Physics-Based Modeling

- Deformable models

- “Deformable models”,
  D. Terzopoulos and K. Fleischer
  The Visual Computer, 4, 306-331, 1988

- Related publications: Terzopoulos’s Siggraph’87 and Siggraph’88 papers!

- These papers present fundamental concepts and techniques!

- For an overview of geometric modeling, especially parametric representation for curves and surfaces, please refer to

- “A survey of curves and surfaces in CAGD”,
  W. Bohm et al.
  Computer Aided Geometric Design, 1(1), 1-60

Deformable Models

- Curves, surfaces, solids
- Shapes can change over time!
- Physics-based modeling features
  - physical laws
  - dynamic differential equations
  - various real-world behaviors
- Numerical Integration
  - finite-element method
  - numerical algorithms
- Applications: graphics, animation, simulation, etc.
- Advantages
  - active, dynamic, realistic
  - natural, intuitive, interactive
  - unify shapes and motions

- integrate material, physics with geometry

- **Challenges**
  - computational complexity
  - numerical issues

Deformation

- Elastic
- Inelastic
- Viscoelastic
- Plastic
- Fracture
- Depend on
  - material property
  - external force

Formulation

- **Deformable models**

$$\mathbf{s}(\mathbf{u},t) = \left[\begin{array}{ccc} x(\mathbf{u},t) & y(\mathbf{u},t) & z(\mathbf{u},t) \end{array}\right]^\top$$

  where $\mathbf{u} = (u,v,w)$ for a solid,
  $\mathbf{u} = (u,v)$ for a surface,
  and $\mathbf{u} = (u)$ for a curve

- **Natural shape**

$$\mathbf{s}^0(\mathbf{u}) = \left[\begin{array}{ccc} x^0(\mathbf{u}) & y^0(\mathbf{u}) & z^0(\mathbf{u}) \end{array}\right]^\top$$

- **Motion equation**

$$\mu\frac{\partial^2\mathbf{s}}{\partial t^2} + \gamma\frac{\partial\mathbf{s}}{\partial t} + \delta_{\mathbf{s}}\mathcal{E}(\mathbf{s}) = \mathbf{f}$$

- **Boundary & initial-value conditions**

- **Energy functional: $\mathcal{E}(\mathbf{s})$**

- **Energy requirements:**

- the energy for the natural shape should be 0
- the energy increases as the deformation increases
- invariant to rigid motions

- **How should we define deformation energy?**
- **Answer: fundamental theorems from differential geometry**

Differential Geometry Fundamentals

- Differential distance and angle

$$\mathrm{G}_{ij}(\mathbf{s}(\mathbf{u},t)) = \frac{\partial \mathbf{s}}{\partial \mathbf{u}_i} \cdot \frac{\partial \mathbf{s}}{\partial \mathbf{u}_j}$$

- Two solids differ only by a rigid motion if their metric tensors G are identical

- How about surfaces?

- Two surfaces have the same shape if
  - their metric tensors are the same
  - their curvature tensors are the same
  - curvature tensor

$$\mathrm{B}_{ij}(\mathbf{s}(\mathbf{u},t)) = \mathbf{n}(\mathbf{u},t) \cdot \frac{\partial^2 \mathbf{s}}{\partial \mathbf{u}_i \partial \mathbf{u}_j}$$

    where

$$\mathbf{n}(\mathbf{u},t) = \frac{\frac{\partial \mathbf{s}}{\partial \mathbf{u}_1} \times \frac{\partial \mathbf{s}}{\partial \mathbf{u}_2}}{\|\frac{\partial \mathbf{s}}{\partial \mathbf{u}_1} \times \frac{\partial \mathbf{s}}{\partial \mathbf{u}_2}\|}$$

- **How about curves?**
- **Arc-length, curvature, and torsion**
  - **two curves are identical if all three are the same!**

Energy Functionals

- **Deformable curve** $\mathcal{E}(\mathbf{s})$

$$\int_{\Omega} w^1(l - l^0)^2 + w^2(\kappa - \kappa^0)^2 + w^3(\tau - \tau^0)^2 du$$

- **Deformable surface** $\mathcal{E}(\mathbf{s})$

$$\int_{\Omega} \|\mathbf{G} - \mathbf{G}^0\|^2_{w^1} + \|\mathbf{B} - \mathbf{B}^0\|^2_{w^2} du_1 du_2$$

  where $\|\cdot\|_{w^1}$ **and** $\|\cdot\|_{w^2}$ **are the weighted matrix norms, and** $w^1_{ij}(u_1, u_2)$ **and** $w^2_{ij}(u_1, u_2)$ **are the weighting functions**

- **Deformable solid** $\mathcal{E}(\mathbf{s})$

$$\int_{\Omega} \|\mathbf{G} - \mathbf{G}^0\|^2_{w} du_1 du_2 du_3$$

- **What are the meanings of these weighting functions?**

- **Control stretching, bending, etc.**

- **The simulation of different behaviors**

Hybrid Formulation

- **Local coordinate system**
- **Explicitly express rigid components**

$$\mathbf{q}(\mathbf{u},t) = \mathbf{r}(\mathbf{u},t) + \mathbf{e}(\mathbf{u},t)$$

- **The origin of its local frame is the center of mass**

$$\mathbf{c}(t) = \int_\Omega \mu(\mathbf{u})\mathbf{s}(\mathbf{u},t)du$$

- **Physical quantities**

$$\mathbf{v}(t) = \frac{d\mathbf{c}}{dt}$$

$$\omega(t) = \frac{d\theta}{dt}$$

- **The velocity of the model relative the world coordinate system**

$$\frac{\partial \mathbf{s}(\mathbf{u},t)}{\partial t} = \mathbf{v} + \omega \times \mathbf{q} + \frac{\partial \mathbf{c}}{\partial t}$$

- **Motion equations after the above decomposition**

$$m\frac{d\mathbf{v}}{dt} + \frac{d}{dt}\int_\Omega \mu \frac{\partial \mathbf{e}}{\partial t} du + \int_\Omega \gamma \frac{\partial \mathbf{s}}{\partial t} du = \mathbf{f}^v$$

$$\frac{d}{dt}(\mathbf{I}\omega) + \frac{d}{dt}\int_\Omega \mu \mathbf{q} \times \frac{\partial \mathbf{e}}{\partial t} du + \int_\Omega \gamma \mathbf{q} \times \frac{\partial \mathbf{s}}{\partial t} du = \mathbf{f}^\omega$$

$$\mu \frac{\partial^2 \mathbf{e}}{\partial t^2} + \mu \frac{d\mathbf{v}}{dt} + \mu\omega \times (\omega \times \mathbf{q}) + 2\mu\omega \times \frac{\partial \mathbf{e}}{\partial t}$$

$$+\mu \frac{d\omega}{dt} \times \mathbf{q} + \gamma \frac{\partial \mathbf{s}}{\partial t} + \delta_{\mathbf{s}}\mathcal{E} = \mathbf{f}$$

- **Mass**

$$m = \int_\Omega \mu du$$

- **Inertial tensor I is a 3×3 symmetric matrix**

$$I_{ij} = \int_\Omega \mu(\delta_{ij}\mathbf{q}^2 - q_i q_j) du$$

- **The applied force $\mathrm{f}(\mathrm{u}, t)$ contributes to**
  - **net translational force**

$$\mathbf{f}^{\mathbf{v}}(t) = \int_\Omega \mathbf{f} du$$

- net torque

$$\mathbf{f}^{\omega}(t) = \int_{\Omega} \mathbf{q} \times \mathbf{f} du$$

- elastic deformation force

- Translational motion
  - moving mass of the body
  - mass motion w.r.t. its local reference
  - damping of the moving mass

- Rotational motion
  - rotating mass
  - rotational vibration
  - damping of the rotating mass

- Deformational force
  - basis inertial force
  - inertial force due to linear acceleration
  - centrifugal force due to rotation
  - coriolis force
  - transverse force due to angular acceleration
  - damping force

– **internal elastic force**

- **Deformation functional** $\mathcal{E}(\mathrm{e})$

$$\frac{1}{2}\int_{\Omega}\sum_{m=0}^{p}\sum_{|j|=m}\frac{m!}{j_1!\ldots j_d!}w_j\|\partial_j^m \mathrm{e}\|^2$$

where $j=(j_1,\ldots,j_d)$, $|j|=j_1+\ldots+j_d$,
and $\partial_j^m=\frac{\partial^m}{\partial u_1^{j_1}\ldots\partial_d^{j_d}}$

- **Weighted sum of the deformation and its partial derivatives!**

Forces

- **Gravity**

$$\mathbf{f_g}(\mathbf{u}) = \mu(\mathbf{u})\mathbf{g}$$

- **Point attraction via spring**

$$\mathbf{f_p}(\mathbf{u}, t) = k(\mathbf{p}_0 - \mathbf{s}(\mathbf{u}_0, t))\delta(\mathbf{u} - \mathbf{u}_0)$$

- **Aerodynamic forces**

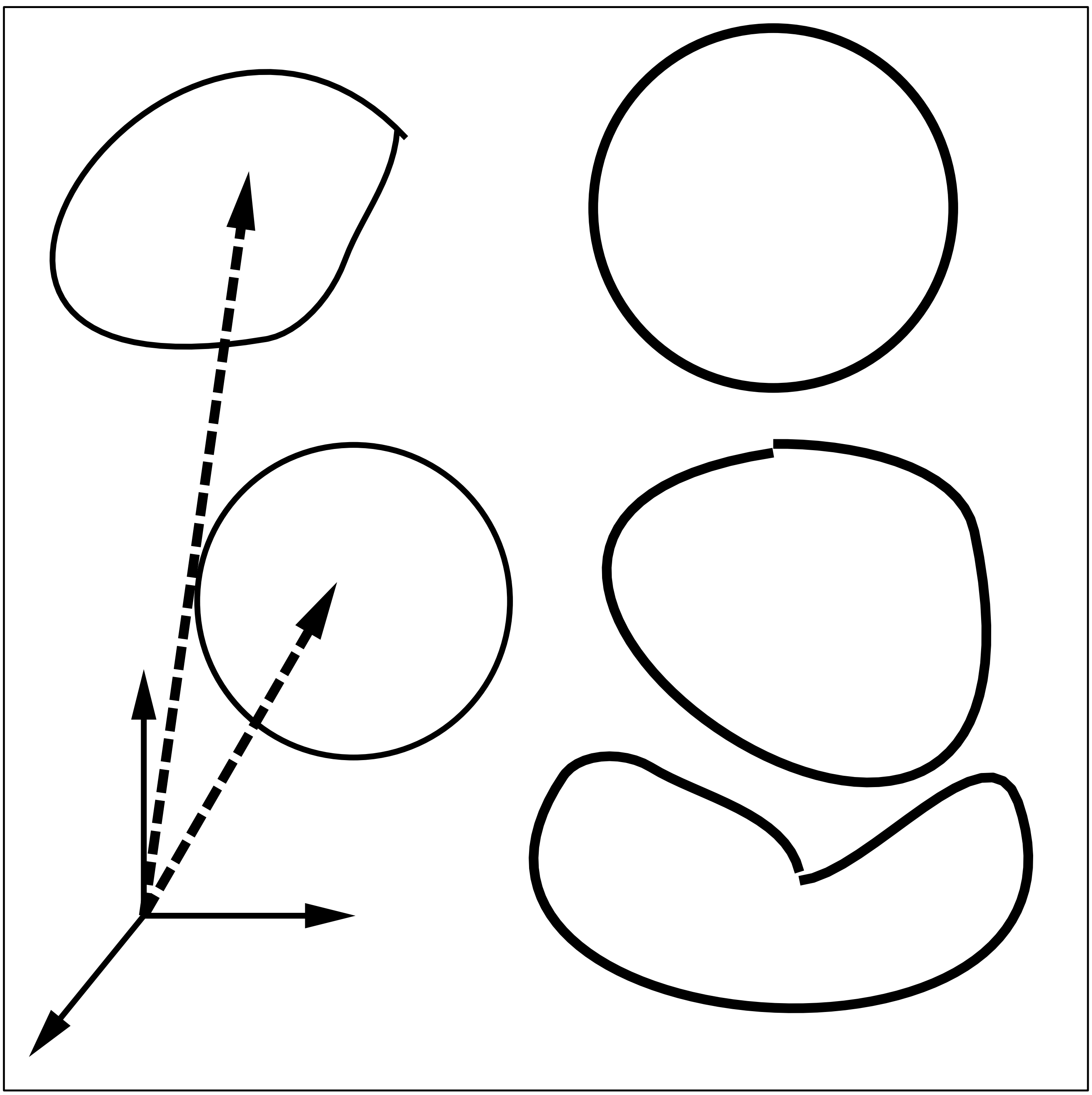
$$\mathbf{f}_v(\mathbf{u}, t) = c(\mathbf{n} \cdot (\mathbf{a}(\mathbf{s}, t) - \frac{\partial \mathbf{s}}{\partial t}))\mathbf{n}$$

- **Repulsive forces**

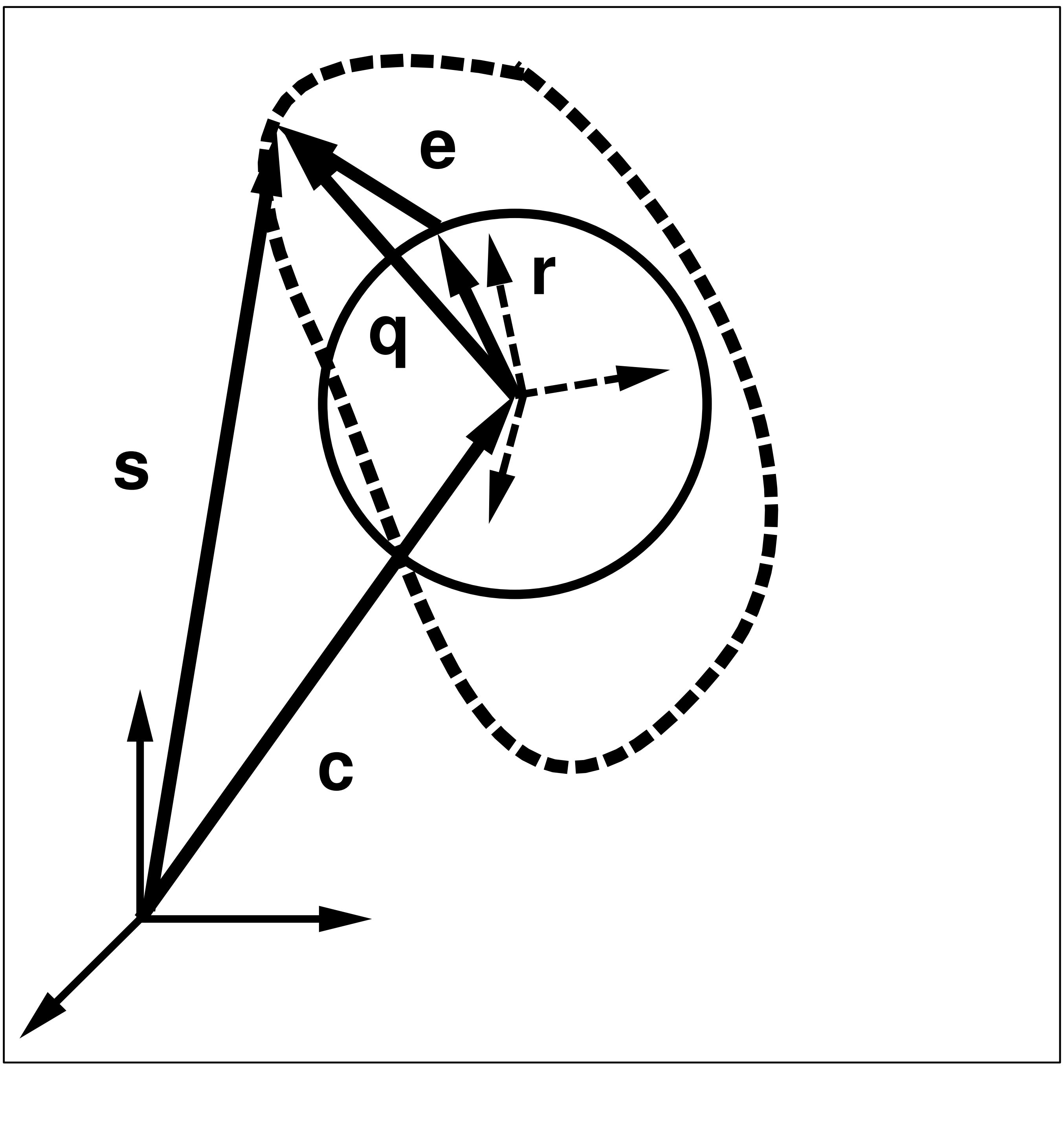
$$\mathbf{f}_c(\mathbf{u}, t) = -c((\nabla f(\mathbf{s})/\varepsilon)exp(-f(\mathbf{s})/\varepsilon) \cdot \mathbf{n})\mathbf{n}$$

- **Friction**

Deformation



Hybrid Formulation



Elastic and Inelastic Deformation

- **Elastic unit**

$$\mathrm{f} = k\mathrm{e}$$

- **Viscous unit**

$$\mathrm{f} = k\dot{\mathrm{e}}$$

- **Plastic unit**

$$\mathrm{f} = c$$

- **Viscoelasticity: elasticity + viscosity**

$$a_2\ddot{e} + a_1\dot{e} + a_0 e = b_2\ddot{f} + b_1\dot{f} + b_0 f$$

- **Elastoplastic**
- **Viscoplastic**
- **Fracture**
- **How to achieve various behaviors**

– **material properties**
– **control functions**

Numerical Implementation

- Discretization
  - finite-difference
  - finite-element
- Time integration

Variational Analysis

- **Energy functional** $\mathcal{E}(\mathrm{s})$

$$\int_{\Omega} \|\mathbf{G} - \mathbf{G}^0\|_{w^1}^2 + \|\mathbf{B} - \mathbf{B}^0\|_{w^2}^2 du_1 du_2$$

where $\|\cdot\|_{w^1}$ **and** $\|\cdot\|_{w^2}$ **are the weighted matrix norms, and** $w_{ij}^1(u_1, u_2)$ **and** $w_{ij}^2(u_1, u_2)$ **are the weighting functions**

- **The first variational derivative** $\delta_{\mathrm{s}}\mathcal{E}(\mathrm{s})$

$$\sum_{i,j=1}^{2} \left(-\frac{\partial}{\partial u_i}\left(\alpha_{ij}\frac{\partial \mathbf{s}}{\partial u_j}\right) + \frac{\partial^2}{\partial u_i \partial u_j}\left(\beta_{ij}\frac{\partial^2 \mathbf{s}}{\partial u_i \partial u_j}\right)\right)$$

where control functions

$$\alpha(\mathbf{u}, \mathbf{s}) = w_{ij}^1(\mathbf{u})(\mathbf{G}_{ij} - \mathbf{G}_{ij}^0)$$

$$\beta_{ij}(\mathbf{u}, \mathbf{s}) = w_{ij}^2(\mathbf{u})(\mathbf{B}_{ij} - \mathbf{B}_{ij}^0)$$

Discretization

- The continuous domain Ω is $0 <= u_1, u_2 <= 1$

- The surface is a regular $M \times N$ discrete grid of nodes: $\mathrm{s}(mh_1, nh_2, t)$, rewrite $\mathrm{s}(\mathbf{u}, t)$ as $\mathrm{s}[m, n]$ where $h_1 = 1/(M-1)$, and $h_2 = 1/(N-1)$

- Spatial derivatives (forward or backward difference operators)

$$\mathrm{s}_{u_1}[m, n] = (\mathrm{s}[m+1, n] - \mathrm{s}[m, n])/h_1$$

$$\mathrm{s}_{u_2}[m, n] = (\mathrm{s}[m, n+1] - \mathrm{s}[m, n])/h_2$$

$$\mathrm{s}_{u_i u_j}[m, n] = \ldots$$

- The discretization of $\alpha_{ij}[m, n]$, $\beta_{ij}[m, n]$, w_{ij}^1, and w_{ij}^2

- The discretization of the first variational derivative

- Boundary conditions!

- **The discretization of the mass and damping density functions**

- **Motion equation (in matrix form)**

$$\mathbf{M}\frac{d^2\mathbf{s}}{dt^2} + \mathbf{C}\frac{d\mathbf{s}}{dt} + \mathbf{k}(\mathbf{s})\mathbf{s} = \mathbf{f}(\mathbf{s})$$

Simulation

- **Time derivatives**

$$\frac{d^2\mathbf{s}}{dt^2} = \frac{\mathbf{s}(t+\delta t) - 2\mathbf{s}(t) + \mathbf{s}(t-\delta t)}{\delta t^2}$$

$$\frac{d\mathbf{s}}{dt} = \frac{\mathbf{s}(t+\delta t) - \mathbf{s}(t-\delta t)}{2\delta t}$$

- **Discretized motion equation**

$$\mathbf{A}(t)\mathbf{s}(t+\delta t) = \mathbf{g}(t)$$

where

$$\mathbf{A}(\mathbf{s}(t)) = \mathbf{K}(\mathbf{s}(t)) + \frac{1}{\delta t^2}\mathbf{M} + \frac{1}{2\delta t}\mathbf{C}$$

$$\mathbf{g}(t) = \mathbf{f}(t) + (\frac{1}{\delta t^2}\mathbf{M} + \frac{1}{2\delta t}\mathbf{C})\mathbf{s}(t) +$$

$$(\frac{1}{\delta t}\mathbf{M} - \frac{1}{2\delta t}\mathbf{C})\frac{\mathbf{s}(t) - \mathbf{s}(t-\delta t)}{\delta t}$$

- **Initial-value conditions!**

Hybrid formulation

- **The variational derivative** $\delta_{\mathbf{e}}\mathcal{E}(\mathbf{e})$

$$w_{00}\mathbf{e} - \frac{\partial}{\partial u_1}(w_{10}\frac{\partial \mathbf{e}}{\partial u_1}) - \frac{\partial}{\partial u_2}(w_{01}\frac{\partial \mathbf{e}}{\partial u_2})$$

$$+\frac{\partial^2}{\partial u_1^2}(w_{20}\frac{\partial^2 \mathbf{e}}{\partial u_1^2}) + 2\frac{\partial^2}{\partial u_1 \partial u_2}(w_{11}\frac{\partial^2 \mathbf{e}}{\partial u_1 \partial u_2})$$

$$+\frac{\partial^2}{\partial u_2^2}(w_{02}\frac{\partial^2 \mathbf{e}}{\partial u_2^2})$$

- **Spatial discretization for** $\delta_e\mathcal{E}(\mathbf{e})$: $\mathbf{Ke}$

- **Ordinary differential equations**

$$m\frac{d\mathbf{v}}{dt} = \mathbf{g}^{\mathbf{v}}$$

$$\frac{d}{dt}(\mathbf{I}\omega) = \mathbf{g}^{\omega}$$

$$\mathbf{M}\frac{d^2\mathbf{e}}{dt^2} + \mathbf{C}\frac{d\mathbf{e}}{dt} + \mathbf{Ke} = g^{\mathbf{e}}$$

where

$$\mathbf{g}^v = h_1 h_2 (\sum_{m,n} \mathbf{f} - \frac{d}{dt} \sum_{m,n} \mu \frac{d\mathbf{e}}{dt} - \sum_{m,n} \gamma \frac{d\mathbf{s}}{dt})$$

$$\mathbf{g}^\omega = h_1 h_2 (\sum_{m,n} \mathbf{q} \times \mathbf{f} - \frac{d}{dt} \sum_{m,n} \mu \mathbf{q} \times \frac{d\mathbf{e}}{dt} - \sum_{m,n} \gamma \mathbf{q} \times \frac{d\mathbf{s}}{dt})$$

$$\mathbf{g}^e = \mathbf{f} - \mu \frac{d\mathbf{v}}{dt} - \mu\omega \times (\omega \times \mathbf{q}) - 2\mu\omega \times \frac{d\mathbf{e}}{dt} + \mu \frac{d\omega}{dt} \times \mathbf{q}$$

- **Time integration**

$$\mathbf{v}(t + \delta t) = \mathbf{v}(t) + \delta t \mathbf{g}^v(t)/m$$

$$\omega(t + \delta t) = \mathbf{I}^{-1}(t + \delta t)(\mathbf{I}(t)\omega(t) + \delta t \mathbf{g}^\omega(t))$$

$$\mathbf{A}\mathbf{e}(t + \delta t) = \mathbf{g}(t)$$

Computation Issues

- Sparse, linear system
- Data structures
- Direct methods
- Iterative methods
- Efficient algorithms
- Various application examples