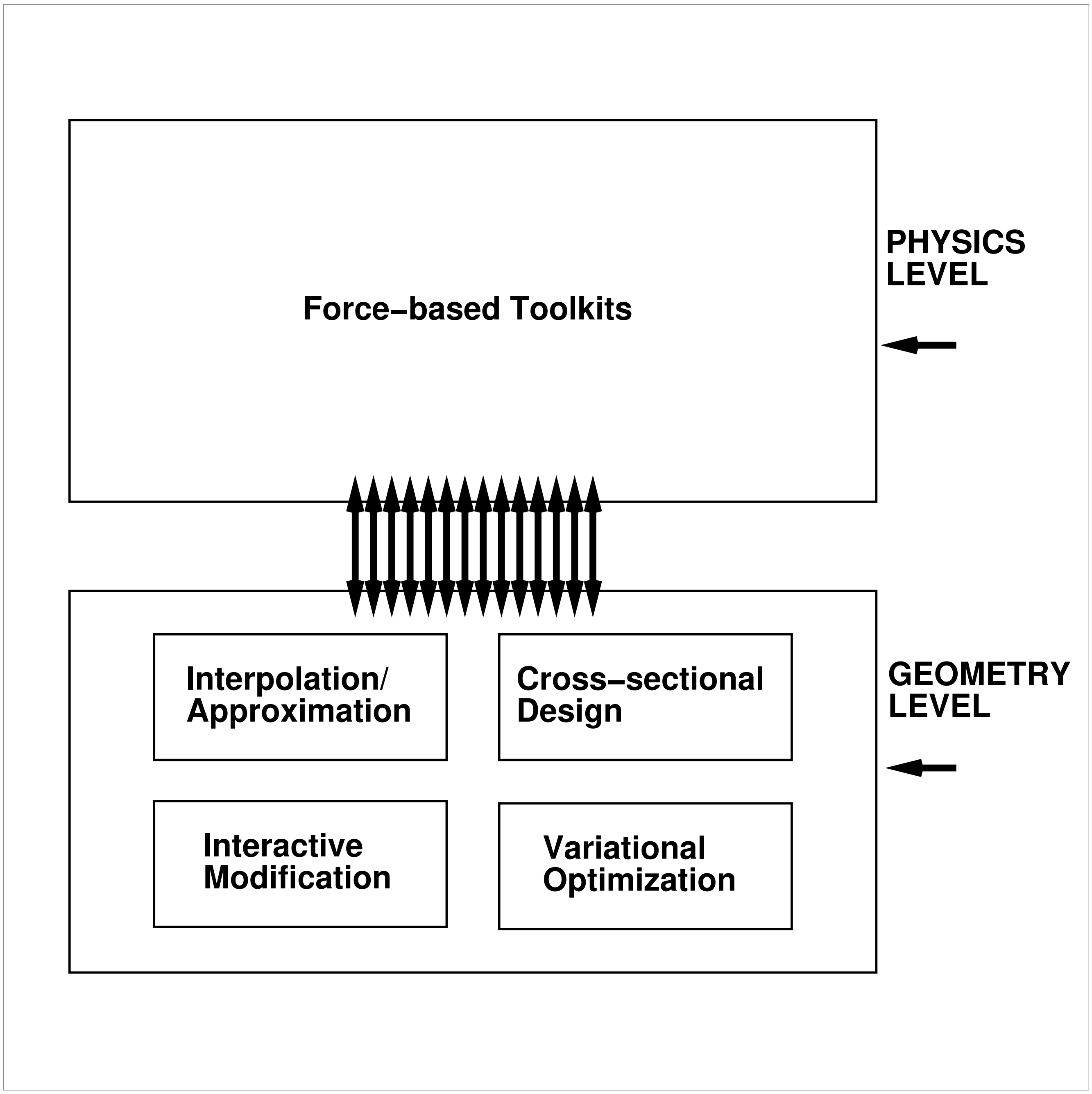
# Automatic DOF Selection

- Systematic mechanism for determining unknown control points & weights
- Intuitive shape control
- Extremely useful for geometric modeling

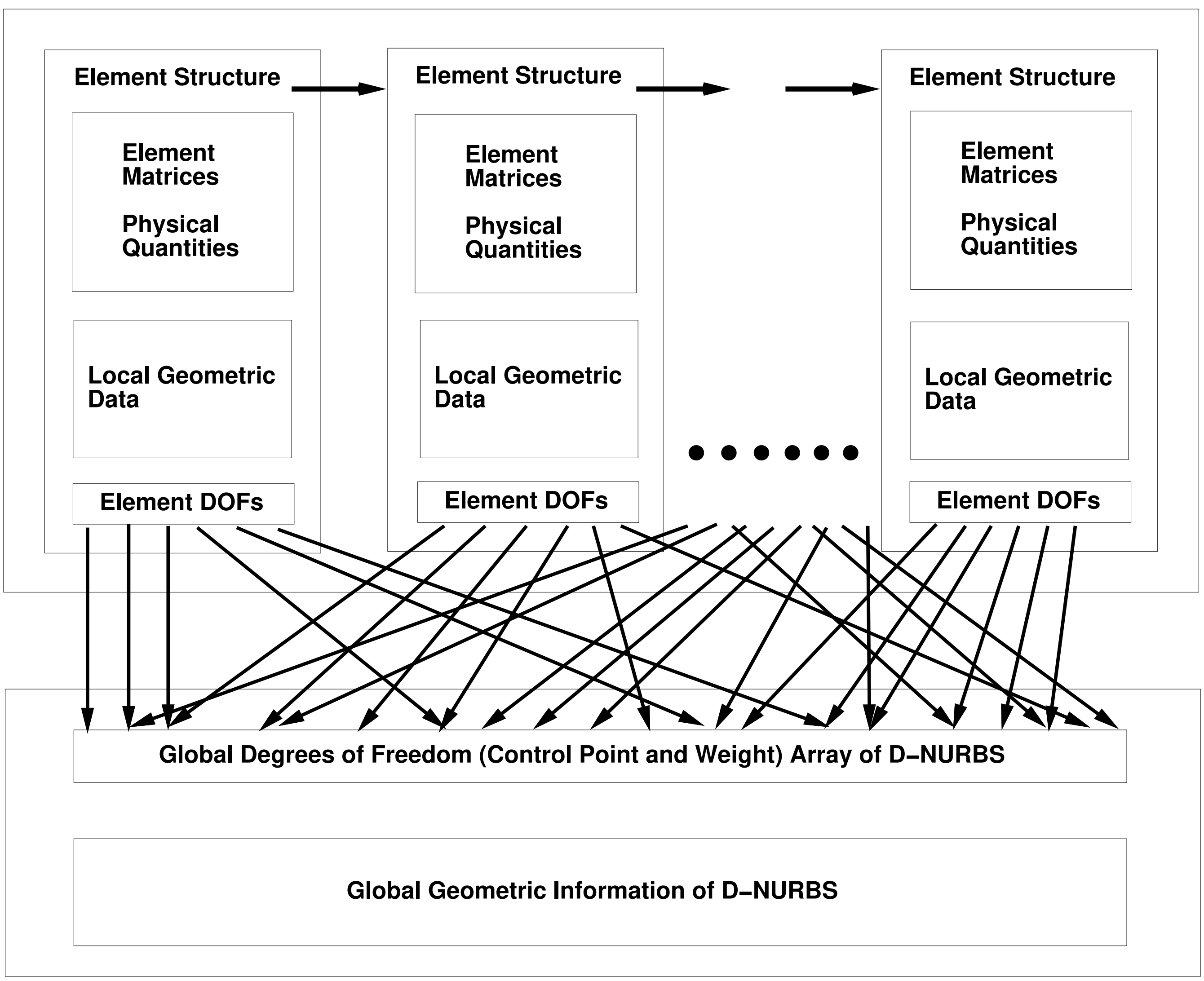
# Physics-Based Geometric Design

- Generalization of geometric design process
- Standard geometric toolkits still usable
- Two-level design framework
- Additional physics-based toolkits
  - sculpting forces
  - elastic energies
  - linear & non-linear constraints
- Integration of traditional design principles
- Enhance geometric design with additional advantages
  - automatic determination of geometric unknowns
  - complicated geometry transparent to designers
  - intuitive shape variation governed by physical properties
  - valuable for non-expert users & engineers
  - relevant to the entire CAD/CAM process

# Physics-Based Shape Design



# Finite Element Data Structure



# Interactive Modeling Environment

- Physics-based design tools
- Various engineering applications
  - solid rounding
  - scattered data fitting
  - shape reconstruction
  - interactive sculpting
  - reverse engineering
  - data visualization
  - hierarchical control
- Unified approach for CAD/CAM
  - variational design
  - user interaction
  - shape control
  - weight selection

# Future Research Focus

- Efficient & robust algorithm design and analysis
- Physics-based sculpting toolkits
- Formulation of new powerful dynamic models
- Advanced user interaction techniques
- Various applications
- Industrial collaboration & support
- Technology transfer to commercial CAD/CAM systems

# Physics-Based Design Paradigm

- Generalization of geometric design process
- Standard geometric toolkits still usable
- Additional physics-based toolkits
  - sculpting forces
  - linear & non-linear constraints

# D-NURBS Applications

- Trimming
- Solid rounding
- Scattered data fitting
- Cross-sectional design
- Shape morphing
- Free-form deformation

# Gaussian Quadrature

- **Parametric domain of the finite element**

$$[u_0, u_1] \times [v_0, v_1]$$

- **Compute entry $k_{ij}$ of the stiffness matrix**

$$k_{ij} = \int_{u_0}^{u_1} \int_{v_0}^{v_1} f_{ij}(u,v)\, du\, dv$$

**where**

$$f_{ij}(u,v) = \alpha_{1,1}(u,v) \frac{\partial \mathbf{j}_i^\top}{\partial u} \frac{\partial \mathbf{j}_j}{\partial u} + \alpha_{2,2}(u,v) \frac{\partial \mathbf{j}_i^\top}{\partial v} \frac{\partial \mathbf{j}_j}{\partial v}$$

$$+\beta_{1,1}(u,v) \frac{\partial^2 \mathbf{j}_i^\top}{\partial u^2} \frac{\partial^2 \mathbf{j}_j}{\partial u^2} + \beta_{1,2}(u,v) \frac{\partial^2 \mathbf{j}_i^\top}{\partial u \partial v} \frac{\partial^2 \mathbf{j}_j}{\partial u \partial v}$$

$$+\beta_{2,2}(u,v) \frac{\partial^2 \mathbf{j}_i^\top}{\partial v^2} \frac{\partial^2 \mathbf{j}_j}{\partial v^2}$$

- **Gaussian quadrature**

$$k_{ij} \approx \sum_{g=1}^{N_g} \sum_{h=1}^{N_h} a_g b_h f_{ij}(u_g, v_h)$$

- **Use well-known NURBS algorithm to evaluate** $f_{ij}(u_g, v_h)$

# Discrete Equations

- **Time discretization (implicit method)**

$$\ddot{\mathbf{p}}^{(t+\Delta t)} = \frac{\mathbf{p}^{(t+\Delta t)} - 2\mathbf{p}^{(t)} + \mathbf{p}^{(t-\Delta t)}}{\Delta t^2}$$

$$\dot{\mathbf{p}}^{(t+\Delta t)} = \frac{\mathbf{p}^{(t+\Delta t)} - \mathbf{p}^{(t-\Delta t)}}{2\Delta t}.$$

- **Time integration**

$$\left(4\mathbf{M} + 2\Delta t\mathbf{D} + 4\Delta t^2\mathbf{K}\right)\mathbf{p}^{(t+\Delta t)} = 4\Delta t^2\mathbf{f}_p +$$

$$8\mathbf{M}\mathbf{p}^{(t)} - (3\mathbf{M} - 2\Delta t\mathbf{D})\mathbf{p}^{(t-\Delta t)} + \iint \mu\mathbf{J}^\top\mathbf{s}\,du\,dv$$

- **Matrices are functions of time**

$$\mathbf{M}^{(t+\Delta t)} = \mathbf{M}^{(t)} + \Delta t\dot{\mathbf{M}}^{(t)} = 2\mathbf{M}^{(t)} - \mathbf{M}^{(t-\Delta t)}$$

- **Simplification**

$$\mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_p$$

$$(\mathbf{D} + \Delta t\mathbf{K})\,\mathbf{p}^{(t+\Delta t)} = \Delta t\mathbf{f}_p + \mathbf{D}\mathbf{p}^{(t)}$$