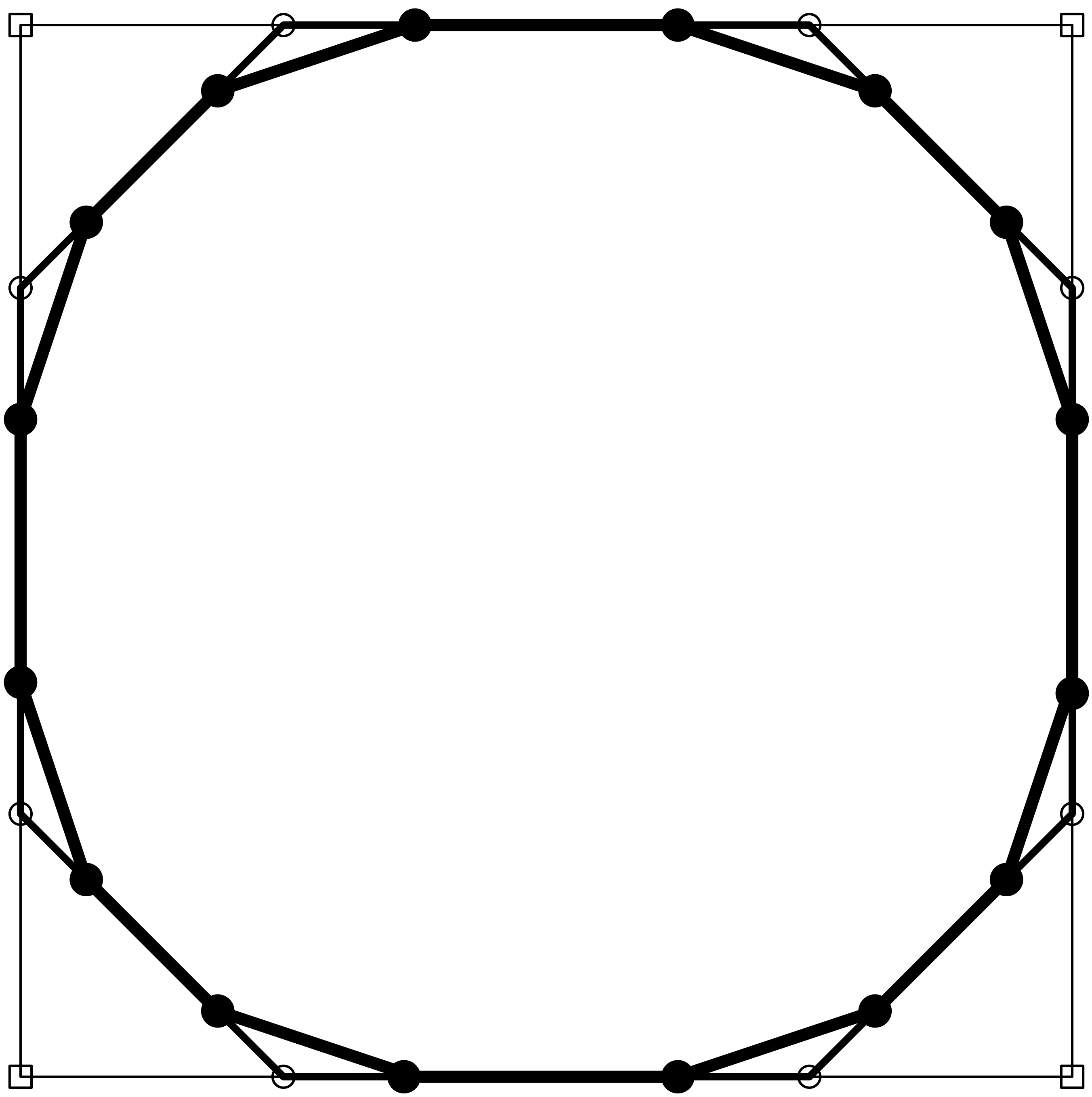
Surface Modeling Techniques

- Fundamental techniques
  - interpolation, approximation
- Limitations
  - the network of data points is regular mesh
  - not useful for solid modeling
  - only interpolate or approximate points, not tangent plane, normal, etc

Recursive Subdivision

- Chaikin's idea (1974)
- A smooth curve from a polygonal shape
- Quadratic B-spline
- Tangent to each edge at its middle point
- Generalize to smooth surface from polyhedron

Subdivision Forms



Chaikin's Algorithm

- **A set of control points**

$$\mathbf{p}_0^0, \mathbf{p}_1^0, \ldots, \mathbf{p}_n^0$$

- **Subdivision process**

$$\mathbf{p}_0^k, \mathbf{p}_1^k, \ldots, \mathbf{p}_{2^k n}^k$$

- **Rules:**

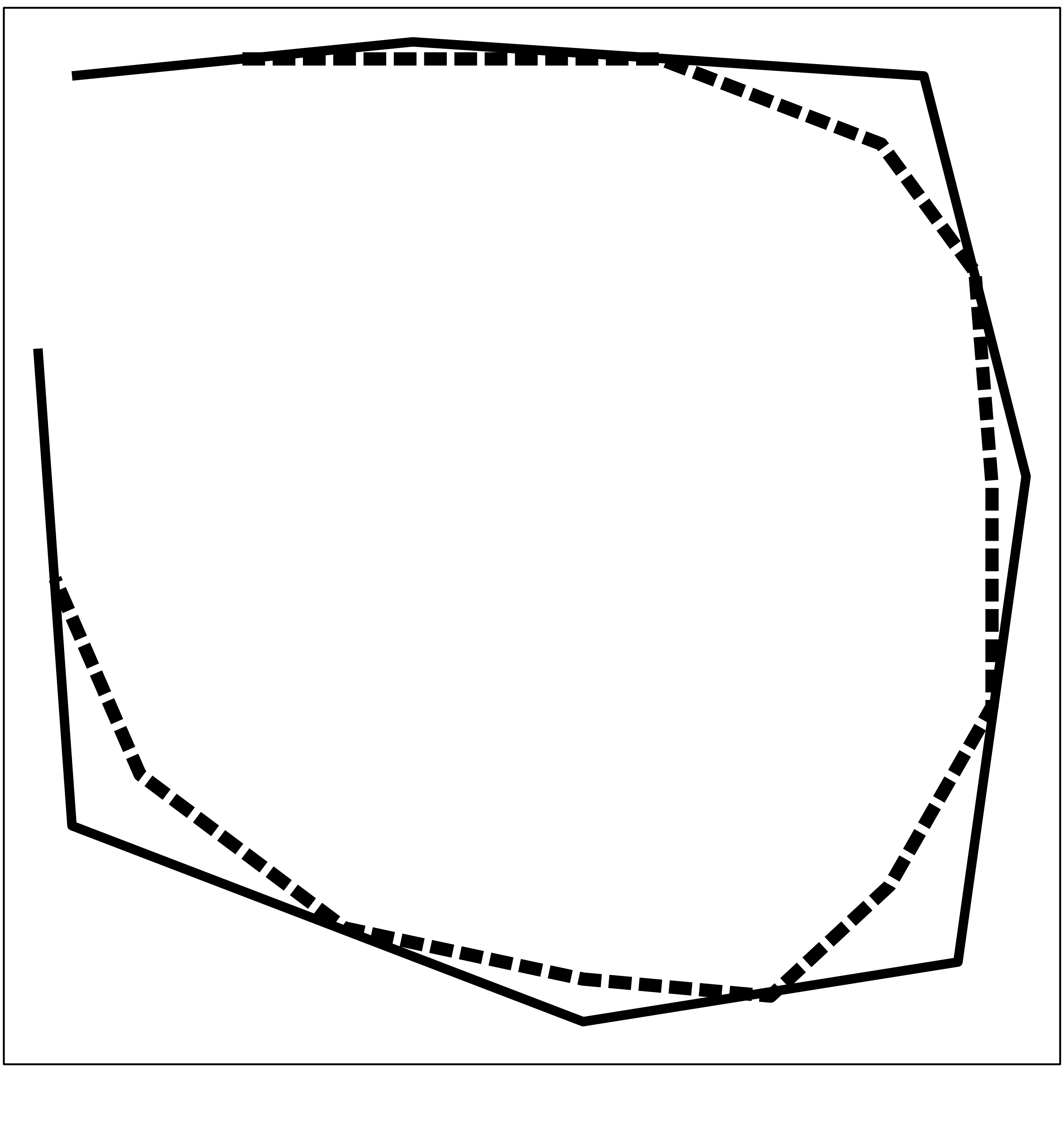
$$\mathbf{p}_{2i}^{k+1} = \frac{3}{4}\mathbf{p}_i^k + \frac{1}{4}\mathbf{p}_{i+1}^k$$

$$\mathbf{p}_{2i+1}^{k+1} = \frac{1}{4}\mathbf{p}_i^k + \frac{3}{4}\mathbf{p}_{i+1}^k$$

- **Properties:**
  - **quadratic B-spline curve**
  - C^1 **continuous**

- **Other rules!!!**

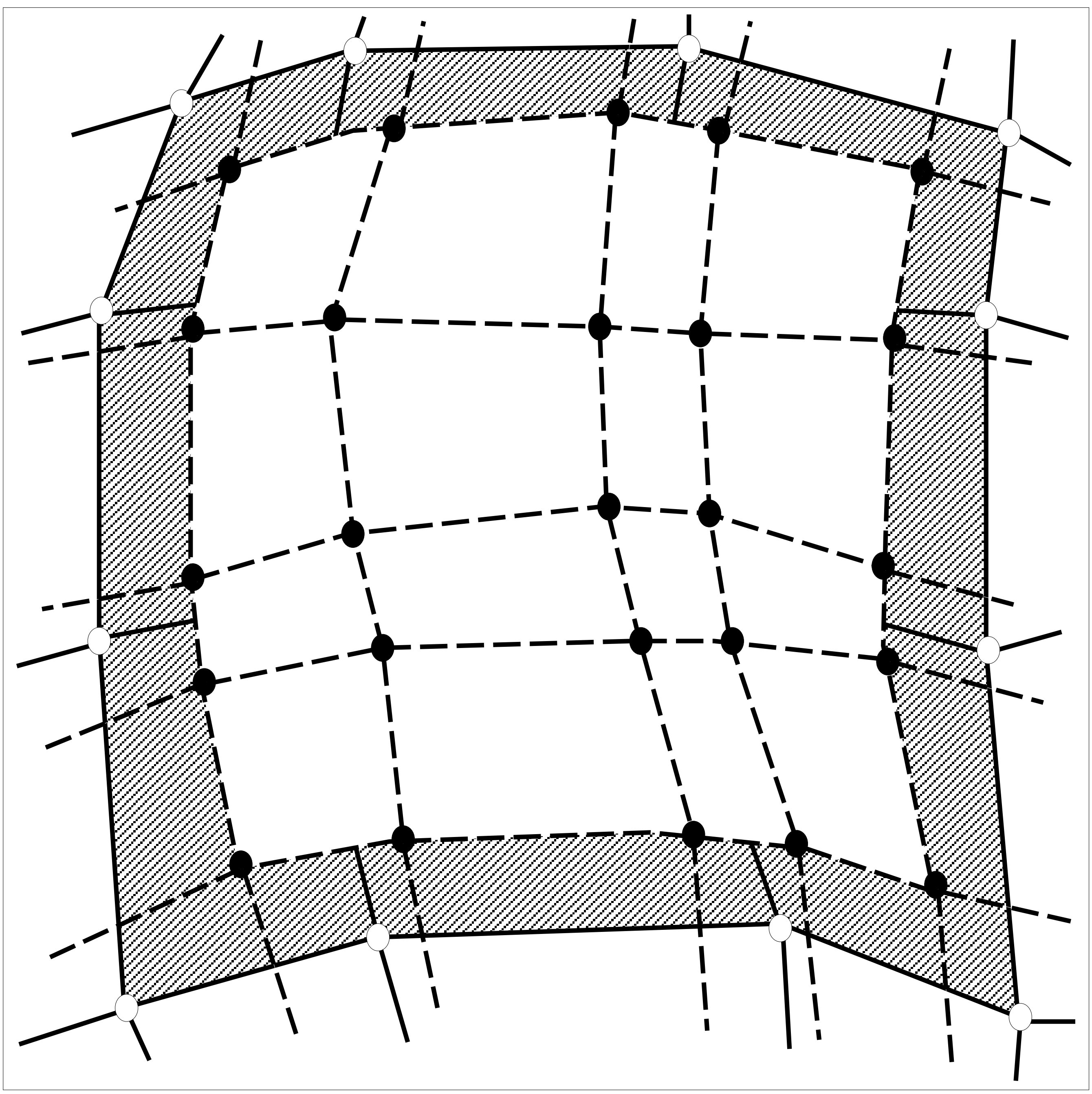
Cubic Spline Subdivision



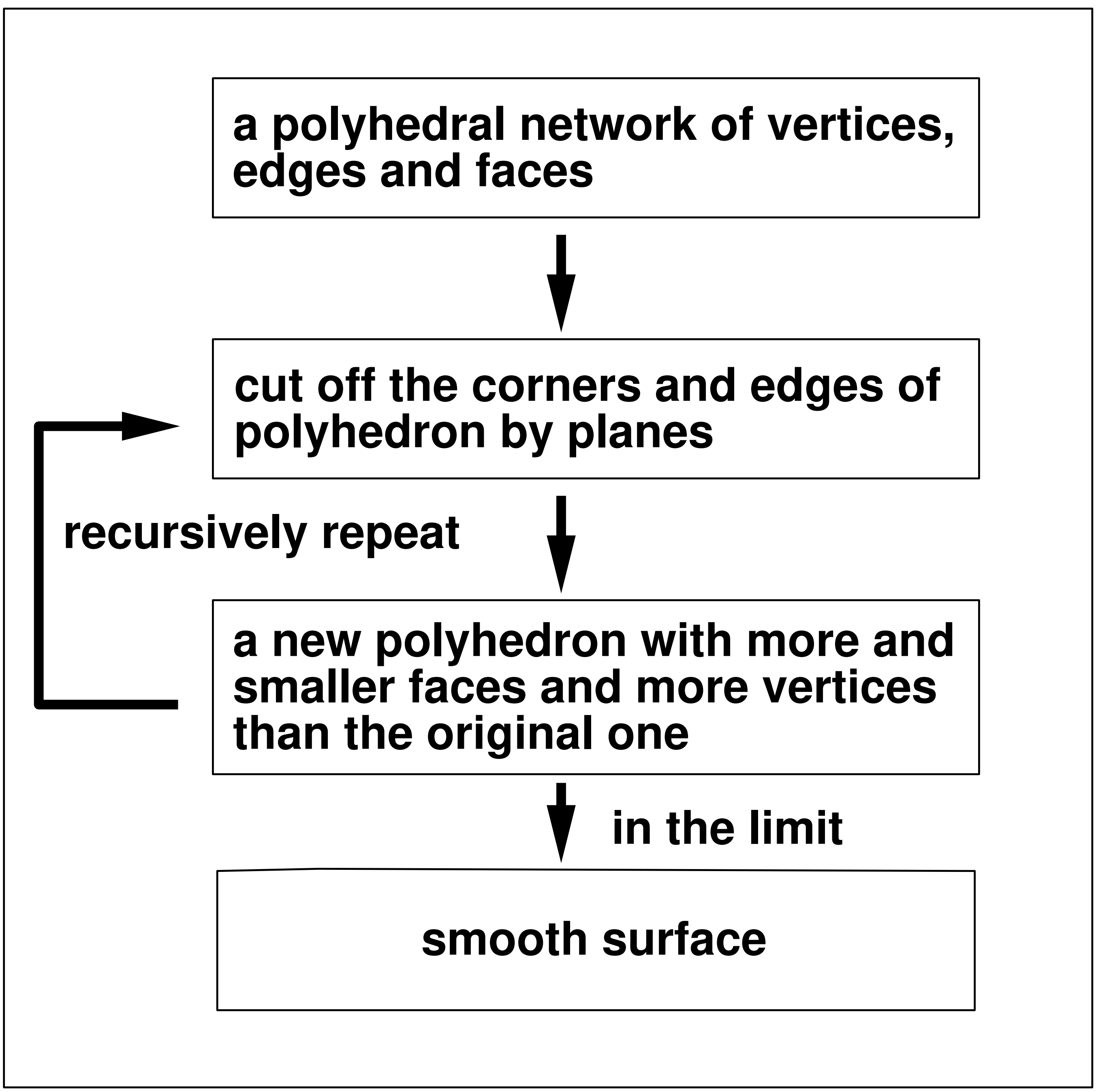
Cubic Spline Subdivision

- $$\mathbf{p}_{2i}^{k+1} = \frac{1}{2}\mathbf{p}_i^k + \frac{1}{2}\mathbf{p}_{i+1}^k$$
- $$\mathbf{p}_{2i+1}^{k+1} = \frac{1}{4}(\frac{1}{2}\mathbf{p}_i^k + \frac{1}{2}\mathbf{p}_{i+2}^k) + \frac{3}{4}\mathbf{p}_{i+1}^k$$
- **Cubic B-spline**
- C^2 **continuous**
- **Corner-chopping!**
- **NOT interpolatory!**

Subdivision Surface



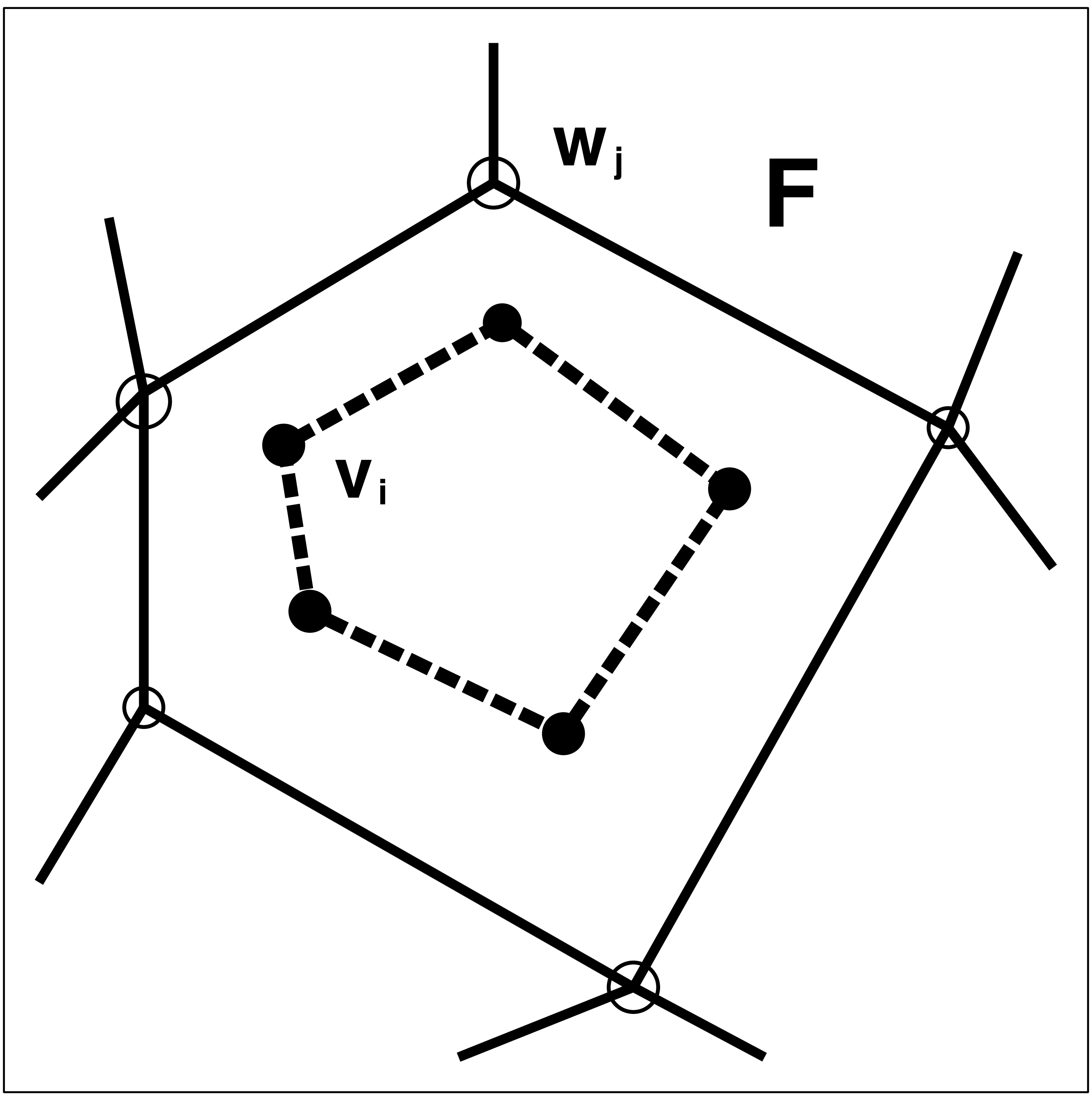
Recursive Subdivision (Idea)



Recursive Subdivision (Rules)

- How NEW vertices are constructed DIFFERS tremendously!!!
- Sabin & Doo approach (1978)
  - extraordinary points corresponding to the vertices when n edges meet (n is not 4)!
  - the limit surface is C^1
  - interpolate the centroids of all faces at every step of subdivision
  - rules: $\mathrm{v}_i = \sum_{j=1}^{n} \alpha_{ij} \mathrm{w}_j$
  - v_i: new vertex
  - w_j: old vertex that defines a face F

Sabin & Doo Algorithm



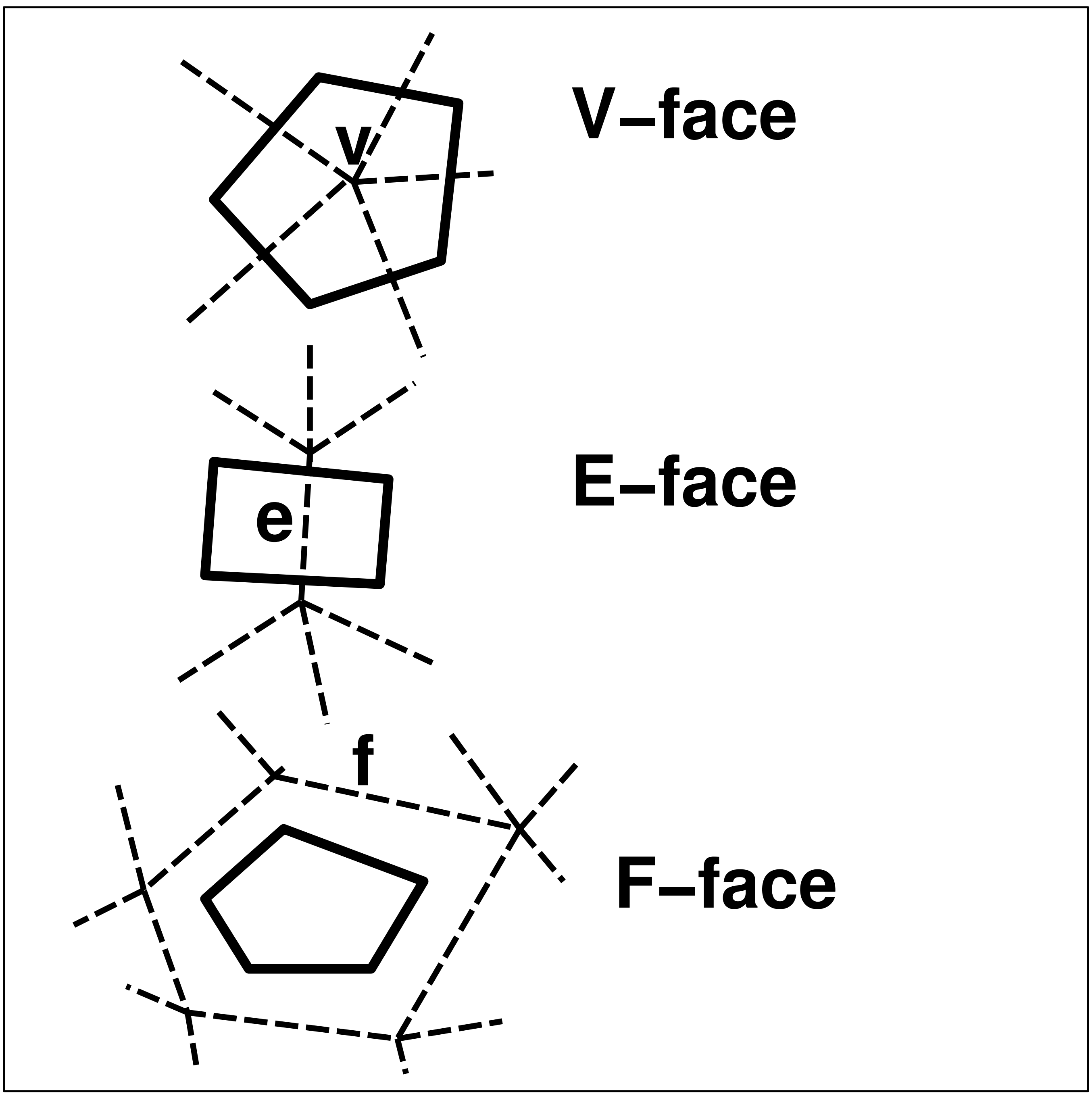
<u>Sabin & Doo Algorithm</u>

- **Subdivision Rules**

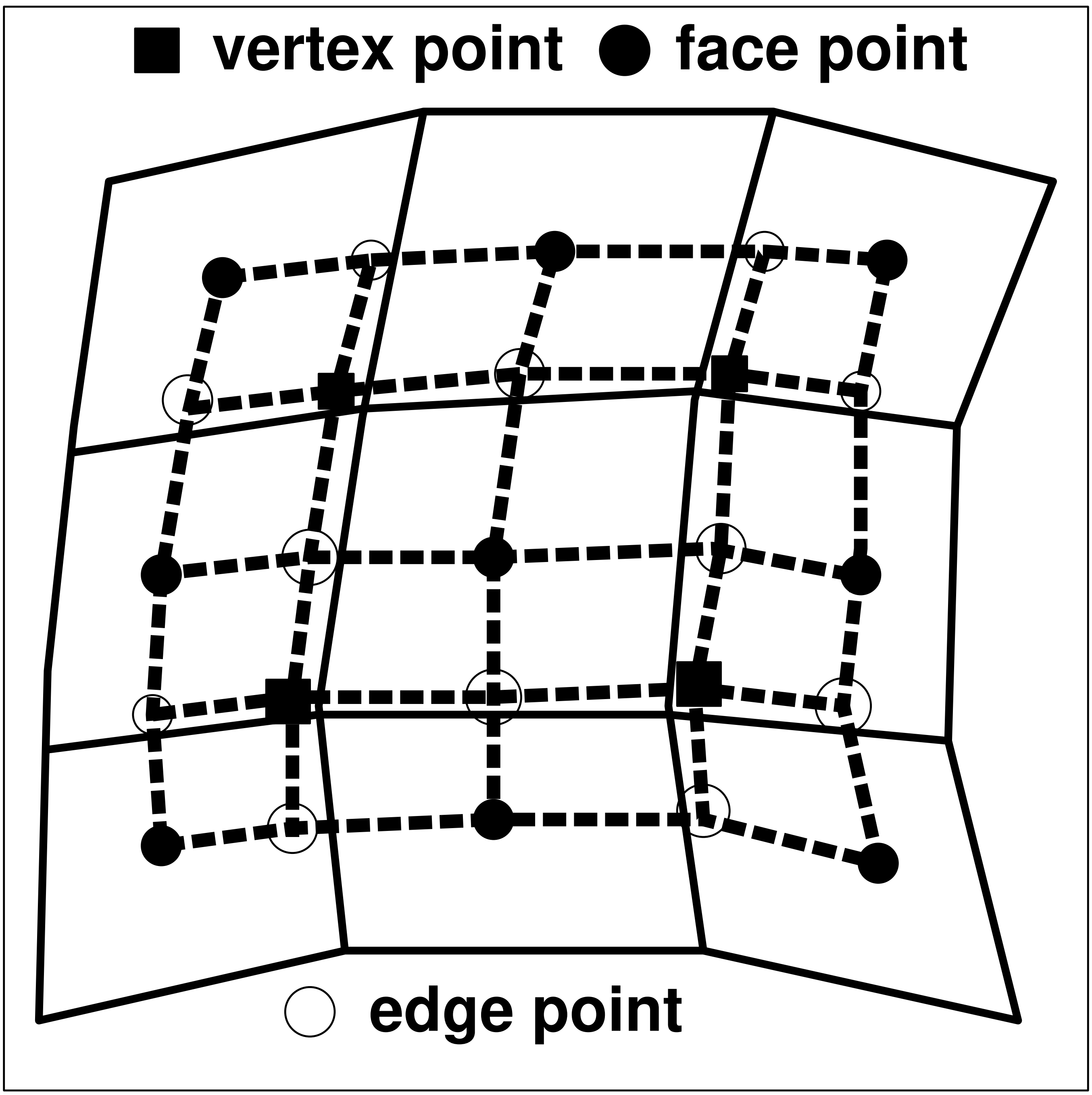
$$\alpha_{ij} = \begin{cases} \frac{n+5}{4n} & i = j \\ \frac{3+2cos(\frac{2(i-j)\pi}{n})}{4n} & otherwise \end{cases}$$

- **connect new vertices and generate new faces and edges**
  - **V-face**
  - **E-face**
  - **F-face**

Sabin & Doo Algorithm



Catmull-Clark Surface



Catmull-Clark Subdivision

- NEW face points — the average of all OLD points defining the face
- NEW edge points — the average of the following four points:
  two OLD vertices defining the edge and two NEW face points of the faces adjacent to the edge
- NEW vertex points — from the average:

  $$\frac{F}{n} + \frac{2E}{n} + \frac{(n-3)V}{n}$$

  where
  F: the average of the NEW face points of all faces adjacent to the OLD vertex point
  E: the average of the midpoints of all edges incident on the OLD vertex
  V: OLD vertex point
  n: the number of the edges incident on the vertex
- NEW edges: (NEW face point, NEW edge point)

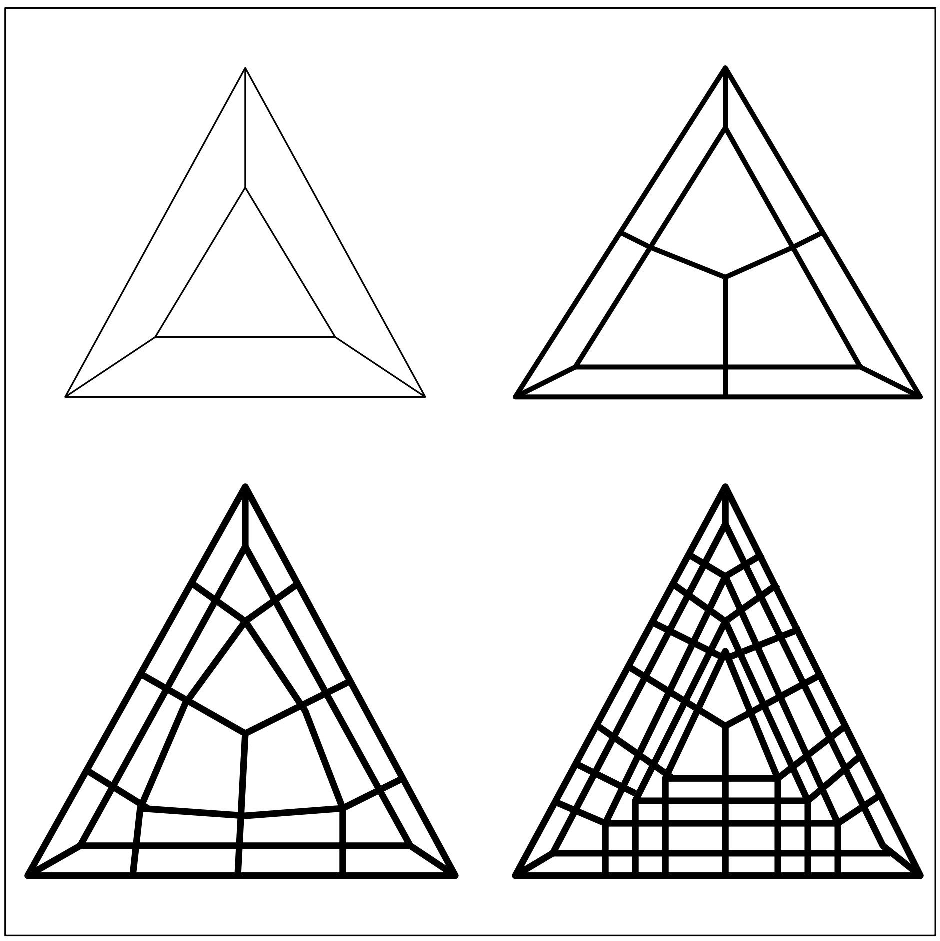
- **NEW edges: (NEW vertex point, NEW edge point)**

- **NEW faces: enclosed by NEW edges**

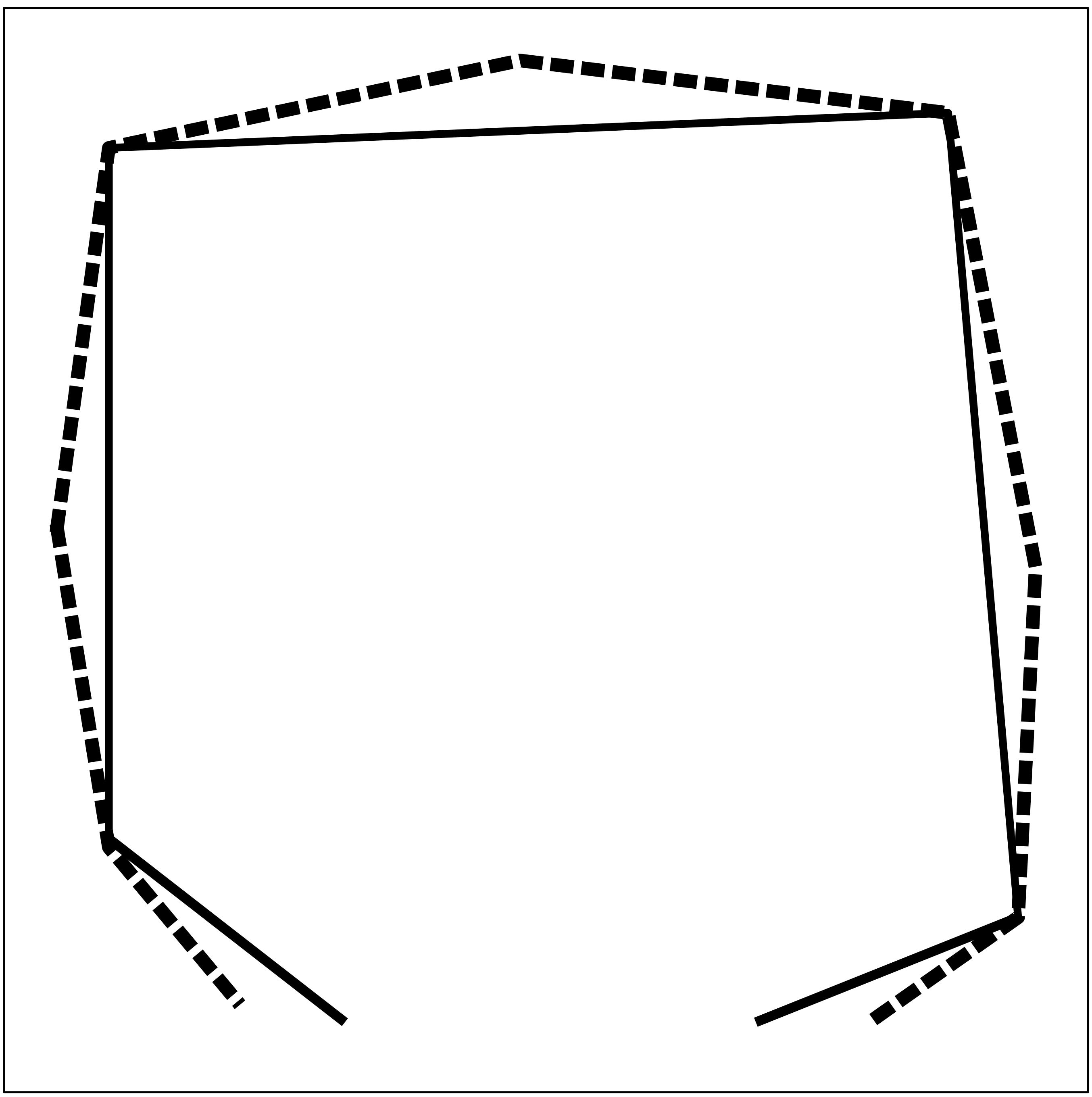
Catmull-Clark Surfaces

- NO parametric domains!
- Geometric objects of arbitrary topology
- Special case: standard bicubic B-spline surfaces
- A finite number of extraordinary points
- Curvature-continuous except at extraordinary vertices
- Tangent-plane-continuous at extraordinary vertices

Catmull-Clark Surface



Interpolation Subdivision



Interpolation Subdivision

- Control points

$$\mathbf{p}^0_{-2}, \mathbf{p}^0_{-1}, \ldots, \mathbf{p}^0_{n+2}$$

- Rules:

$$\mathbf{p}^{k+1}_{2i} = \mathbf{p}^k_i, -1 <= i <= 2^k n + 1$$

$$\mathbf{p}^{k+1}_{2i+1} = (\frac{1}{2} + w)(\mathbf{p}^k_i + \mathbf{p}^k_{i+1}) - w(\mathbf{p}^k_{i-1} + \mathbf{p}^k_{i+2})$$

  where $-1 <= i <= 2^k n$

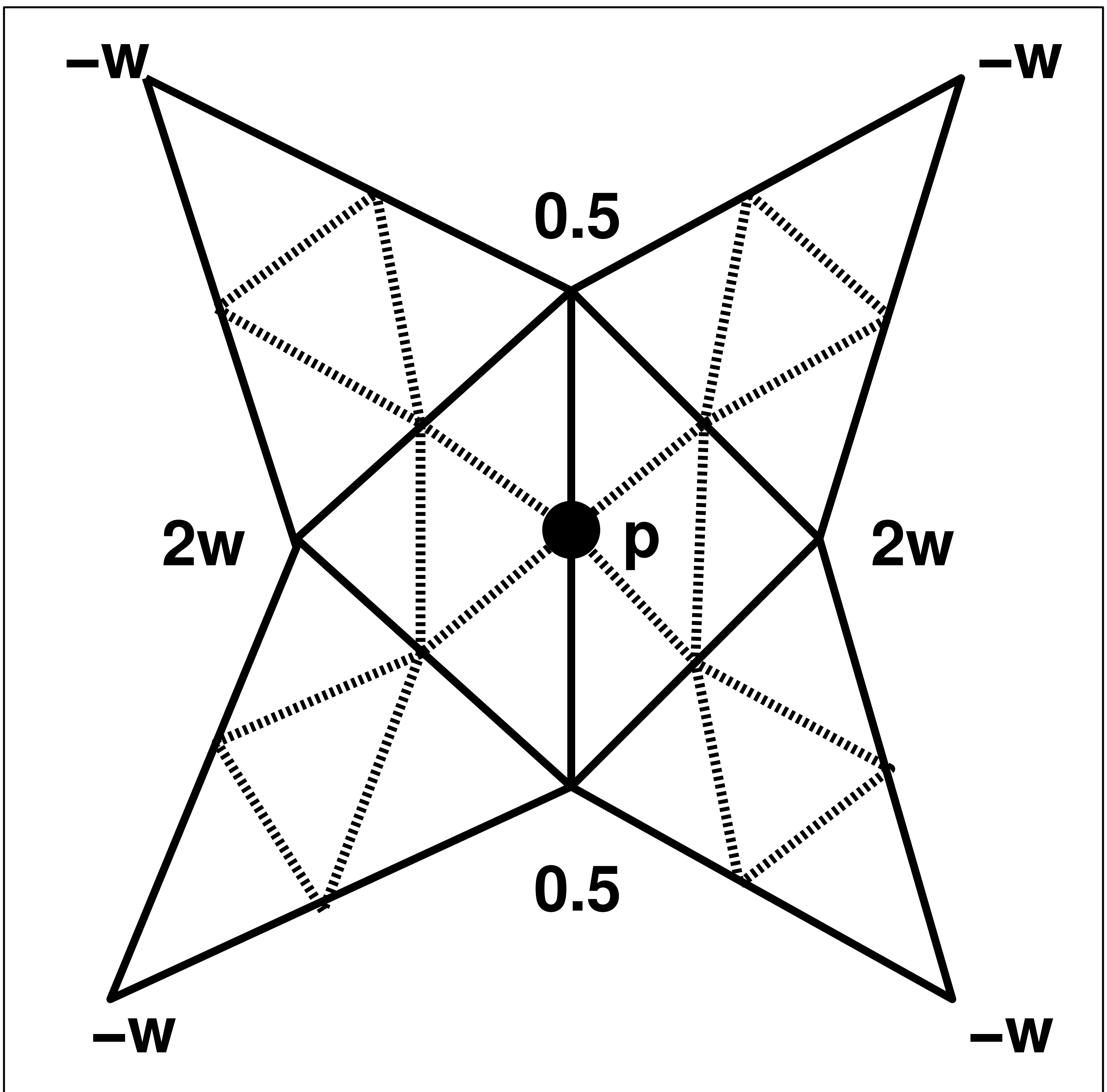
- At each stage, we keep all the OLD points and insert NEW points "in between" the OLD ones

- Interpolation!

- The behaviors and properties of the limit curve depend on the parameter w

- Generalize to SIX-point interpolatory scheme!

Butterfly Subdivision



Motivation

- Why splines (instead of pixels or voxels)
- Much higher and easy managable level of abstraction
- We are dealing with features (tangent, orientation, curvature, corners, etc)
- Features are invariant under standard geometric transformation

Surface Construction

- Generator curves
- Transfinite patch
- Multiple patch
- “Carpet” method
- Subdivision
- Algebraic patch
- Irregular surface
- Multivariate B-splines
- Other representations

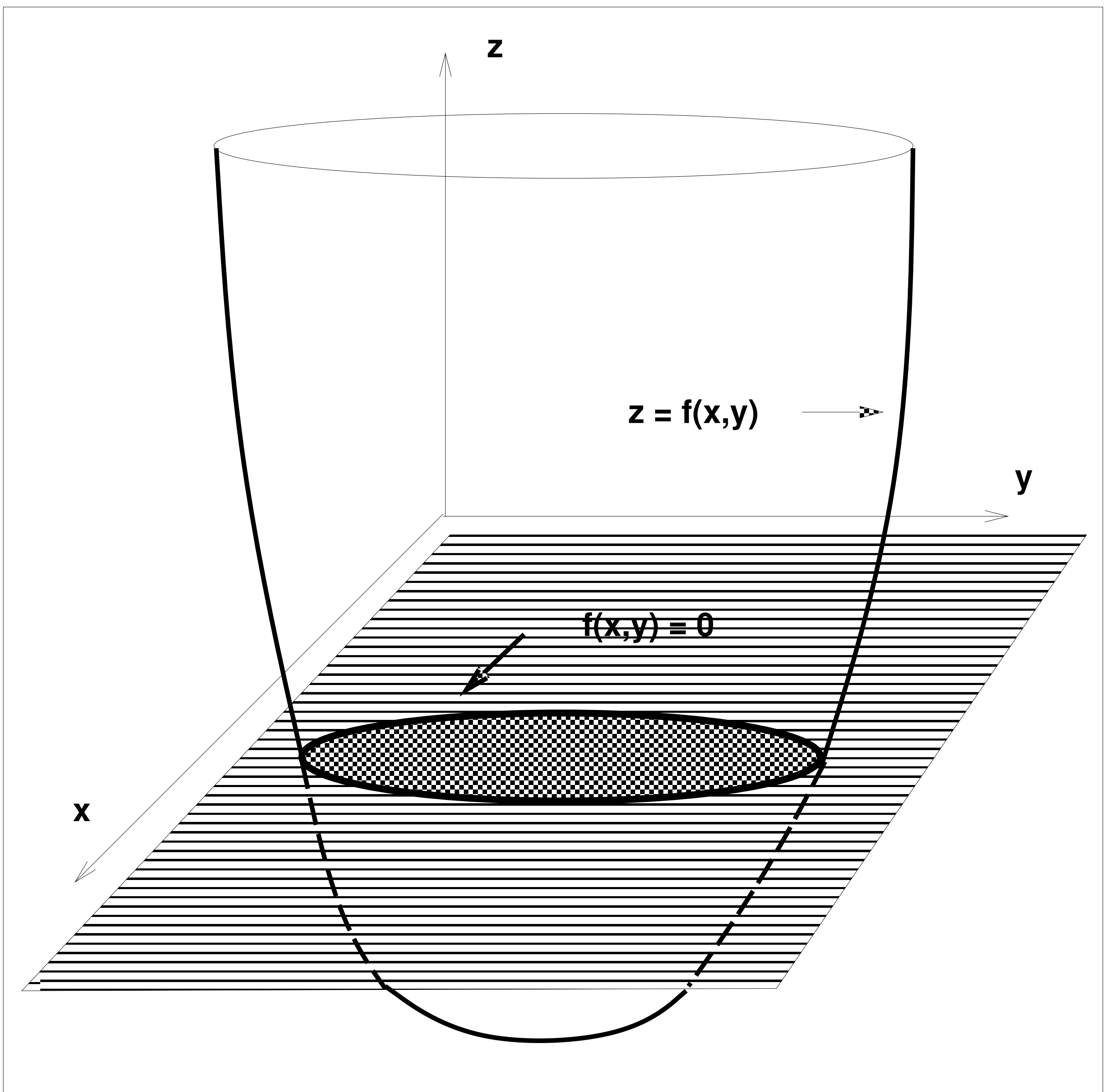
Algebraic Function

- **Parametric representation is popular, but...**
- **Formulation:**

$$\sum_{i,j,k,i+j+k=n} a_{ijk} x^i y^j z^k = 0$$

- **Properties...**
  - **powerful, but lack of modeling tools**

Implicit Function



Algebraic Patch

- **A tetrahedron with noncoplanar vertices**

$$\mathbf{v}_{n000}, \mathbf{v}_{0n00}, \mathbf{v}_{00n0}, \mathbf{v}_{000n}$$

- **Trivariate barycentric coordinate (r, s, t, u) for $\mathbf{p}$**

$$\mathbf{p} = r\mathbf{v}_{n000} + s\mathbf{v}_{0n00} + \mathbf{v}_{00n0} + u\mathbf{v}_{000n}$$

where $r + s + t + u = 1$

- **A regular lattice of control points and weights**

$$\mathbf{p}_{ijkl} = \frac{i\mathbf{v}_{n000} + j\mathbf{v}_{0n00} + k\mathbf{v}_{00n0} + l\mathbf{v}_{000n}}{n}$$

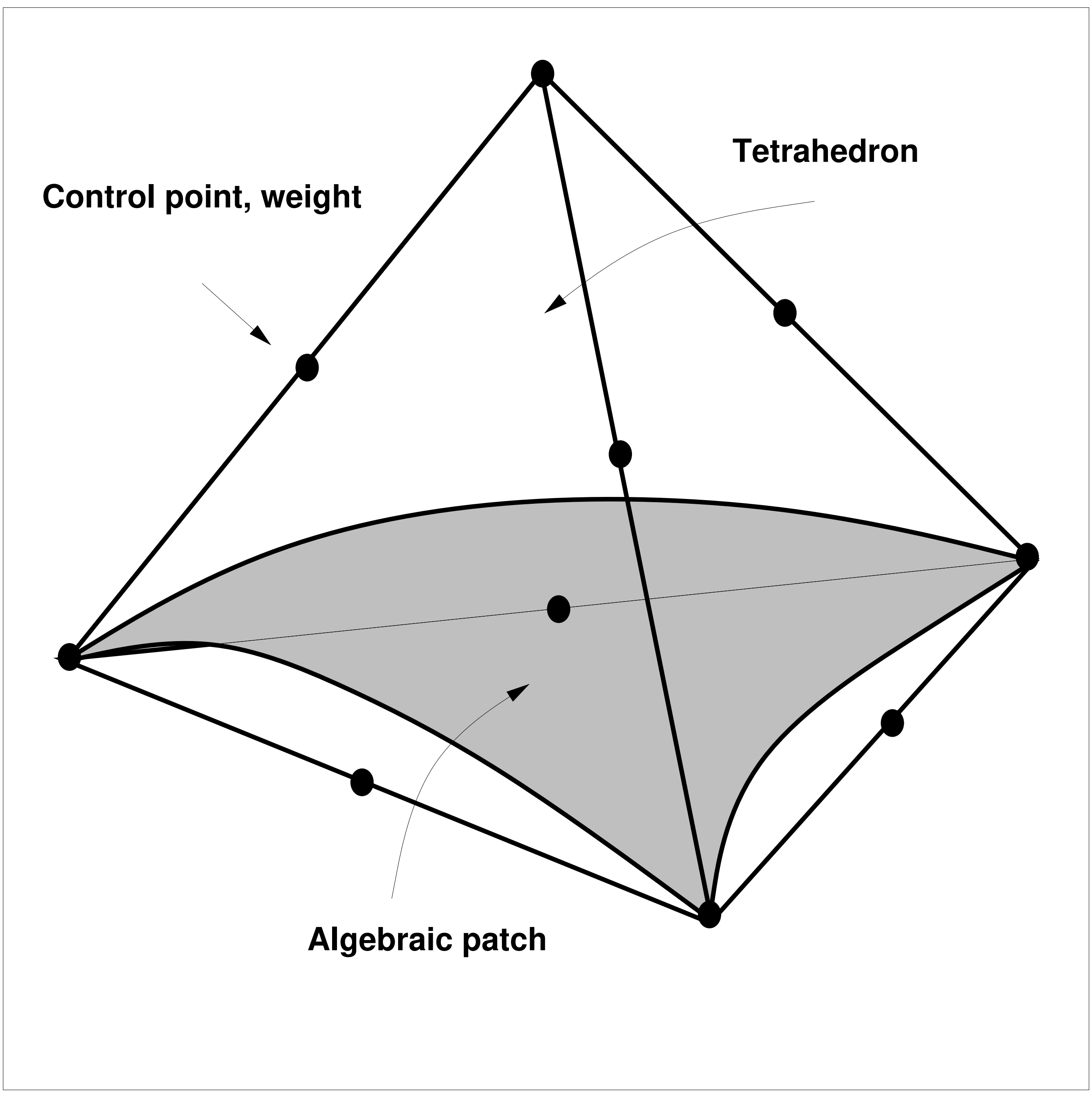
where $i, j, k, l \geq 0$, and $i + j + k + l = n$,
there are $(n + 1)(n + 2)(n + 3)/6$ control points
A weight w_{ijkl} is assigned to each control point.

- **Algebraic patch formulation**

$$\sum_i \sum_j \sum_k \sum_{l=n-i-j-k} w_{ijkl} \frac{n!}{i!j!k!l!} r^i s^j t^k u^l = 0$$

- **properties**
  - **meaningful control**
  - **local control**
  - **boundary interpolation**
  - **gradient control**
  - **self-intersection avoidance**
  - **continuity condition across the boundaries**
  - **subdivision**

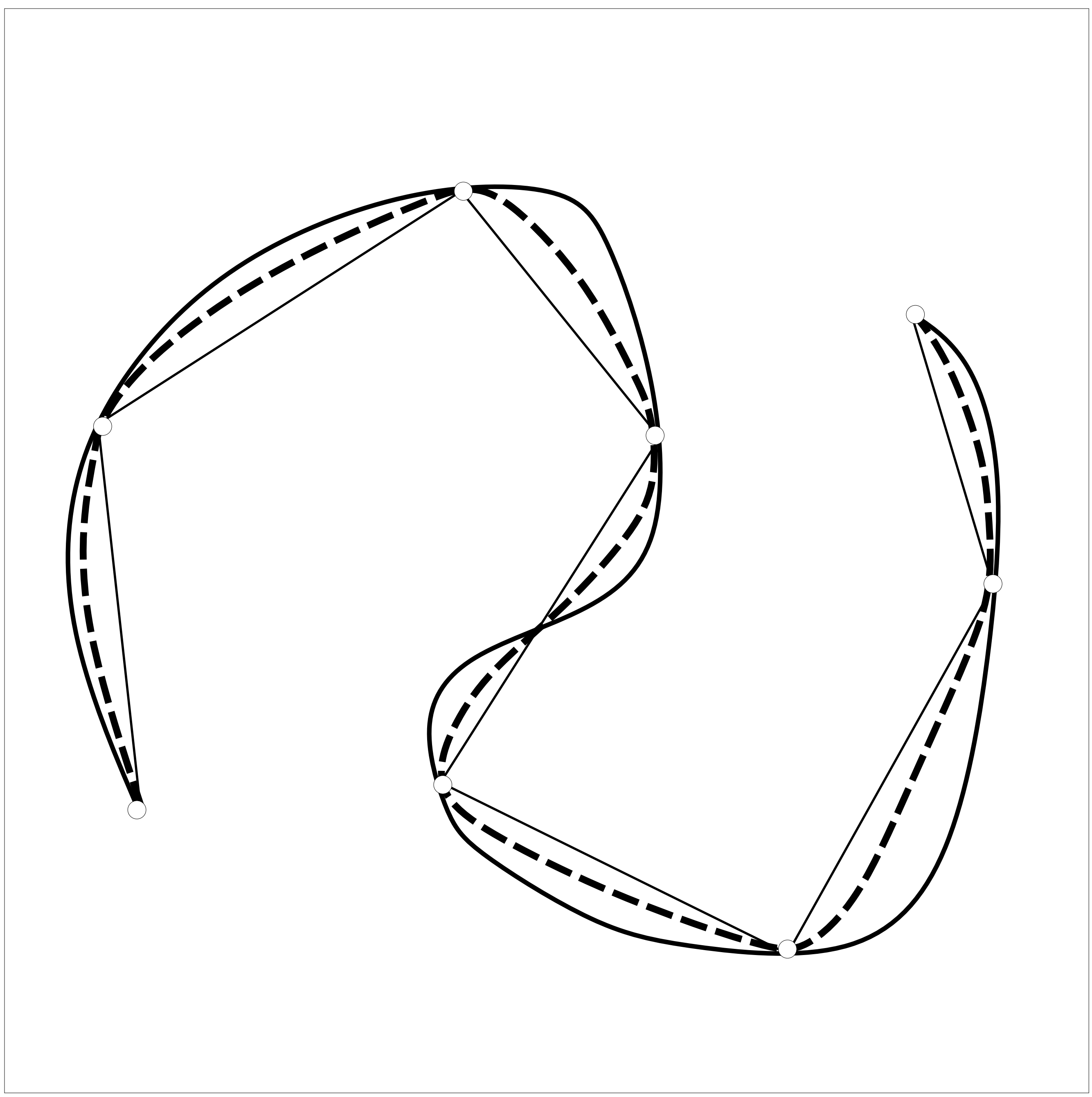
Algebraic Patch



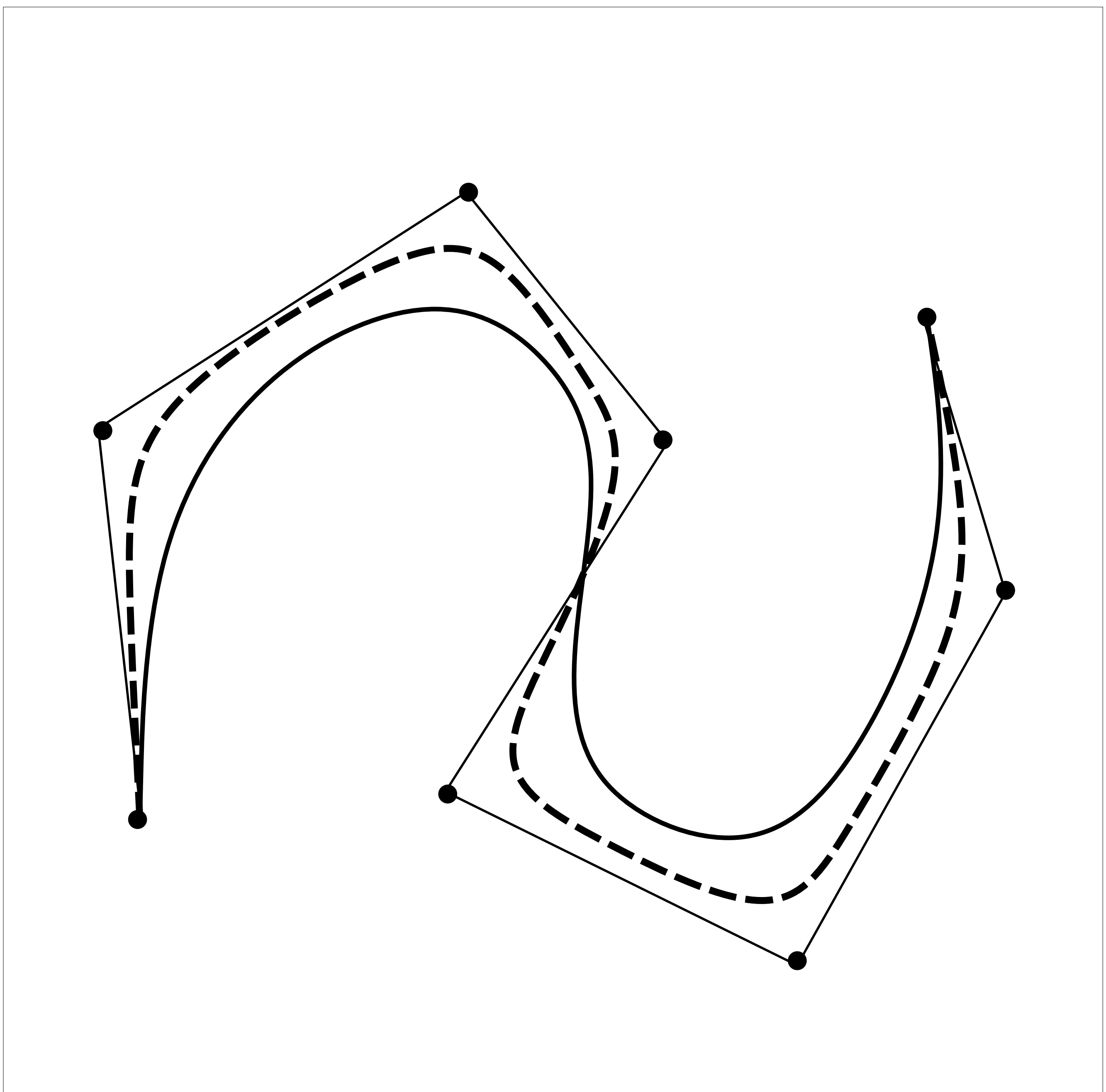
Geometric and Variational Splines

- Smoothness requirements at conjunction points
- Parametric continuity is TOO restrictive!
- Geometric continuity via reparameterization
- G^n: C^n continuous under arc-length parameterization
- G^1 and G^2
- Geometric continuity is less restrictive!
- Shape parameters: tension, bias, etc.
- Many special splines

Tension Control



Tension Control



Special Splines

- **Beta-spline: generalization of cubic B-spline**

- **Catmull-Rom splines:** $\mathrm{c}(u) = \sum_i \mathrm{a}_i(u) W_i(u)$

- **Nu-spline as a G^2 (continuous curvature) piecewise polynomial minimizing the following functional over $[t_0, t_n]$:**

$$\int_{t_0}^{t_n} \left\| \mathrm{f}''(u) \right\|^2 du + \sum_{i=0}^{n} \nu_i \left\| \mathrm{f}'(u_i) \right\|^2$$

  subject to the interpolation $\mathrm{f}(t_i) = \mathrm{d}_i$ and necessary end conditions.

- **Tau-spline: a quintic spline that maintains both curvature and torsion continuity by minimizing**

$$\int_{t_0}^{t_n} \left\| \mathrm{f}^{(k)}(u) \right\|^2 du + \sum_{i=0}^{n} \sum_{j=1}^{k-1} \nu_{i,j} \left\| \mathrm{f}^{(j)}(u_i) \right\|^2$$

  subject to interpolatory constraints.

- **Weighted Nu-spline by minimizing**

$$\sum_{i=0}^{n-1} w_i \int_{t_i}^{t_{i+1}} \left\| \mathrm{f}''(u) \right\|^2 du + \sum_{i=0}^{n} \nu_i \left\| \mathrm{f}'(u_i) \right\|^2,$$

subject to the interpolatory conditions $\mathrm{f}(t_i) = \mathrm{d}_i$

- **Others!!!**

Geometric Design Paradigms

- Interpolation
- Approximation
- Interactive manipulation
- Cross-sectional design
- Optimization
- Others!

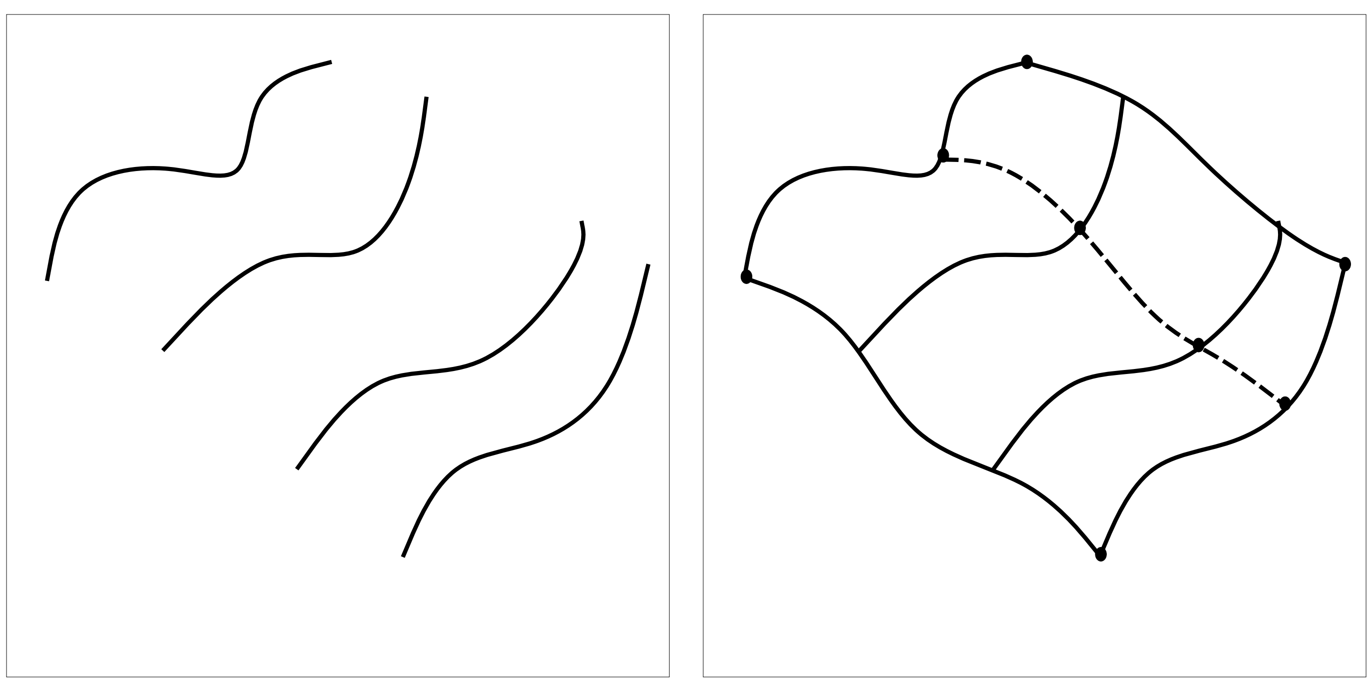
Interpolation

- A B-spline curve with $n+1$ control points
- Interpolate independent $n+1$ data points d_i
- The parametric values $u_0, \ldots, u_n$ are provided
- solve the linear equation

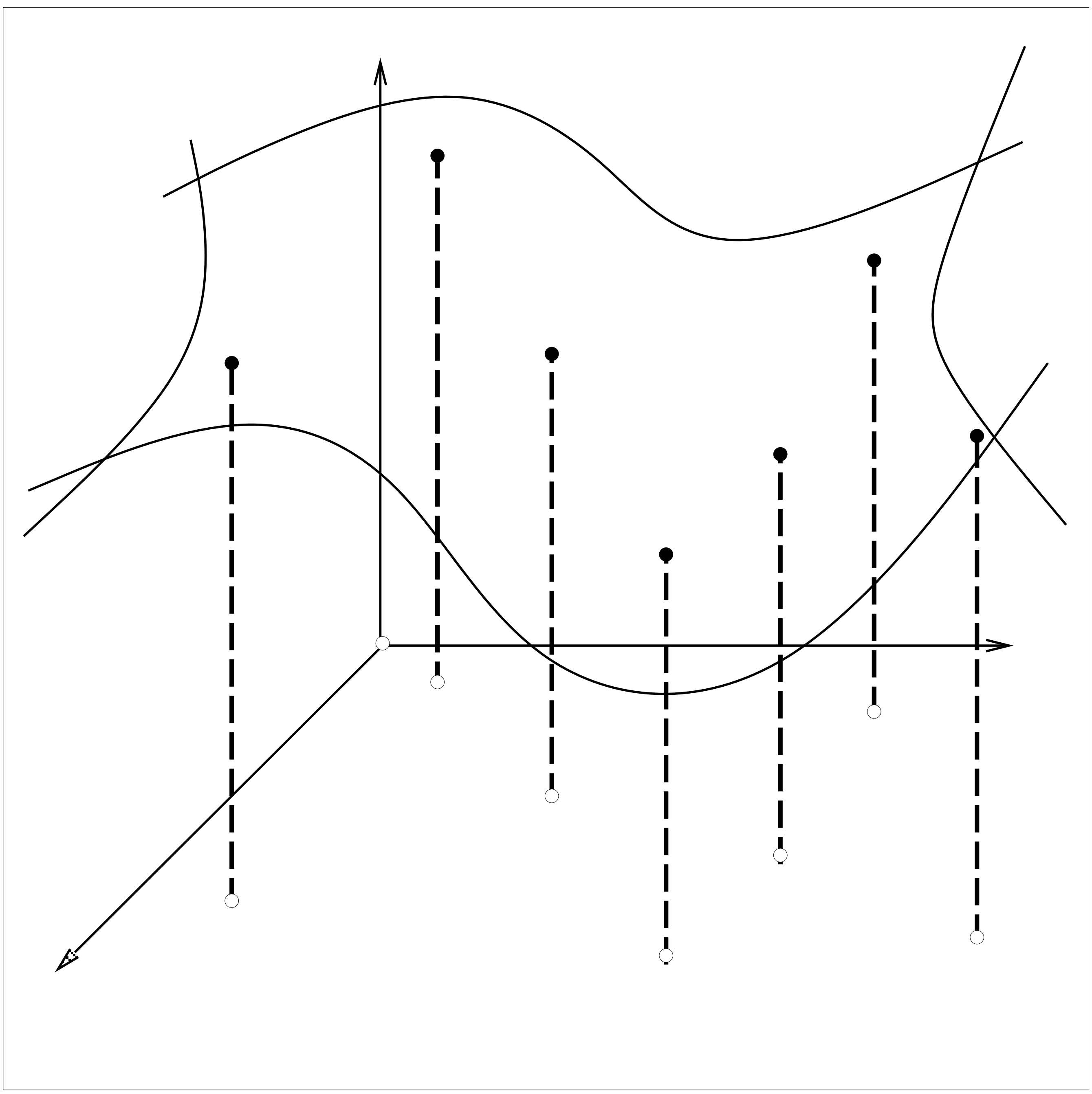
$$\left[\begin{array}{ccc} B_{0,k}(u_0) & \cdots & B_{n,k}(u_0) \\ \vdots & \ddots & \vdots \\ B_{0,k}(u_n) & \cdots & B_{n,k}(u_n) \end{array}\right] \left[\begin{array}{c} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_n \end{array}\right] = \left[\begin{array}{c} \mathbf{d}_0 \\ \vdots \\ \mathbf{d}_n \end{array}\right]$$

- How about the parameterization is unknown
- Non-linear system
- Generalize to tensor-product surfaces
- data points conform to a rectangular grid
- “Lofting”: a smooth surface passes through a set of cross-sectional curves

Cross-Sectional Design



Scattered Data Interpolation



Approximation

- **Rationales**
  - **data exchange**
  - **data reduction**
  - **decomposition**
  - **multi-level analysis and hierarchical model**
  - **special shapes**
- **The system is overdetermined**

$$\mathbf{C}^{\top}\mathbf{C}\mathbf{p} = \mathbf{C}^{\top}\mathbf{d}$$

- **Approximation minimizes the error functional**

$$E = \sum_{i=0}^{m} \|\mathbf{c}(u_i) - \mathbf{d}_i\|^2$$

Interactive Manipulation

- Control points
- Weights
- Knot vector
- Shape parameters
- Free-form deformation
- High-level deformation
- Others!

Variational Design

- Functional requirements
  - interpolation, continuity, etc
- Visually-pleasing or fair shapes implied in data points
  - monotonic, convex, etc
- Shape-preserving interpolant
- Qualitative and subjective criteria
- Partial information
- Objective function for energy functional
- Optimization subject to constraints
- Less user input
- Automatic determination via optimization solver

Shape-Preserving Interpolation