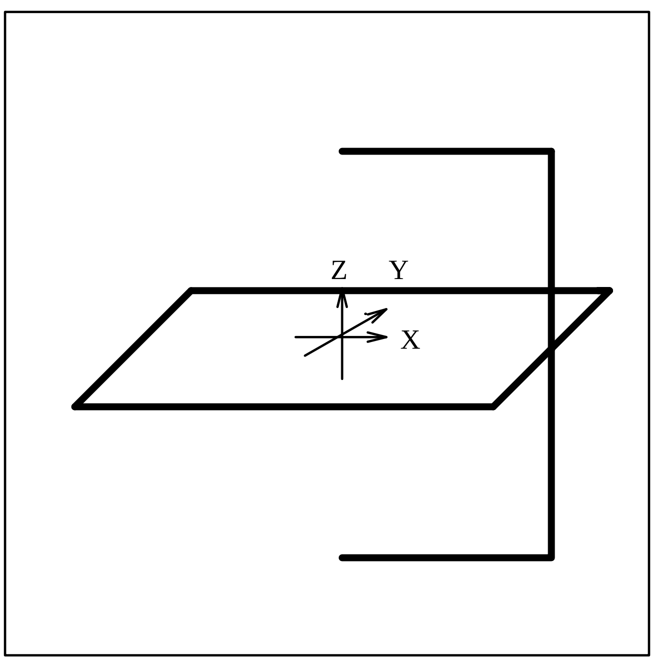
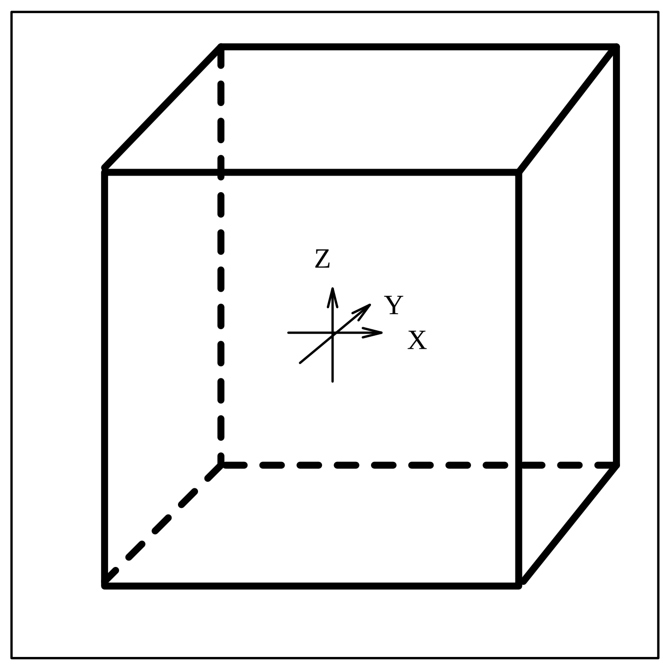
## Geometry of NURBS Swung Surface

- Two NURBS generator curves
  - a profile curve on $x$-$z$ plane
  - a trajectory curve on $x$-$y$ plane
- Symmetries and topological variability
- Broad geometric coverage
- Fewer degrees of freedom: $O(m + n)$
  - compared to general surface: $O(m * n)$

# Cube as a NURBS Swung Surface



# Geometric Design with Swung Surface

- Cross-sectional design—NURBS curve design
- Disadvantages
  - indirect design process
  - DOFs-based requirements
  - restricted functional criteria

## D-NURBS Curve Formulation

- **Curve:** $\mathbf{c}(u,t) = \frac{\sum_{i=0}^{n} \mathbf{p}_i(t) w_i(t) B_{i,k}(u)}{\sum_{i=0}^{n} w_i(t) B_{i,k}(u)}$
- **DOF vector:** $\mathbf{p}(t) = \left[ \begin{array}{cccccc} \mathbf{p}_0^\top & w_0 & \cdots & \mathbf{p}_n^\top & w_n \end{array} \right]^\top$
- **Kinematics:** $\dot{\mathbf{c}}(u,\mathbf{p}) = \mathbf{J}\,\dot{\mathbf{p}}; \quad \mathbf{c}(u,\mathbf{p}) = \mathbf{J}\,\mathbf{p}$
  - **Jacobian matrix:** $\mathbf{J}$

# Curve Jacobian

- **Curve Jacobian is a $3 \times 4(n+1)$ matrix**

$$\mathbf{J}(u, \mathbf{p}) = \begin{bmatrix} \cdots & \begin{bmatrix} \frac{\partial \mathbf{c}_x}{\partial \mathbf{p}_{i,x}} & 0 & 0 \\ 0 & \frac{\partial \mathbf{c}_y}{\partial \mathbf{p}_{i,y}} & 0 \\ 0 & 0 & \frac{\partial \mathbf{c}_z}{\partial \mathbf{p}_{i,z}} \end{bmatrix} & \frac{\partial \mathbf{c}}{\partial w_i} & \cdots \end{bmatrix}$$

**where**

$$\frac{\partial \mathbf{c}_x}{\partial \mathbf{p}_{i,x}} = \frac{\partial \mathbf{c}_y}{\partial \mathbf{p}_{i,y}} = \frac{\partial \mathbf{c}_z}{\partial \mathbf{p}_{i,z}} = \frac{w_i B_{i,k}}{\sum_{j=0}^{n} w_j B_{j,k}};$$

$$\frac{\partial \mathbf{c}}{\partial w_i} = \frac{\sum_{j=0}^{n} (\mathbf{p}_i - \mathbf{p}_j) w_j B_{i,k} B_{j,k}}{(\sum_{j=0}^{n} w_j B_{j,k})^2}$$

# D-NURBS Swung Surface Formulation

- **NURBS generator curves**

$$\mathbf{c}_1(u, \mathbf{a}) = \mathbf{J}_1\mathbf{a}; \qquad \dot{\mathbf{c}}_1(u, \mathbf{a}) = \mathbf{J}_1\dot{\mathbf{a}}$$

$$\mathbf{c}_2(v, \mathbf{b}) = \mathbf{J}_2\mathbf{b}; \qquad \dot{\mathbf{c}}_2(v, \mathbf{b}) = \mathbf{J}_2\dot{\mathbf{b}}$$

- **Curve DOF vectors**

$$\mathbf{a}(t) = [\mathbf{a}_0^\top, w_0^a, \ldots, \mathbf{a}_m^\top, w_m^a]^\top$$

$$\mathbf{b}(t) = [\mathbf{b}_0^\top, w_0^b, \ldots, \mathbf{b}_n^\top, w_n^b]^\top$$

- **Jacobian Matrices**

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \\ \mathbf{Z}_1 \end{bmatrix}; \qquad \mathbf{J}_2 = \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{Y}_2 \\ \mathbf{Z}_2 \end{bmatrix}$$

# D-NURBS Swung Surface Formulation

- **Swung Surface**

$$\mathbf{s}(u,v,t) = \left[ \begin{array}{c} \alpha(t)\mathbf{c}_{1,x}\mathbf{c}_{2,x} \\ \alpha(t)\mathbf{c}_{1,x}\mathbf{c}_{2,y} \\ \mathbf{c}_{1,z} \end{array} \right]$$

- **DOF vector**

$$\mathbf{p} = \left[ \begin{array}{c} \alpha \\ \mathbf{a} \\ \mathbf{b} \end{array} \right]$$

  – **dimensionality:** $M = 1 + 4(m+1) + 4(n+1)$

- **Matrix Form**

$$\mathbf{s}(u,v,\mathbf{p}) = \left[ \begin{array}{c} \alpha(t)(\mathbf{X}_1\mathbf{a})(\mathbf{X}_2\mathbf{b}) \\ \alpha(t)(\mathbf{X}_1\mathbf{a})(\mathbf{Y}_2\mathbf{b}) \\ \mathbf{Z}_1\mathbf{a} \end{array} \right]$$

## Kinematics of D-NURBS Swung Surface

- **Velocity:** $\dot{\mathbf{s}}(u, v, \mathbf{p}) = \mathbf{L}\dot{\mathbf{p}}$
  - **Jacobian matrix:** $\mathbf{L}$
- **Position:** $\mathbf{s}(u, v, \mathbf{p}) = \mathbf{H}\mathbf{p}$
  - **basis function matrix:** $\mathbf{H}$

## <u>Surface Jacobian Matrix</u>

- $\mathbf{L}(u, v, \mathbf{p})$ **is the Jacobian matrix w.r.t.** $\mathbf{p}$**.**
- $\mathbf{L}$ **comprises** $\partial \mathbf{s}/\partial \alpha$**,** $\partial \mathbf{s}/\partial \mathbf{a}$**, and** $\partial \mathbf{s}/\partial \mathbf{b}$
- $$\frac{\partial \mathbf{s}}{\partial \alpha} = \left[ \begin{array}{c} (\mathbf{X}_1\mathbf{a})(\mathbf{X}_2\mathbf{b}) \\ (\mathbf{X}_1\mathbf{a})(\mathbf{Y}_2\mathbf{b}) \\ 0 \end{array} \right] = \mathbf{A}\mathbf{c}_2 = (\mathbf{B} - \mathbf{C})\mathbf{c}_1$$
- $$\frac{\partial \mathbf{s}}{\partial \mathbf{a}} = \left[ \begin{array}{c} \alpha(\mathbf{X}_2\mathbf{b})\mathbf{X}_1 \\ \alpha(\mathbf{Y}_2\mathbf{b})\mathbf{X}_1 \\ \mathbf{Z}_1 \end{array} \right] = \mathbf{B}_\alpha \mathbf{J}_1,$$
- $$\frac{\partial \mathbf{s}}{\partial \mathbf{b}} = \left[ \begin{array}{c} \alpha(\mathbf{X}_1\mathbf{a})\mathbf{X}_2 \\ \alpha(\mathbf{X}_1\mathbf{a})\mathbf{Y}_2 \\ \mathbf{0} \end{array} \right] = \mathbf{A}_\alpha \mathbf{J}_2,$$

## Jacobian Matrix Computation

- **Structure**

$$\mathbf{L} = \left[\begin{array}{ccc} \mathbf{A}\mathbf{c}_2 & \mathbf{B}_\alpha \mathbf{J}_1 & \mathbf{A}_\alpha \mathbf{J}_2 \end{array}\right]$$

  - **auxiliary matrices**

$$\mathbf{A}(u, \mathbf{a}) = \begin{bmatrix} \mathbf{X}_1\mathbf{a} & 0 & 0 \\ 0 & \mathbf{X}_1\mathbf{a} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \mathbf{A}_\alpha(\alpha, u, \mathbf{a}) = \alpha\mathbf{A}$$

$$\mathbf{B}(v, \mathbf{b}) = \begin{bmatrix} \mathbf{X}_2\mathbf{b} & 0 & 0 \\ \mathbf{Y}_2\mathbf{b} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_\alpha(\alpha, v, \mathbf{b}) = \alpha\mathbf{B} + (1 - \alpha)\mathbf{C}$$

# Basis Function Matrix Computation

- **Auxiliary matrices**

$$\mathbf{H}_1 = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{B}_\alpha \mathbf{J}_1 & \mathbf{0} \end{array}\right]; \quad \mathbf{H}_2 = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{C}\mathbf{J}_1 & \mathbf{A}_\alpha \mathbf{J}_2 \end{array}\right]$$

$$\mathbf{H}_3 = \left[\begin{array}{ccc} \mathbf{A}\mathbf{c}_2 & \mathbf{C}\mathbf{J}_1 & \mathbf{0} \end{array}\right]; \quad \mathbf{H}_4 = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{C}\mathbf{J}_1 & \mathbf{0} \end{array}\right]$$

- **Equation**

$$3\mathrm{s}(u, v, \mathbf{p}) = \mathbf{H}_1\mathbf{p} + \mathbf{H}_2\mathbf{p} + \mathbf{H}_3\mathbf{p} = \mathbf{L}\mathbf{p} + 2\mathbf{H}_4\mathbf{p}$$

- **Structure**

$$\mathbf{H} = \frac{(\mathbf{L} + 2\mathbf{H}_4)}{3}$$

## D-NURBS Swung Surface Matrices

- **Mass matrix**

$$\mathbf{M}(\mathbf{p}) = \iint \mu \mathbf{L}^\top \mathbf{L} \, du \, dv$$

- **Damping matrix**

$$\mathbf{D}(\mathbf{p}) = \iint \gamma \mathbf{L}^\top \mathbf{L} \, du \, dv$$

- **Stiffness matrix**

$$\mathbf{K}(\mathbf{p}) = \iint \alpha_{1,1} \mathbf{L}_u^\top \mathbf{H}_u + \alpha_{2,2} \mathbf{L}_v^\top \mathbf{H}_v +$$

$$\beta_{1,1} \mathbf{L}_{uu}^\top \mathbf{H}_{uu} + \beta_{1,2} \mathbf{L}_{uv}^\top \mathbf{H}_{uv} + \beta_{2,2} \mathbf{L}_{vv}^\top \mathbf{H}_{vv} \, du \, dv$$

## Dynamics of D-NURBS Swung Surface

- **Motion equation**

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}_p + \mathbf{g}_p$$

  - **generalized forces**

$$\mathbf{f}_p(\mathbf{p}) = \iint \mathbf{L}^\top \mathbf{f}(u,v,t)\,du\,dv$$

    $\mathbf{f}(u,v,t)$ **is applied force distribution**

  - **generalized inertial forces**

$$\mathbf{g}_p(\mathbf{p}) = -\iint \mu \mathbf{L}^\top \dot{\mathbf{L}}\dot{\mathbf{p}}\,du\,dv$$

## Matrix Structure & Computation

- **Block matrix expression**

$$\mathbf{L}^\top \mathbf{L} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \mathbf{F}_{1,3} \\ & \mathbf{F}_{2,2} & \mathbf{F}_{2,3} \\ \text{symmetric} & & \mathbf{F}_{3,3} \end{bmatrix}$$

  where
  $\mathbf{F}_{1,1} = \mathbf{c}_2^\top \mathbf{A}^\top \mathbf{A} \mathbf{c}_2 = (\mathbf{X}_1 \mathbf{a})^2 \|\mathbf{c}_2\|^2$,
  $\mathbf{F}_{1,2} = \mathbf{c}_2^\top \mathbf{A}^\top \mathbf{B}_\alpha \mathbf{J}_1 = \alpha \|\mathbf{c}_2\|^2 (\mathbf{X}_1 \mathbf{a}) \mathbf{X}_1$,
  $\mathbf{F}_{1,3} = \mathbf{c}_2^\top \mathbf{A}^\top \mathbf{A}_\alpha \mathbf{J}_2 = \alpha (\mathbf{X}_1 \mathbf{a})^2 \mathbf{c}_2^\top \mathbf{J}_2$,
  $\mathbf{F}_{2,2} = \mathbf{J}_1^\top \mathbf{B}_\alpha^\top \mathbf{B}_\alpha \mathbf{J}_1 = \alpha^2 \|\mathbf{c}_2\|^2 \mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{Z}_1^\top \mathbf{Z}_1$,
  $\mathbf{F}_{2,3} = \mathbf{J}_1^\top \mathbf{B}_\alpha^\top \mathbf{A}_\alpha \mathbf{J}_2 = \alpha^2 \mathbf{X}_1^\top (\mathbf{X}_1 \mathbf{a}) \mathbf{c}_2^\top \mathbf{J}_2$, and
  $\mathbf{F}_{3,3} = \mathbf{J}_2^\top \mathbf{A}_\alpha^\top \mathbf{A}_\alpha \mathbf{J}_2 = \alpha^2 (\mathbf{X}_1 \mathbf{a})^2 \mathbf{J}_2^\top \mathbf{J}_2$

- **How about stiffness matrix?!**

# D-NURBS Surface Formulation

- **Surface:** $\mathrm{s}(u,v,t) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{q}_{i,j}(t)w_{i,j}(t)B_{i,k}(u)B_{j,l}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n}w_{i,j}(t)B_{i,k}(u)B_{j,l}(v)}$

- **DOF vector:** $\mathbf{q}(t) = \left[\begin{array}{cccc} \cdots & \mathbf{q}_{i,j}^{\top} & w_{i,j} & \cdots \end{array}\right]^{\top}$
  - **dimensionality:** $N = 4(m+1)(n+1)$

- **Kinematics:** $\dot{\mathrm{s}}(u,v,\mathbf{q}) = \mathbf{J}\dot{\mathbf{q}}$; $\mathsf{v}_{s,(}u,v,\mathbf{q}) = \mathbf{J}\mathbf{q}$

- **Dynamics:** $\mathbf{M}_q\ddot{\mathbf{q}} + \mathbf{D}_q\dot{\mathbf{q}} + \mathbf{K}_q{=}\mathbf{f}_q + \mathbf{g}_q$
  - $\mathbf{M}_q$, $\mathbf{D}_q$, **and** $\mathbf{K}_q$ **are** $N \times N$ **matrices**

# Nonlinear Constraints

- **Position**

$$\begin{aligned} \mathbf{q}_{i,j} &= \begin{bmatrix} \alpha \mathbf{a}_{i,x} \mathbf{b}_{j,x} \\ \alpha \mathbf{a}_{i,x} \mathbf{b}_{j,y} \\ \mathbf{a}_{i,z} \end{bmatrix} \\ w_{i,j} &= w_i^a w_j^b \end{aligned}$$

- **Velocity**

$$\begin{aligned} \dot{\mathbf{q}}_{i,j} &= \begin{bmatrix} \dot{\alpha} \mathbf{a}_{i,x} \mathbf{b}_{j,x} + \alpha \dot{\mathbf{a}}_{i,x} \mathbf{b}_{j,x} + \alpha \mathbf{a}_{i,x} \dot{\mathbf{b}}_{j,x} \\ \dot{\alpha} \mathbf{a}_{i,x} \mathbf{b}_{j,y} + \alpha \dot{\mathbf{a}}_{i,x} \mathbf{b}_{j,y} + \alpha \mathbf{a}_{i,x} \dot{\mathbf{b}}_{j,y} \\ \dot{\mathbf{a}}_{i,z} \end{bmatrix} \\ \dot{w}_{i,j} &= \dot{w}_i^a w_j^b + w_i^a \dot{w}_j^b \end{aligned}$$

# Constrained D-NURBS

- **DOF vectors**

$$\dot{\mathbf{q}} = \mathbf{G}\dot{\mathbf{p}}; \quad \mathbf{q} = \mathbf{B}\mathbf{p}$$

  - $\mathbf{B}$ **and** $\mathbf{G}$ **are** $N \times M$ **matrices**

- **Matrices**

$$\mathbf{M} = \mathbf{G}^\top \mathbf{M}_q \mathbf{G}; \quad \mathbf{D} = \mathbf{G}^\top \mathbf{D}_q \mathbf{G}; \quad \mathbf{K} = \mathbf{G}^\top \mathbf{K}_q \mathbf{B}$$

- **Generalized forces**

$$\mathbf{f}_p = \mathbf{G}^\top \mathbf{f}_q; \quad \mathbf{g}_p = \mathbf{G}^\top (\mathbf{g}_q \mathbf{M}_q \dot{\mathbf{G}} \dot{\mathbf{p}})$$

# Applications

- Solid rounding
- Surface blending
- Scattered data fitting
- Interactive sculpting

## Conclusion

- Developed dynamic NURBS swung surface
  - two D-NURBS profile curves
  - constrained D-NURBS surface
- Physics-based design methodology
  - direct and physical interaction
  - constraint-based paradigms
  - automatic weight adjustment
- Unified approach for various applications