CSE528 Computer Graphics: Theory, Algorithms, and Applications

Hong Qin

Rm. 151, NEW CS

Department of Computer Science

Stony Brook University (State University of New York at Stony Brook)

Stony Brook, New York 11794-2424

Tel: (631)632-8450; Fax: (631)632-8334

qin@cs.stonybrook.edu

http://www.cs.stonybrook.edu/~qin

Splines and Applications

Functions

- Functions are the basic mathematical tool for describing and analyzing many physical processes of interest
- Frequently in applications, we do not have function itself and have to construct an approximation to it based on limited information about the underlying process
- Such approximation problems are central part of applied Mathematics

Two Major Categories of Approximation Problems

- Data fitting problems

It is required to construct an approximation to unknown function based on finite amount of data (often measurements) on the function

For example, consider a weather map where data are collected at a set of weather stations, but a continuous model of temperature, pressure, etc., is desired

Another Type of Problems

- Operator-equation problems

These simply arise when you have a model for the physical process which involves an equation that either cannot be solved explicitly or cannot be solved at all.

Common Approach to Find Approximation to the Function

- Choose a reasonable *class of functions* in which to look for an approximation

- Devise an appropriate *selection scheme*

Polynomials

- The most convenient class of functions to work with is a class of polynomials:
  - Easy to store on a digital computer
  - Smooth
  - Approximate any continuous function as close as we like (celebrated Weierstrass theorem)

Huge Drawback

- Polynomials are very inflexible

The more points we want to interpolate, the higher degree polynomial we have to use. But high degree polynomials tend to wiggle a lot.

Runge's Phenomenon

- The red curve is the Runge function, the blue curve is a 5th-order polynomial, while the green curve is a 9th-order polynomial. The approximation only gets worse
- Consider the function:

$$f(x) = \frac{1}{1 + 25x^2}$$

- Runge found that if you interpolate this function at equidistant points between -1 and 1 with a polynomial which has a degree smaller or equal with n, the resulting interpolation would oscillate toward the end of the interval, i.e. close to -1 and 1. It can even be proved that the interpolation error tends toward infinity when the degree of the polynomial increases:

$$\lim_{n \to \infty} \left(\max_{-1 \le x \le 1} |f(x) - P_n(x)| \right) = \infty$$

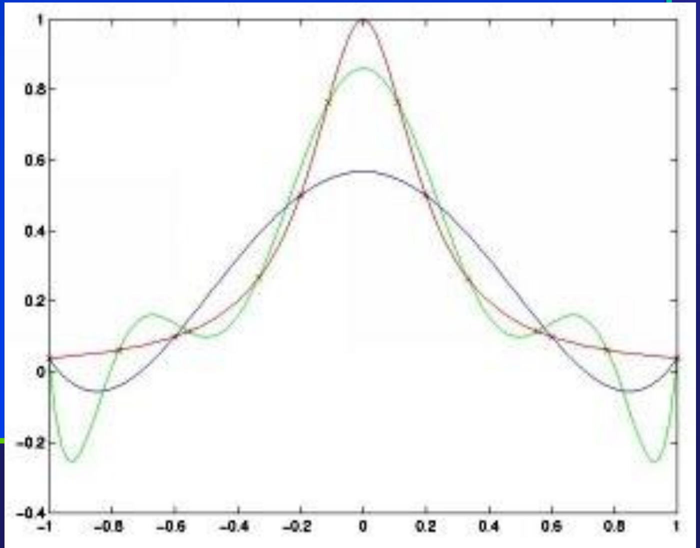
- Runge's phenomenon demonstrates that lower-order polynomials are generally to be preferred instead of raising the degree of the interpolation polynomial.

red curve - Runge function;

blue curve - 5th order polynomial;

green curve - 9th order polynomial.

The approximation only gets worse



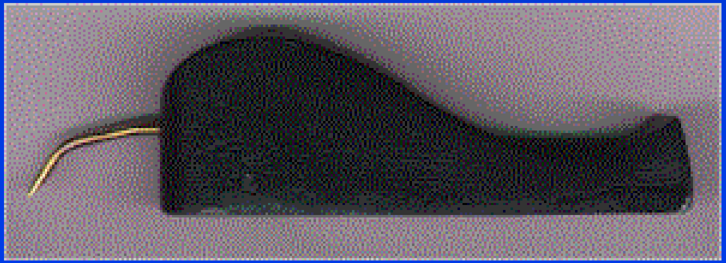
Idea to Resolve This Problem

- Break the big interval into small ones and consider polynomials (different ones!) on each separate interval.

- So we came to the concept of splines. Simply speaking, <u>spline is a piecewise polynomial curve</u> that you all are familiar with.

Draftsman's Spline

Duck



The Beginning

I. J. Schoenberg

April 21, 1903 – Feb. 21, 1990

Quart. Appl Math. 4 (1946), 45—99 and Contributions to the problem of approximation of equidistant data by analytic functions, *112—141.*

The Natural Cubic Splines

1) s is a piecewise cubic polynomial

2) $s \in C^2[a,b]$

3) $s''(a) = s''(b) = 0$

The Great Property

$$\text{Minimize Energy} \approx \int_a^b \kappa(t)^2 \approx \int_a^b [f''(t)]^2\, dt$$

Spline Theory and Applications

- Approximation theory
- Numerical analysis
- Computer science
- Application areas
  - Engineering
  - Biosciences, Chemistry, Physics, Geophysics, Meteorology
  - Medicine
  - Business and Social Sciences
  - Imaging and Visualization
  - Computer-aided design and Manufacture
  - Computer Vision and Robotics

Polynomial Splines

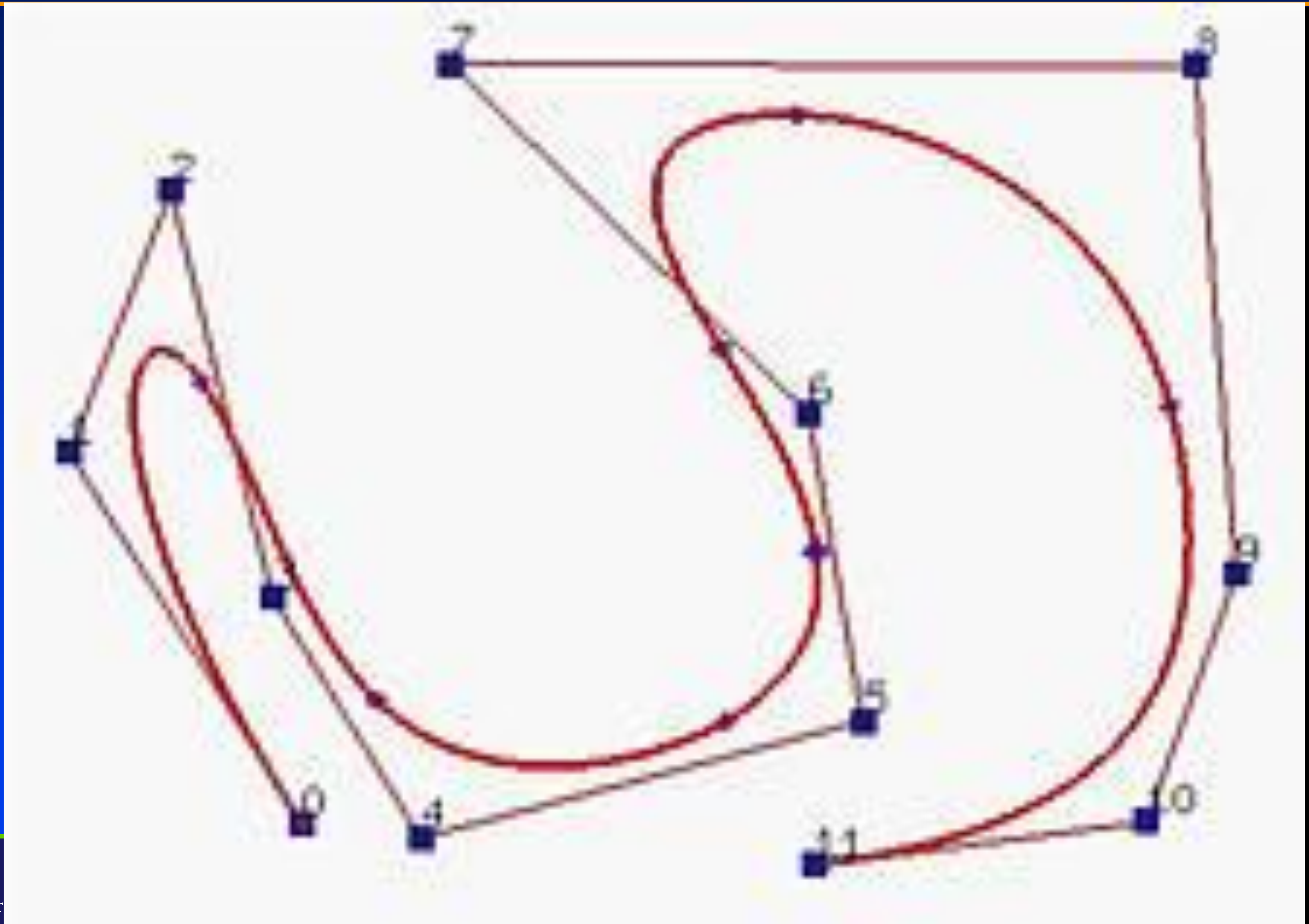
Finite dimensional space

Stable, local basis (B-splines)

Two Main Categories of Splines

- Interpolating spline (passes through all of the control points)
- Approximating spline (passes near all of the control points)

Parametric Splines with Control Polygon



Cubic Spline Smoothness Theorem

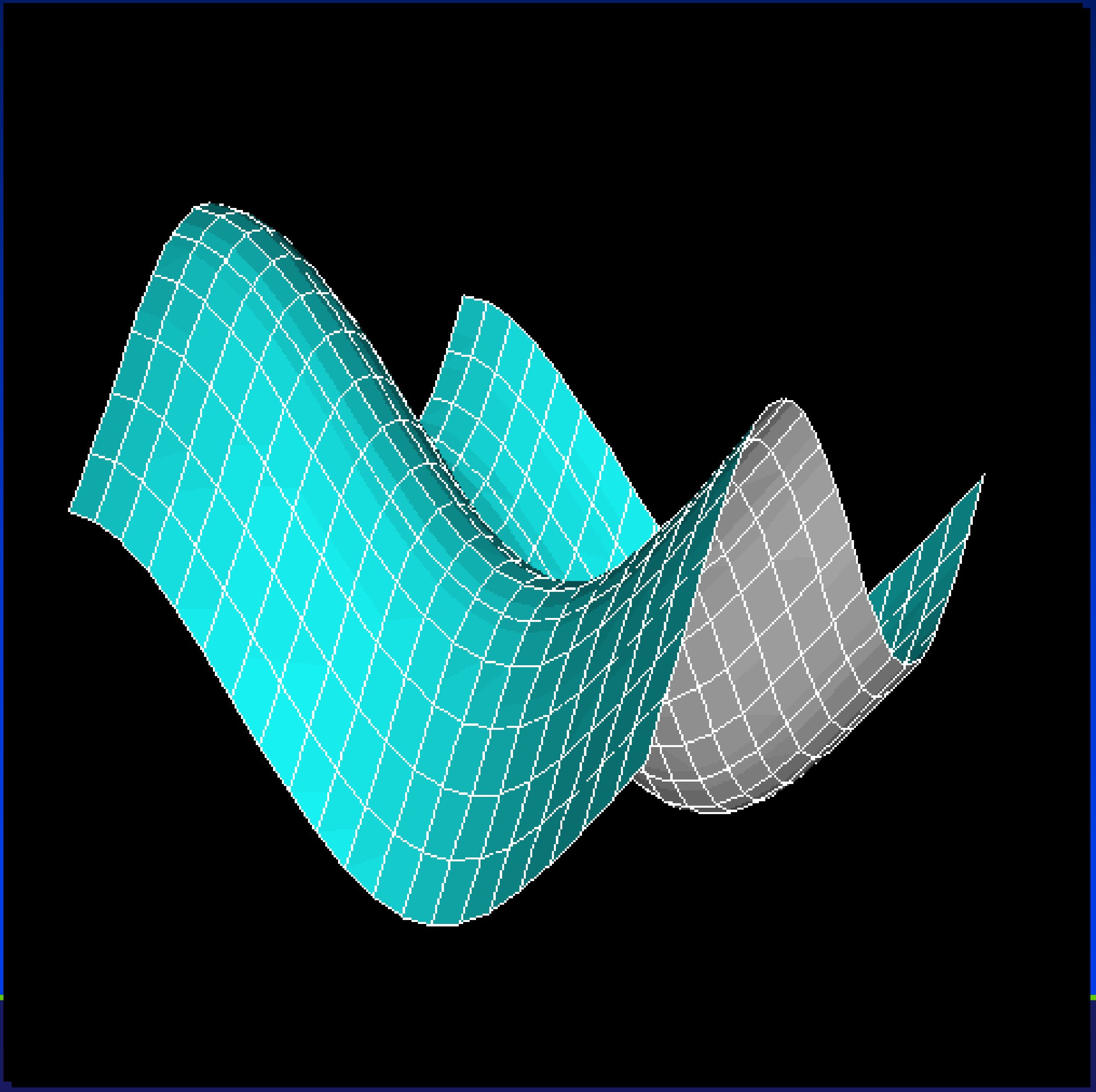
If S is the natural cubic spline function that interpolates a twice-continuously differentiable function f at knots

$$a = t_0 < t_1 < \ldots < t_n = b$$

Then

$$\int_a^b [S''(x)]^2 dx \le \int_a^b [f''(x)]^2 dx$$

Tensor-product Splines

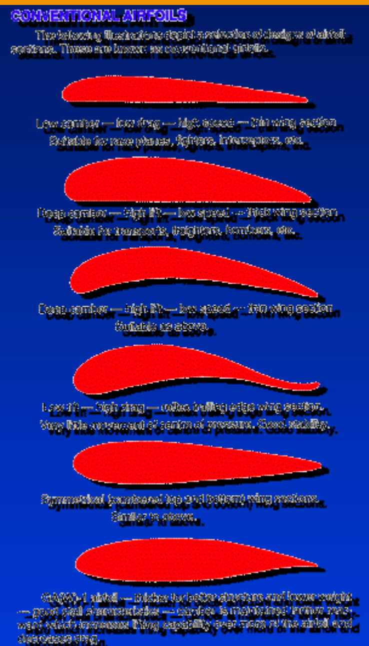


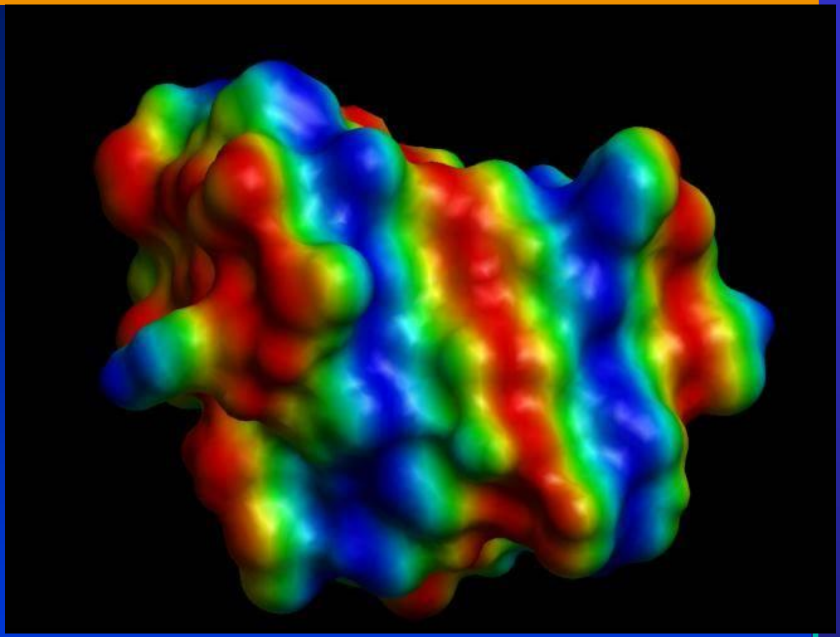
Areas

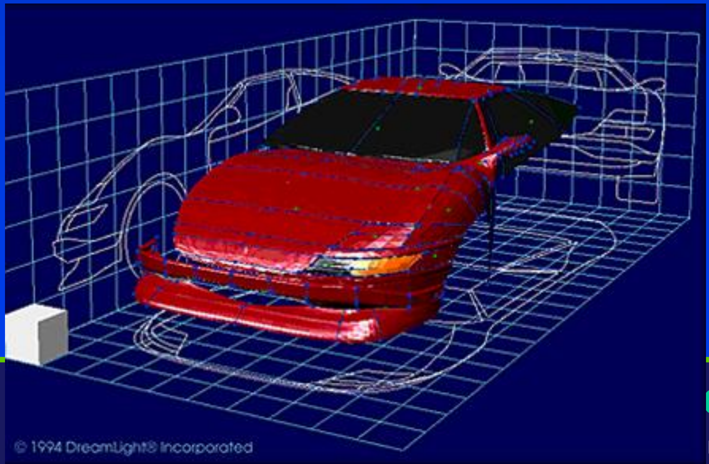
- The study of various classes of approximating functions is precisely the problem of APPROXIMATION THEORY

- The design and analysis of effective algorithms utilizing these approximation classes are a major part of NUMERICAL ANALYSIS.

What Do the Followings Have in Common?







Answer

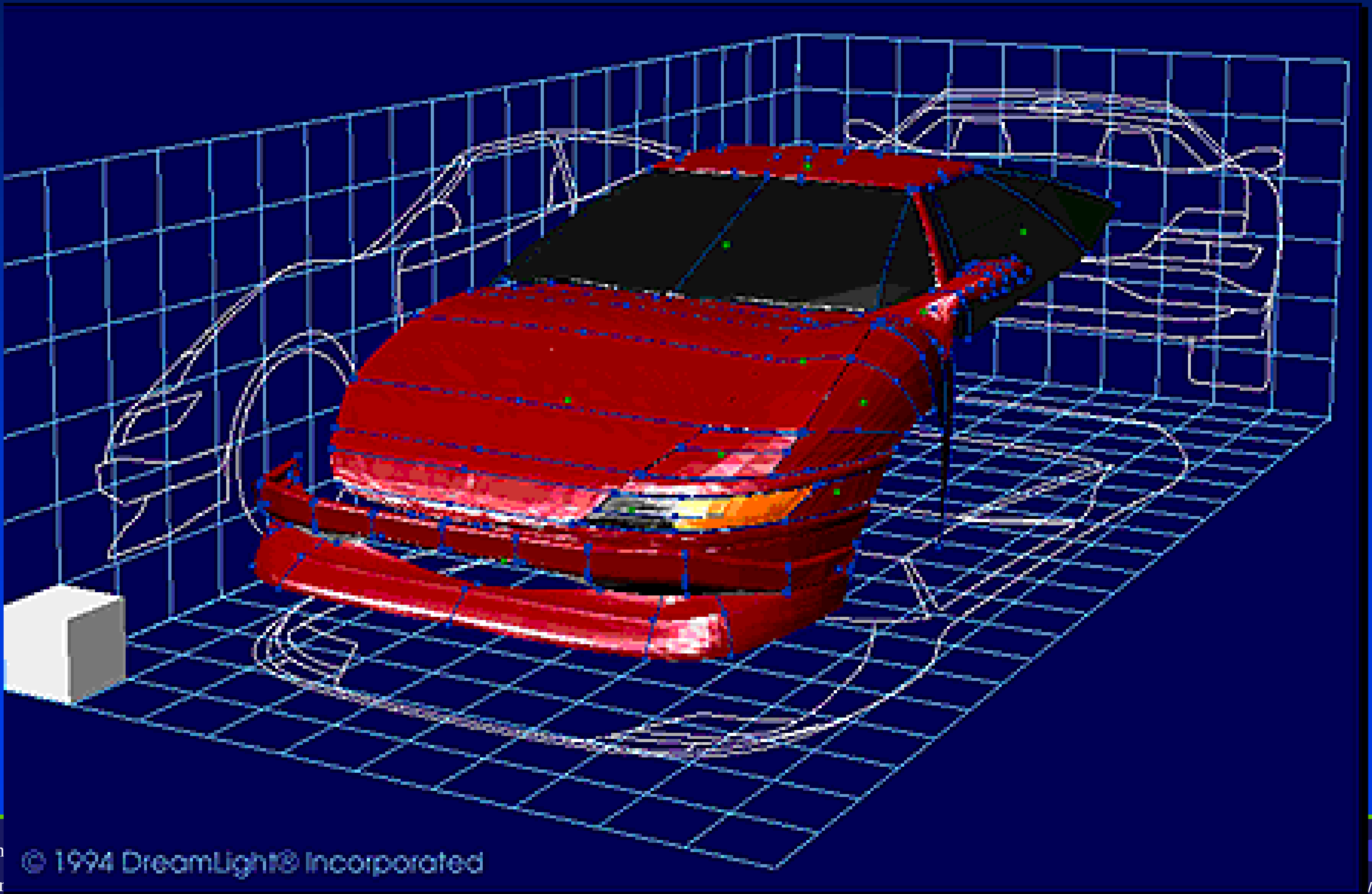
- All these 2D and 3D models are created with the help of <u>splines</u>.

Computer Aided Geometric Design

- Whenever free-form curves and surfaces are represented mathematically, as they are in CAGD, analysis and manufacturing, B-splines (basis splines) are the foundation of an efficient implementations

- B-splines are especially important in the aircraft and automotive industries, where shape is all important. Designers may not see B-splines – they may manipulate a handle on the end of curve to control its curvature instead – but B-splines are likely to be the hidden bearings on which the design engine runs

Car Design

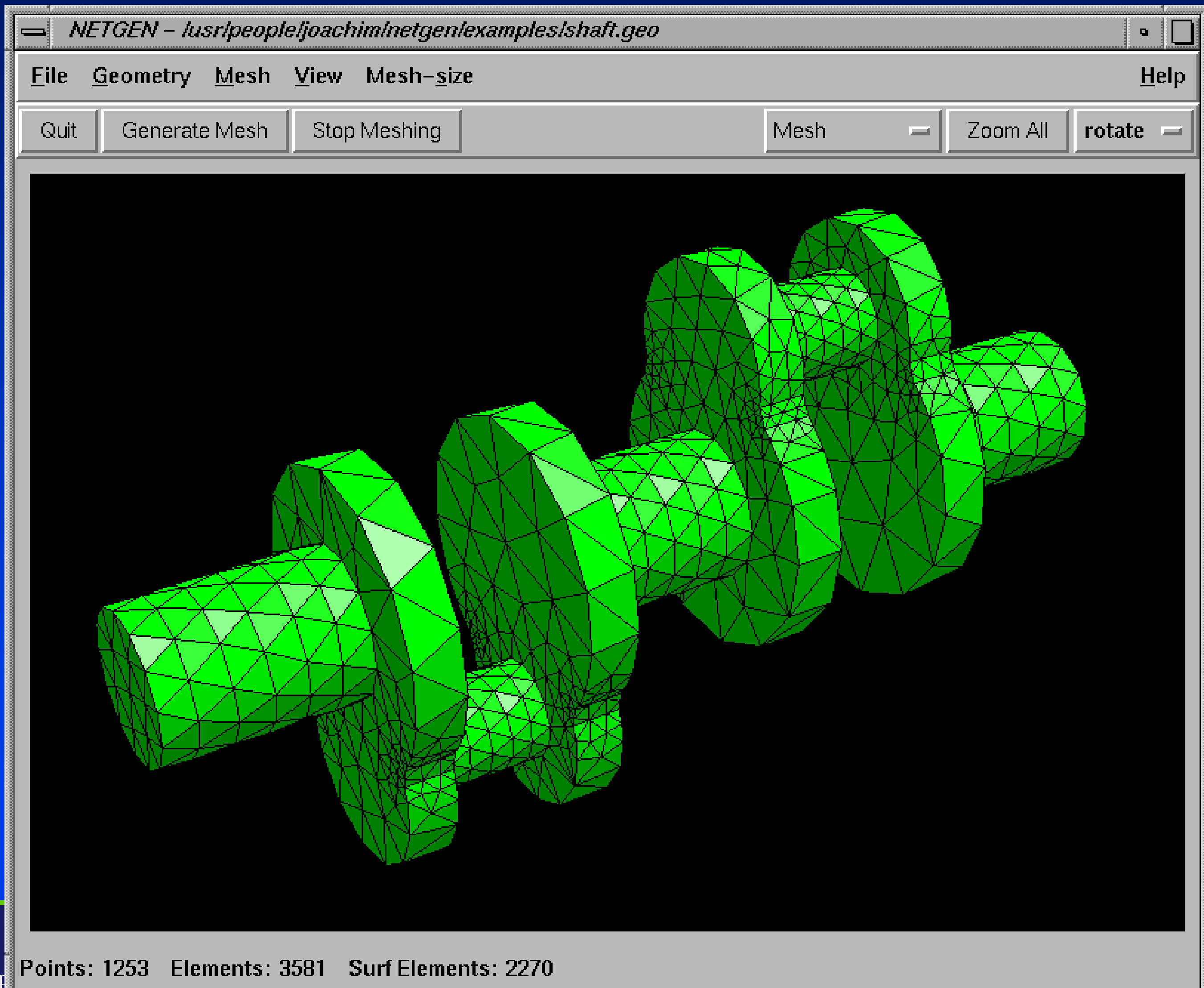


Another Application at Boeing

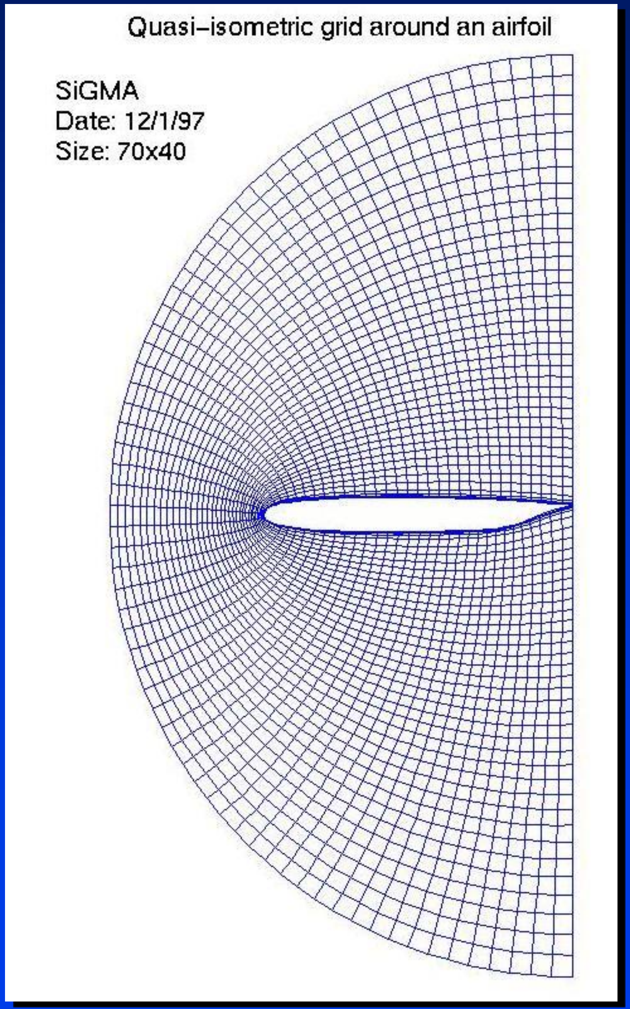
- "Flight test data, like take of distance as a function of temperature, altitude, weight, and many other variables, used to be stored in a succession of look-up tables in a bulky flight manual. Now the 777 pilot enters the flight parameters into a computer, and a spline-based interpolation of that multivariate function immediately returns the take-off distance.

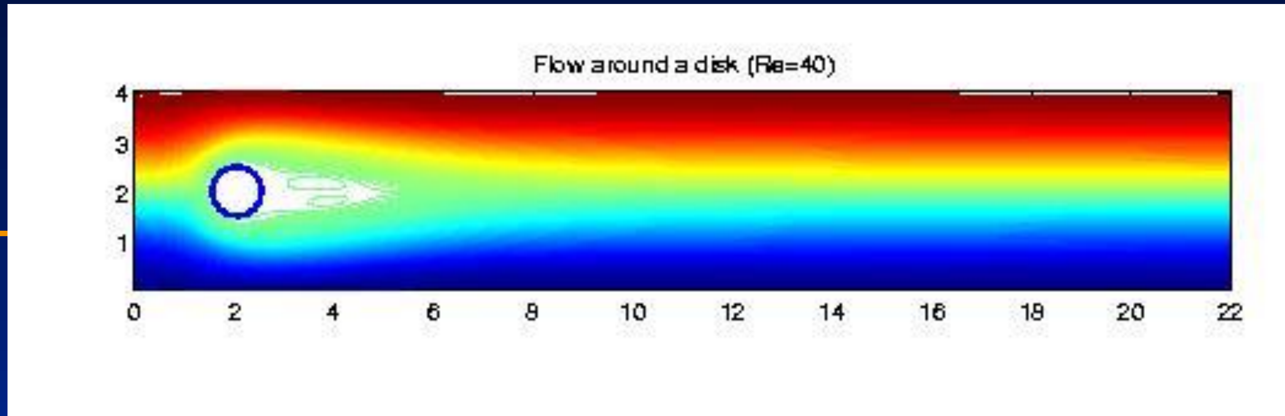
  The look-up process has been completely abandoned".

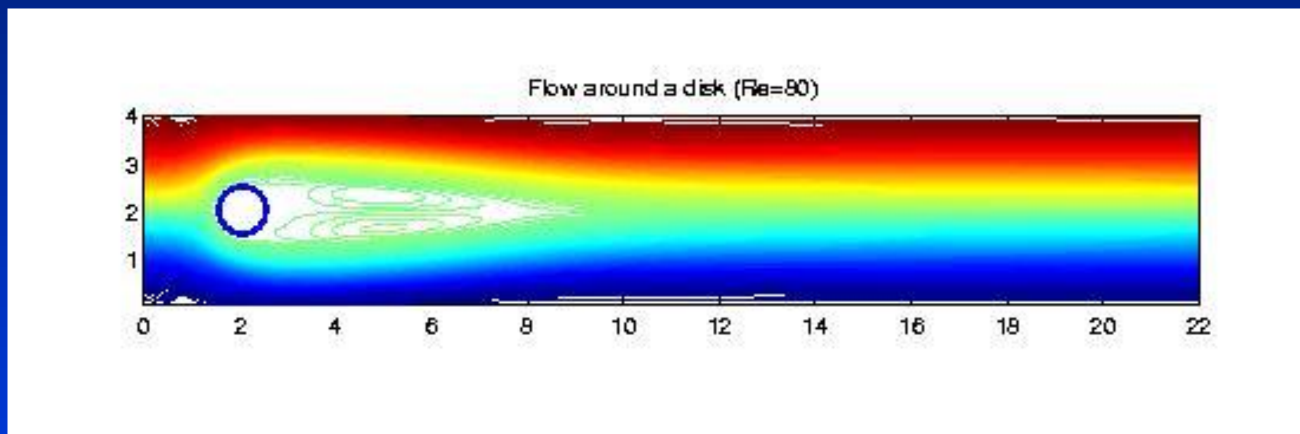
- Another important application of splines at Boeing is in the computation of optimal orbit and flight trajectories. The continuous variables representing the physics of the vehicle and the control mechanism – flaps and thrusters, for example, - are replaced by spline approximation.

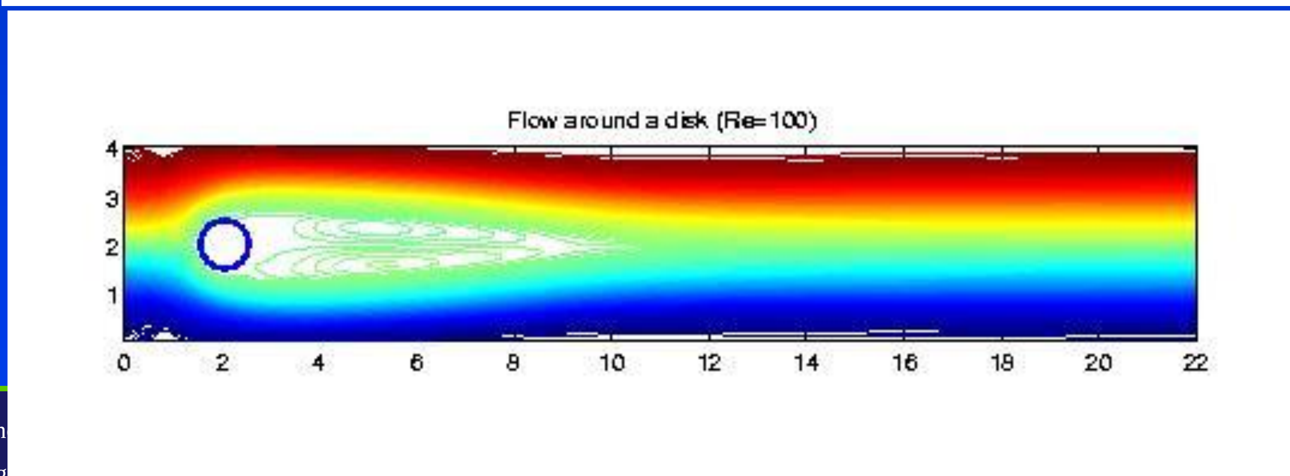

NETGEN – /usr/people/joachim/netgen/examples/shaft.geo
File Geometry Mesh View Mesh–size Help
Quit Generate Mesh Stop Meshing Mesh Zoom All rotate
Points: 1253 Elements: 3581 Surf Elements: 2270

Quadrilateral Grid

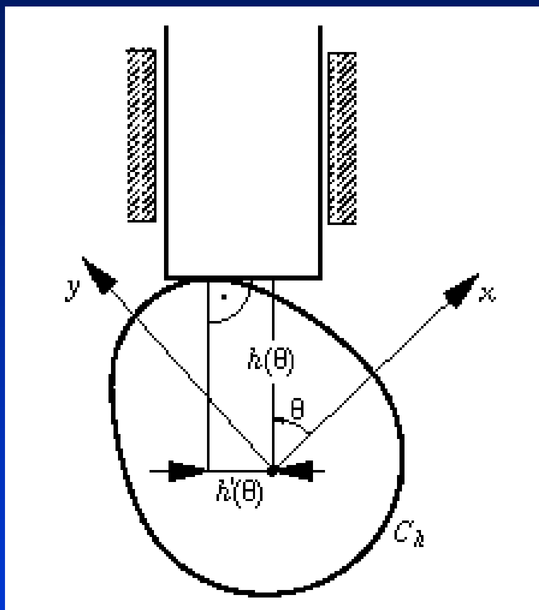


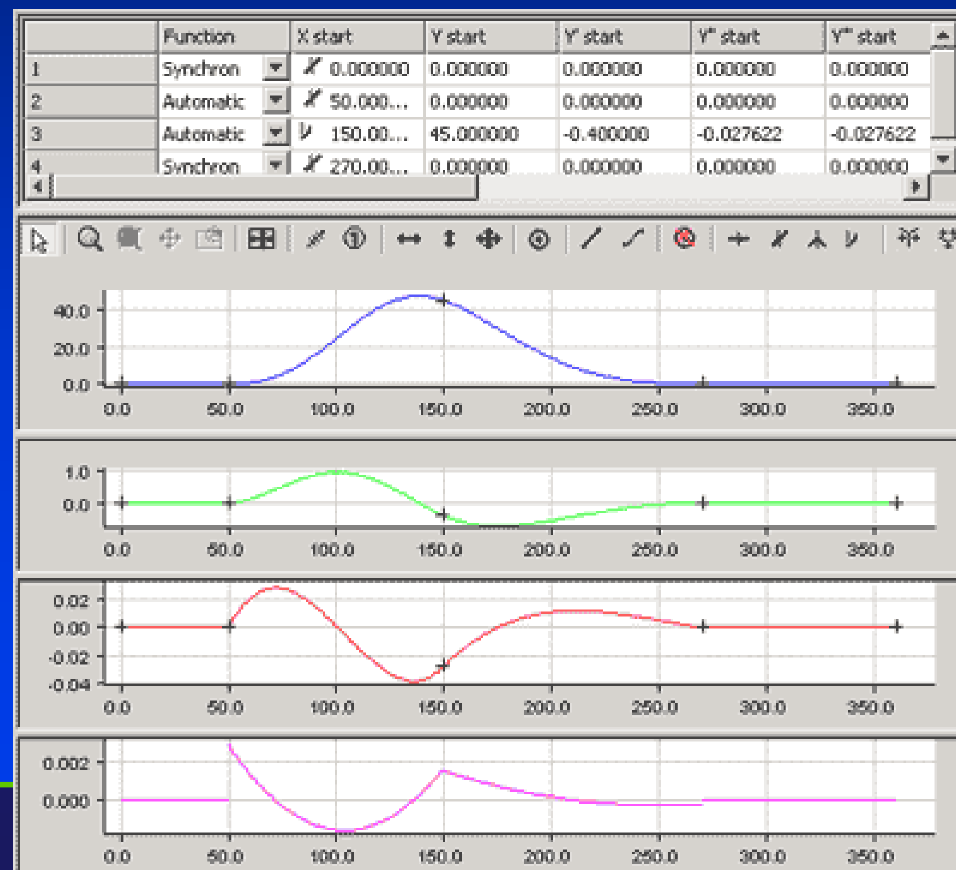

Flow around a disk (Re=40)
4
3
2
1
0 2 4 6 8 10 12 14 16 18 20 22


Flow around a disk (Re=80)
4
3
2
1
0 2 4 6 8 10 12 14 16 18 20 22


Flow around a disk (Re=100)
4
3
2
1
0 2 4 6 8 10 12 14 16 18 20 22

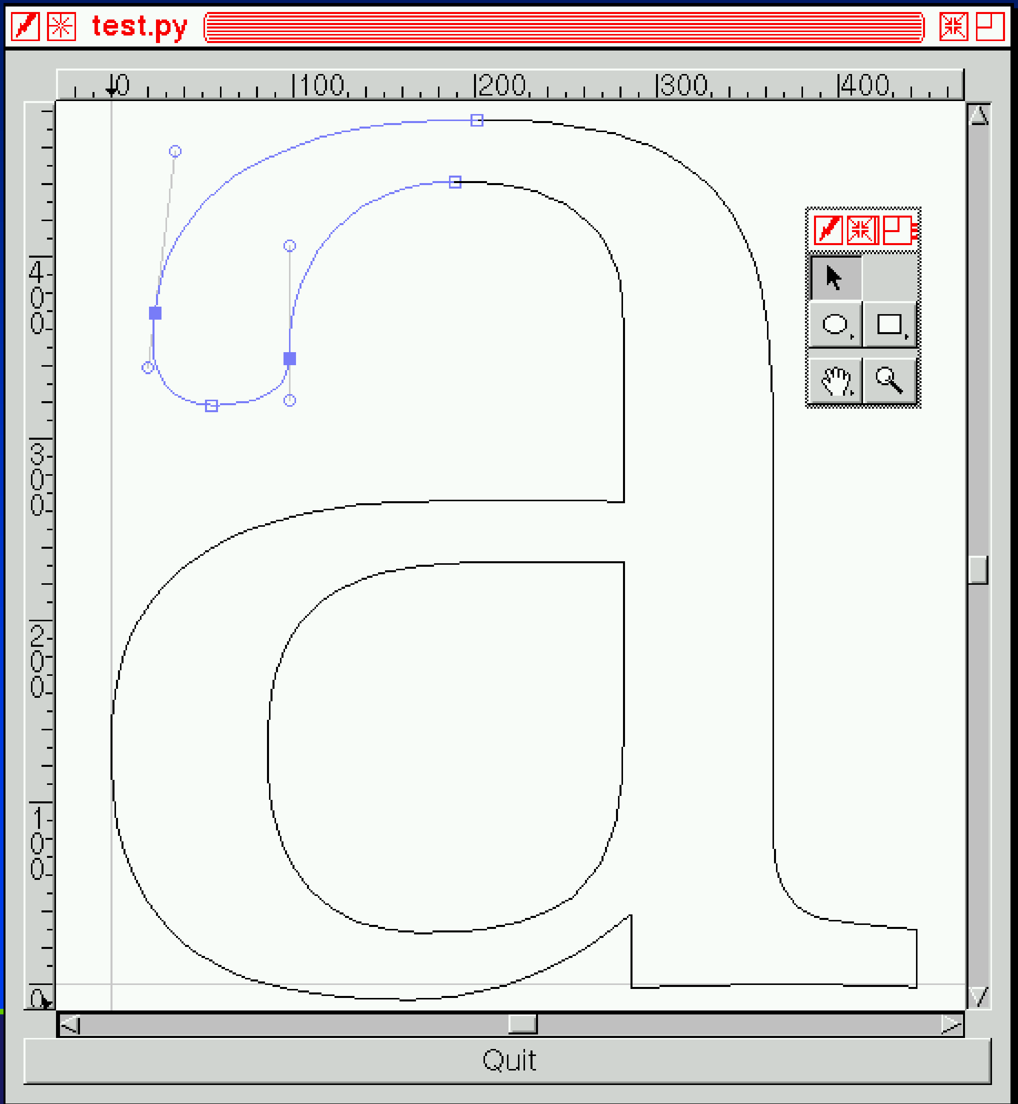
CAM Design





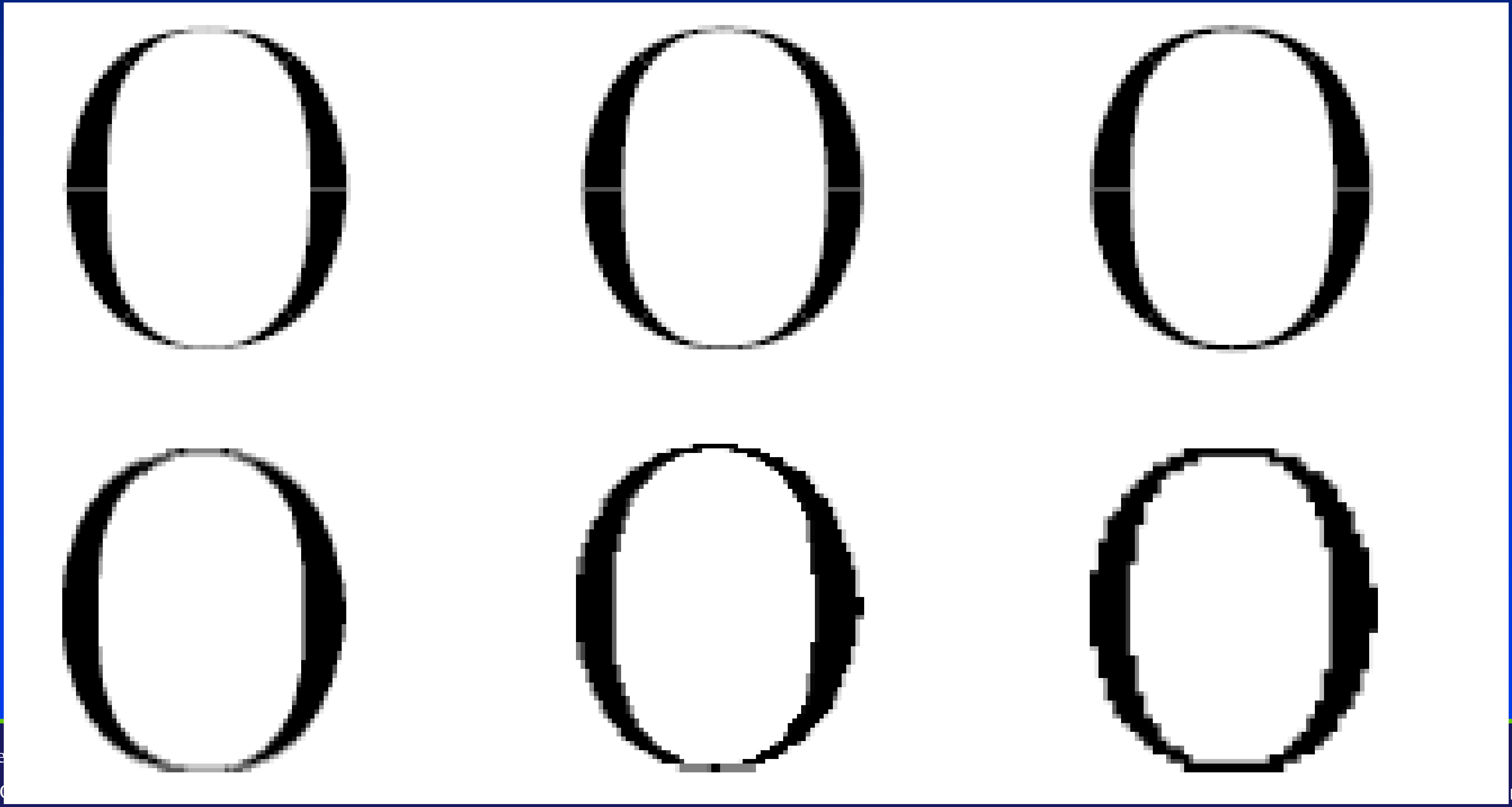
	Function	X start	Y start	Y' start	Y'' start	Y''' start
1	Synchron	0.000000	0.000000	0.000000	0.000000	0.000000
2	Automatic	50.000...	0.000000	0.000000	0.000000	0.000000
3	Automatic	150.00...	45.000000	-0.400000	-0.027622	-0.027622
4	Synchron	270.00...	0.000000	0.000000	0.000000	0.000000

Font Design



TrueType Fonts

- TrueType fonts are outline fonts which means that they can deliver good quality output at any resolution or size:



Applications to Medicine

- "Fly through" models
- Edge detection in ultra sound images
- Modeling of molecule
- Etc.

Molecule Design

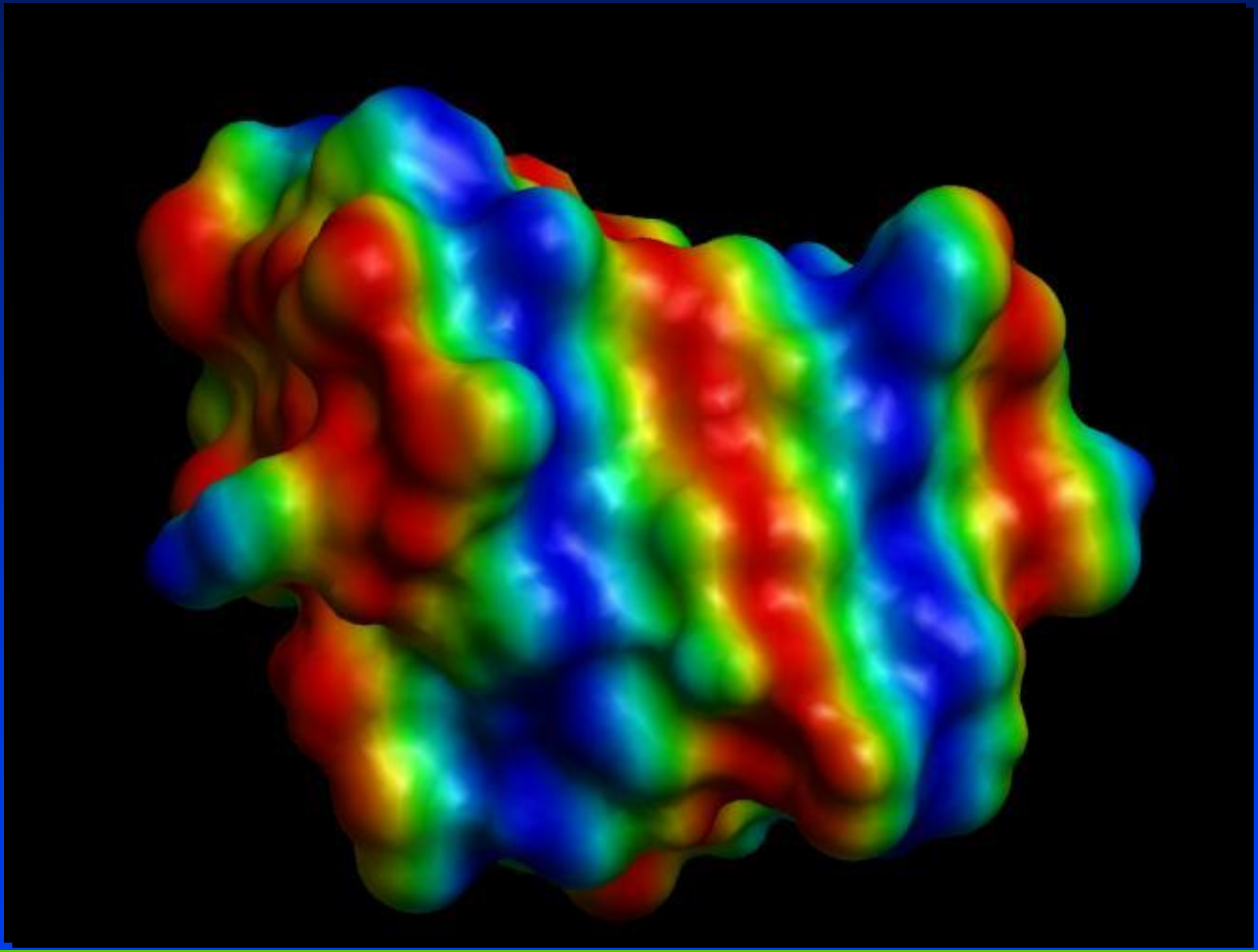
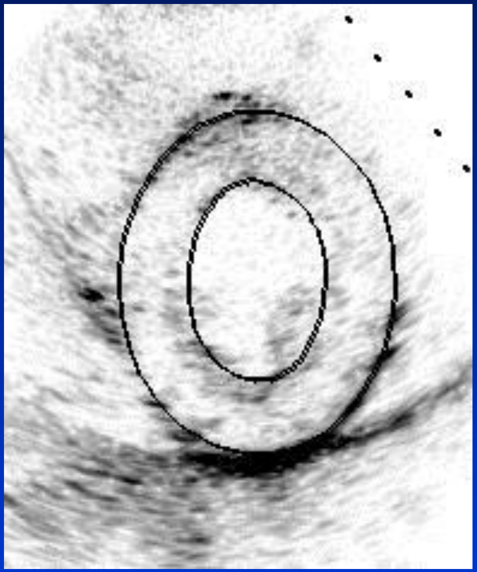
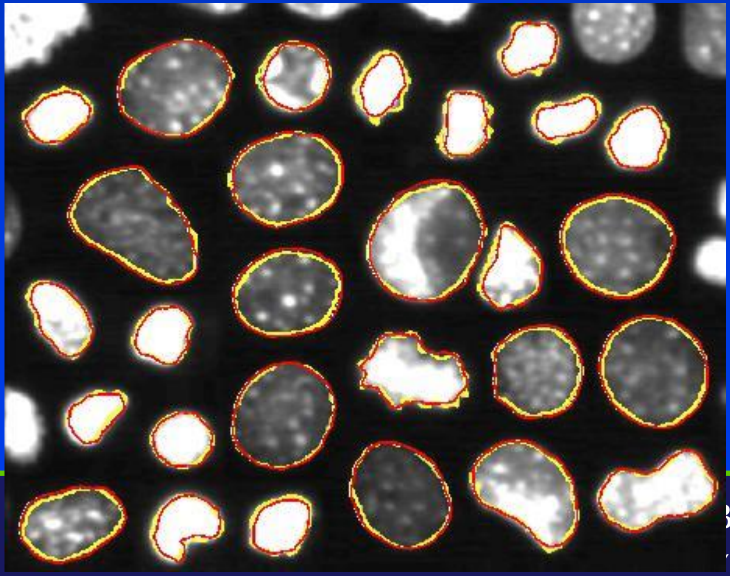


Image Segmentation





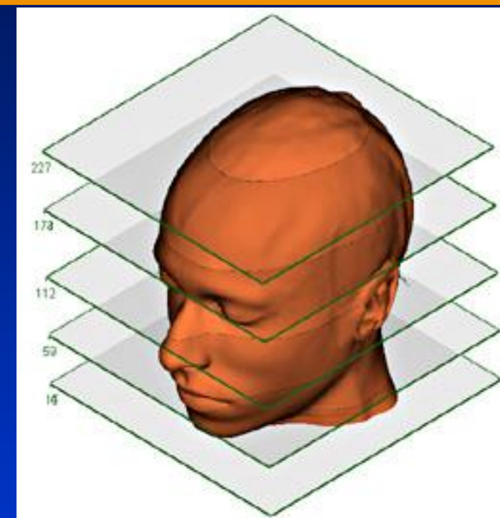
"Fly through" Models

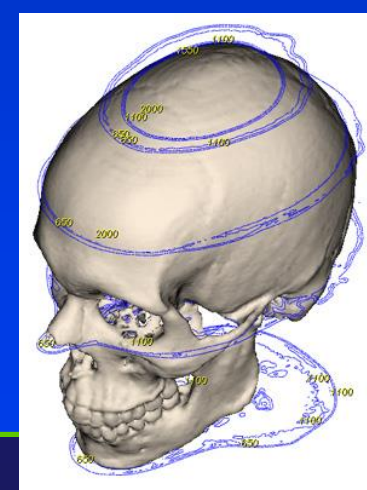
Goal: 3D reconstruction from 2D slices obtained from CT scan.

- The data starts out as slices (images) taken at regular intervals throughout a portion of the body.
- Then slices are segmented to separate the various tissues.
- Using an algorithm based on bivariate B-spline 3D model is created:

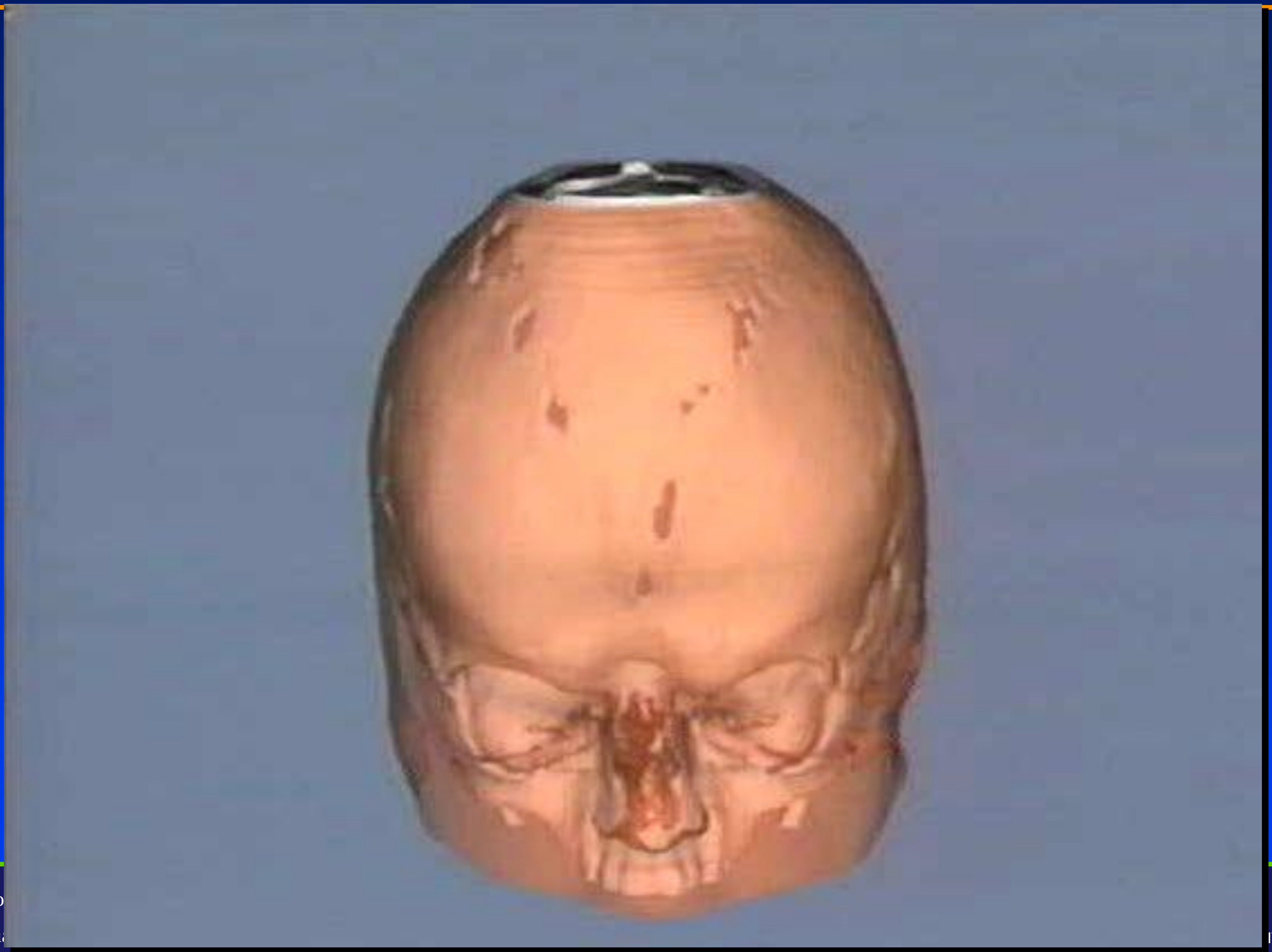
Image interpolation creates a number of new slices between known slices in order to obtain an isotropic volume image.

3D … truction from Slices



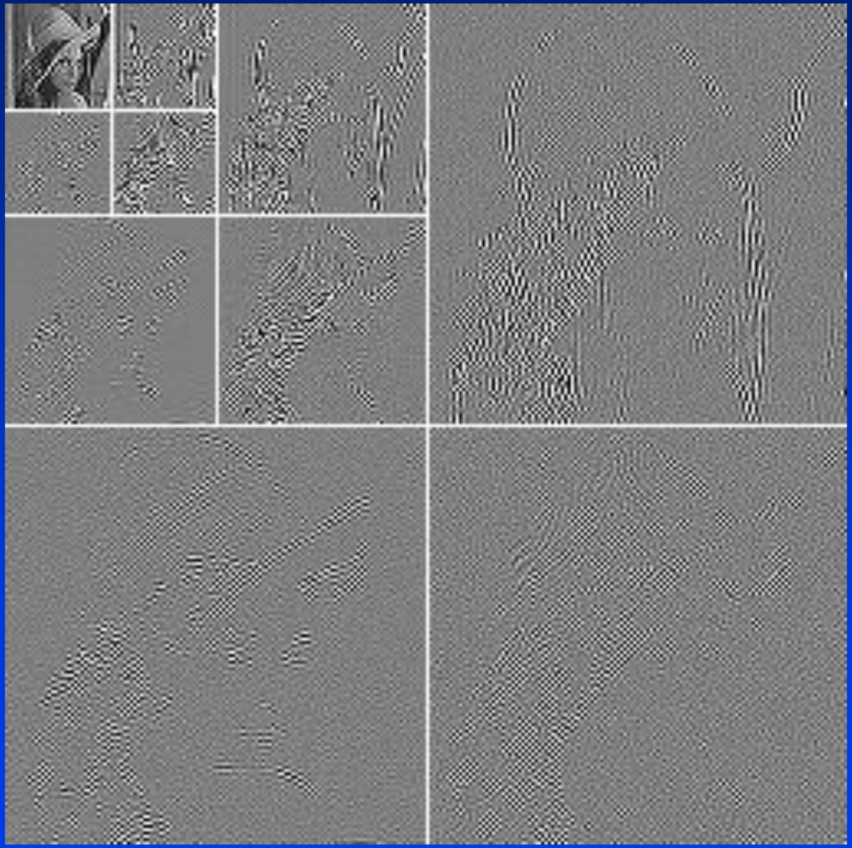


CT Skull Fly Through

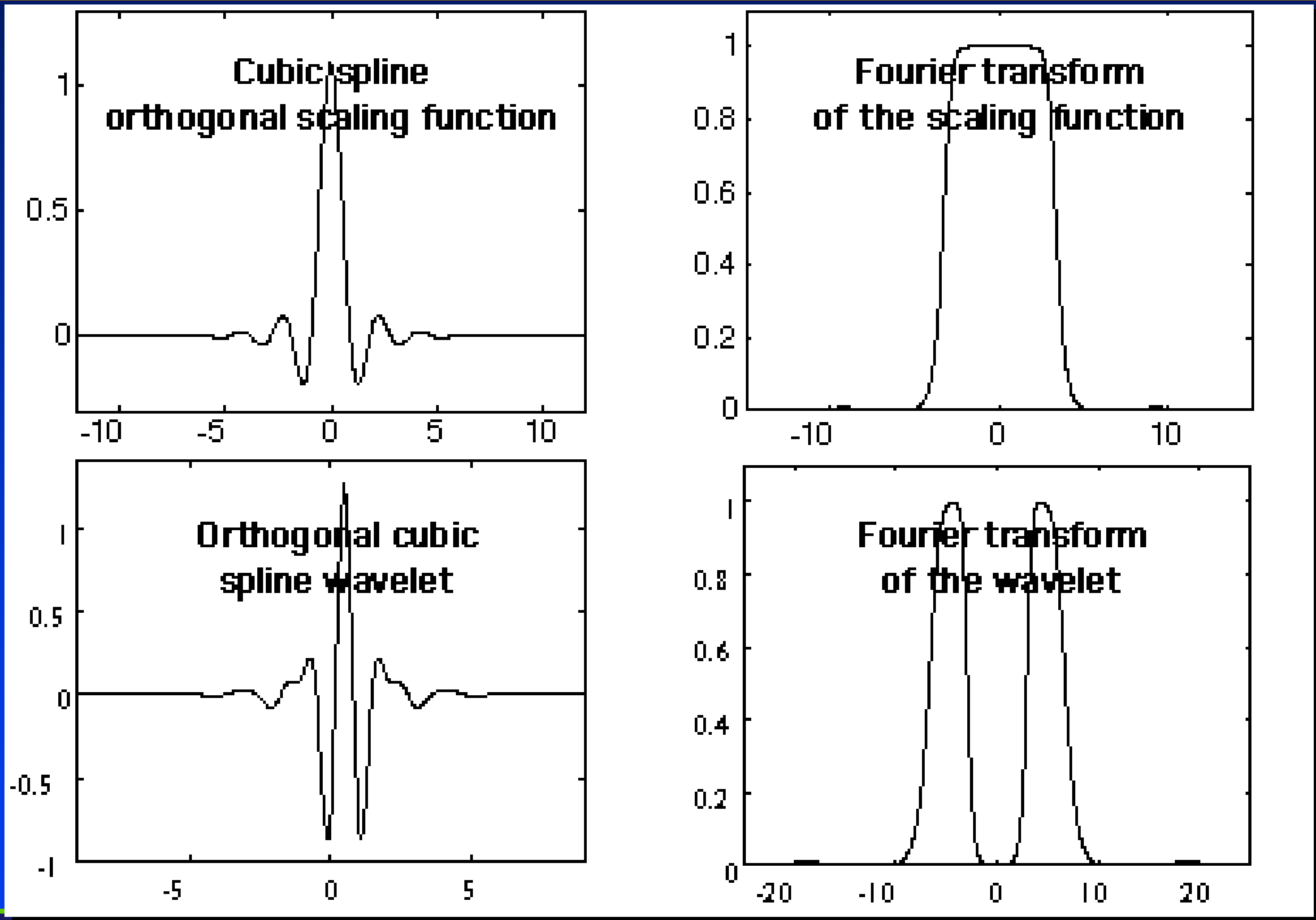


CT Lung Fly Through

Image Compression



Spline Wavelets



Another Applications of Image Interpolation

- Examples of situations when reconstruction of 3D object from 2D needed:
  - prosthesis design
  - surgery planning

Practical Problem

- The number of scans must be limited in order to protect patients from the risks related to X-ray absorption.
- 3D reconstruction accuracy depends on which set of image slices are used

- The main goal is to maximize the density of information minimizing the X-ray absorption.

Iso–Surface Extraction from Volume Data

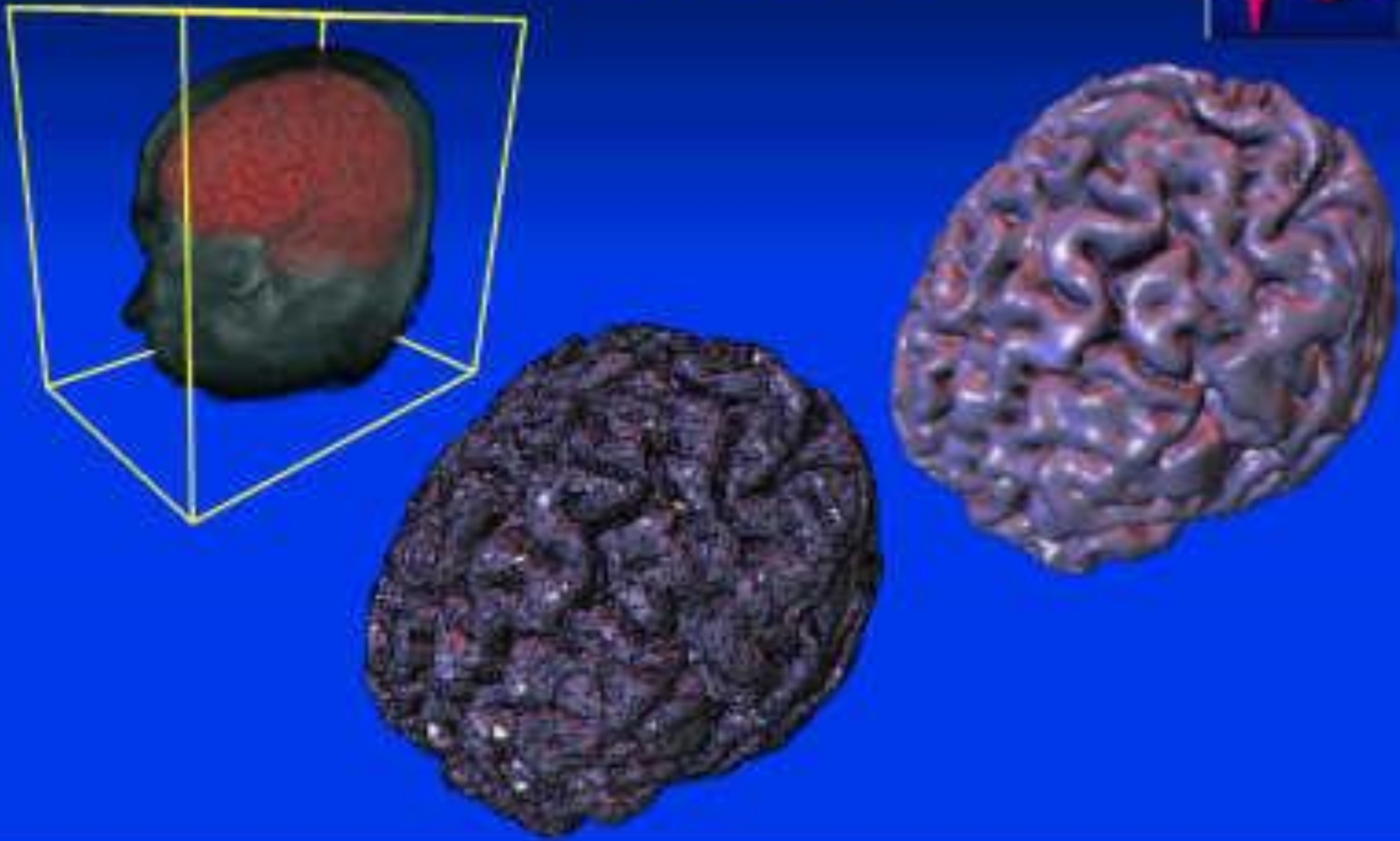


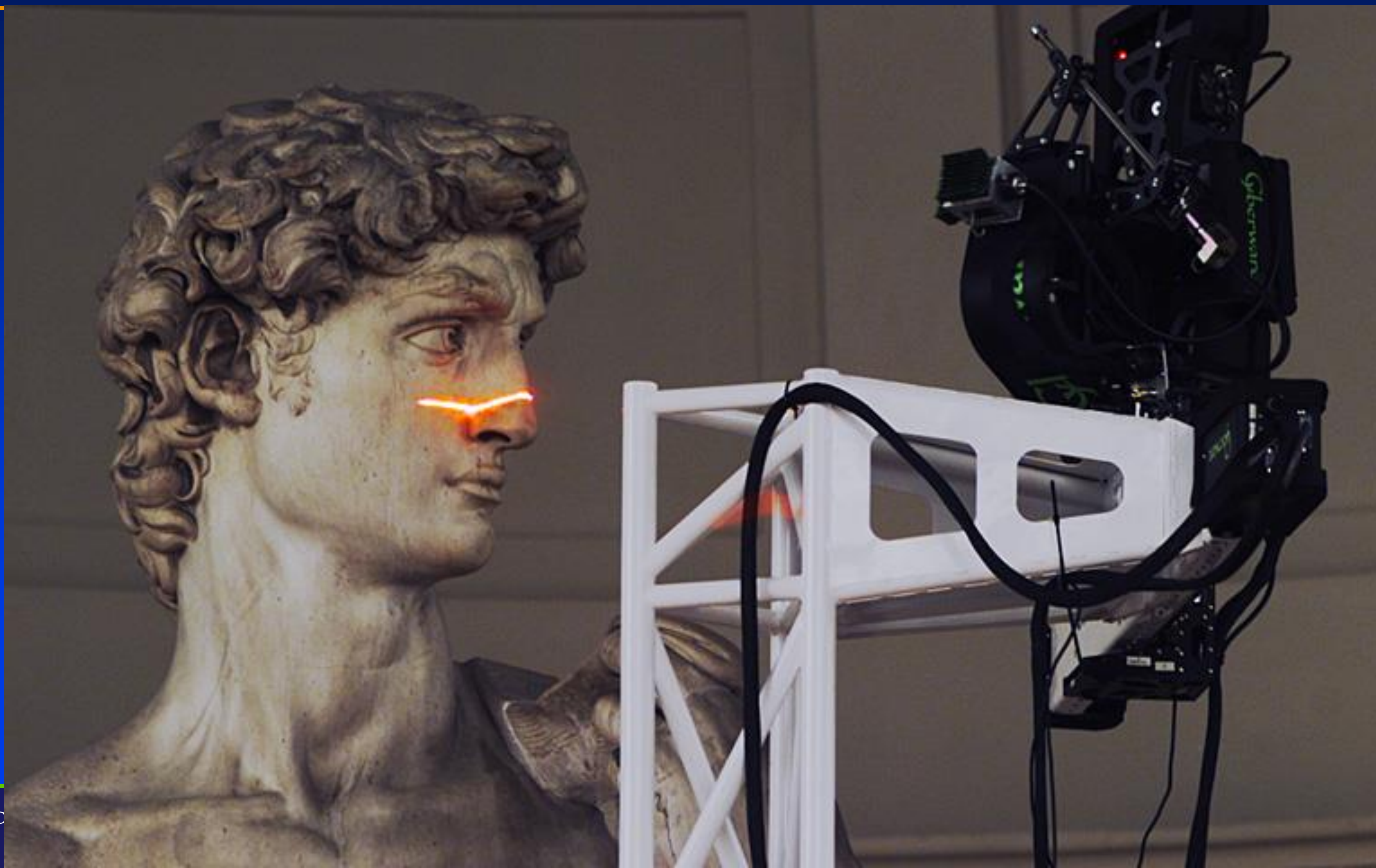
Image Morphing

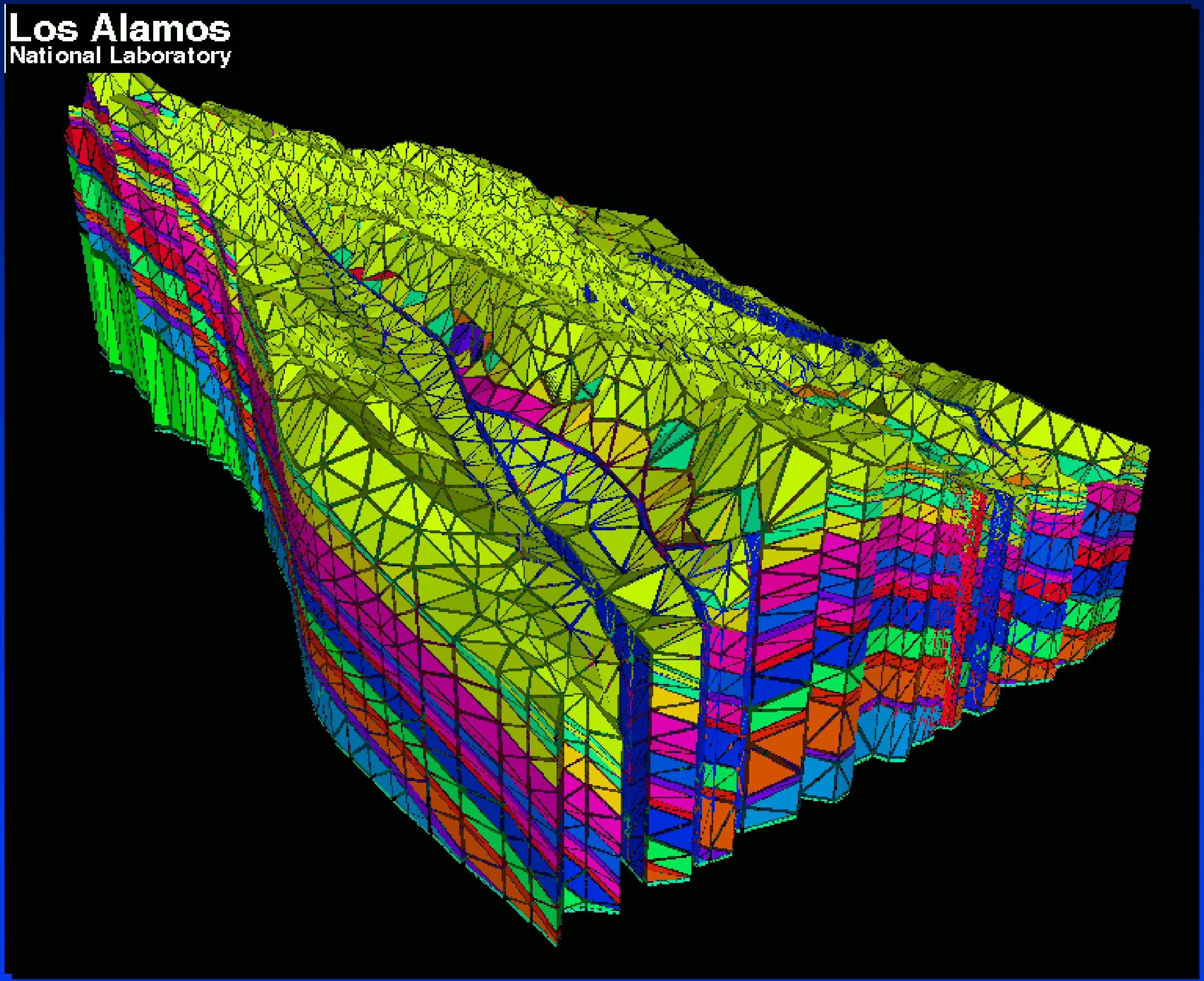
Image Morphing

- Image morphing is the construction of an image sequence depicting a gradual transition between two images.

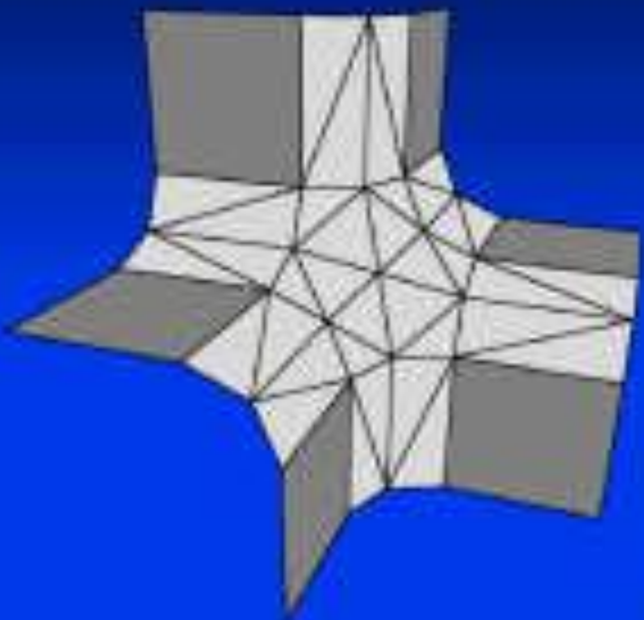
Hole Filling

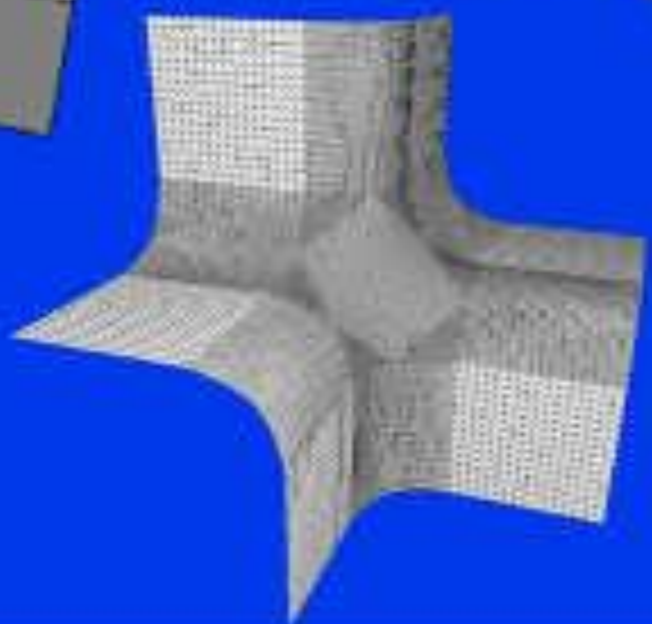
Reverse Engineering

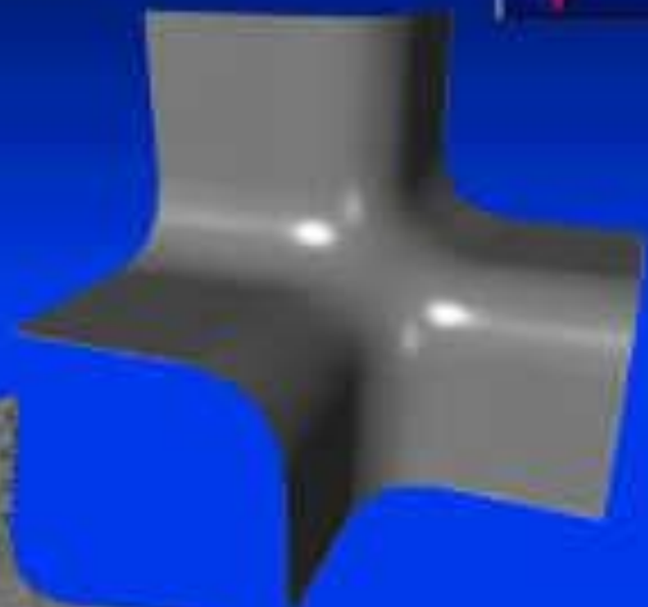



Los Alamos
National Laboratory

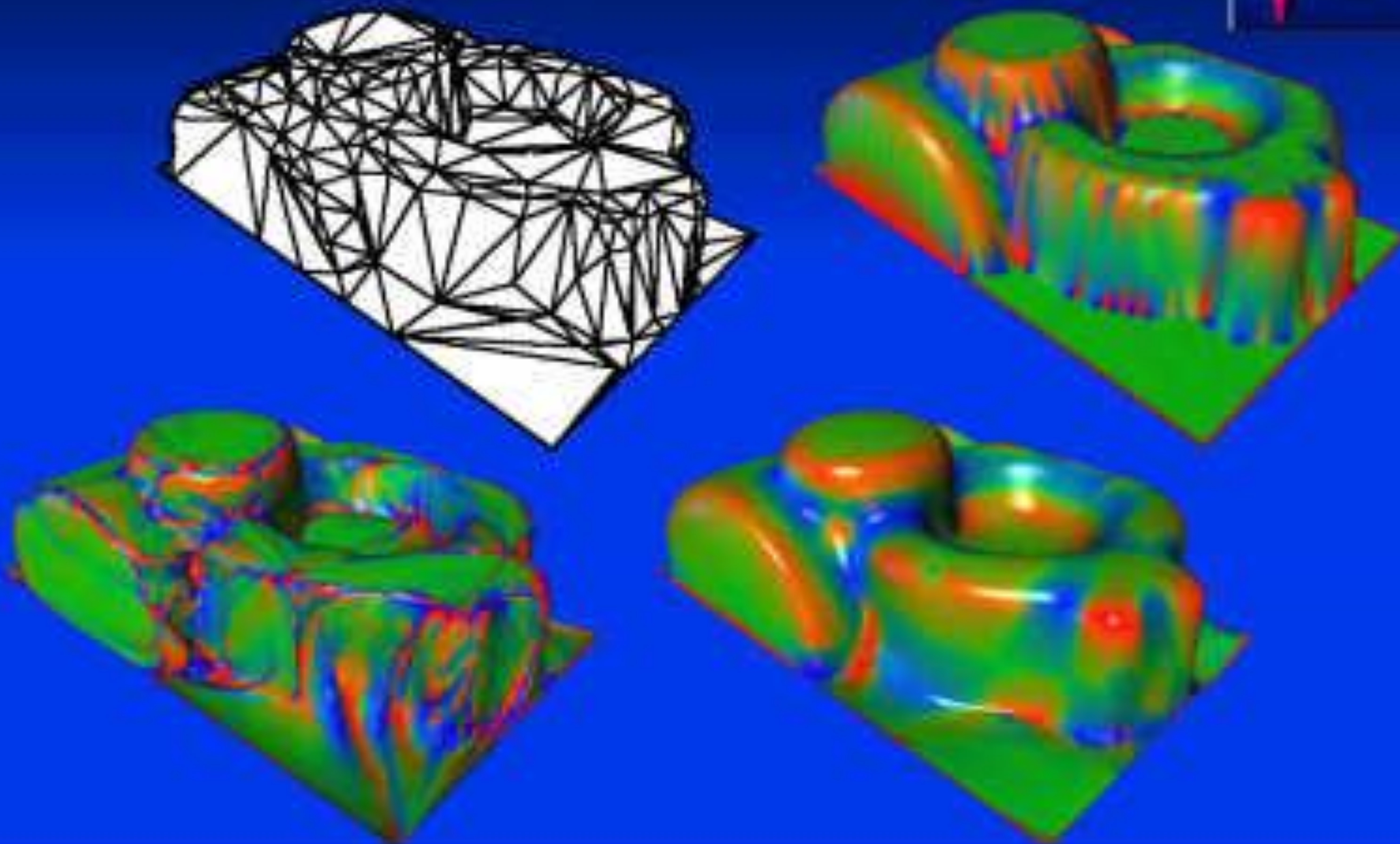
Blending & Hole Filling



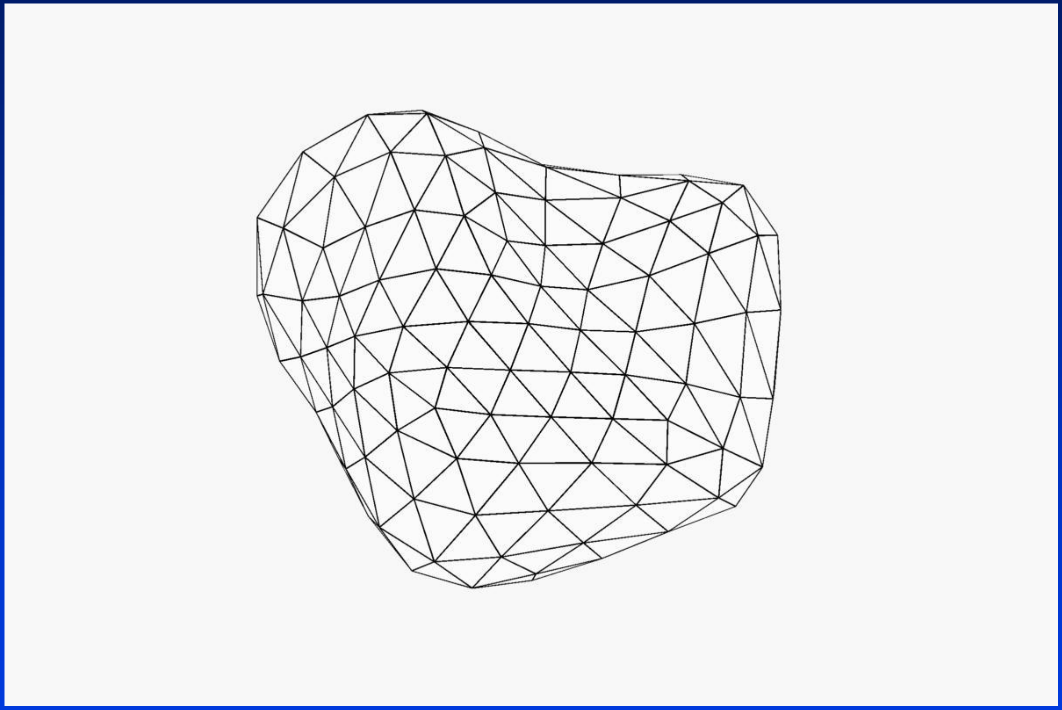




Smooth Shape (locally / globally)

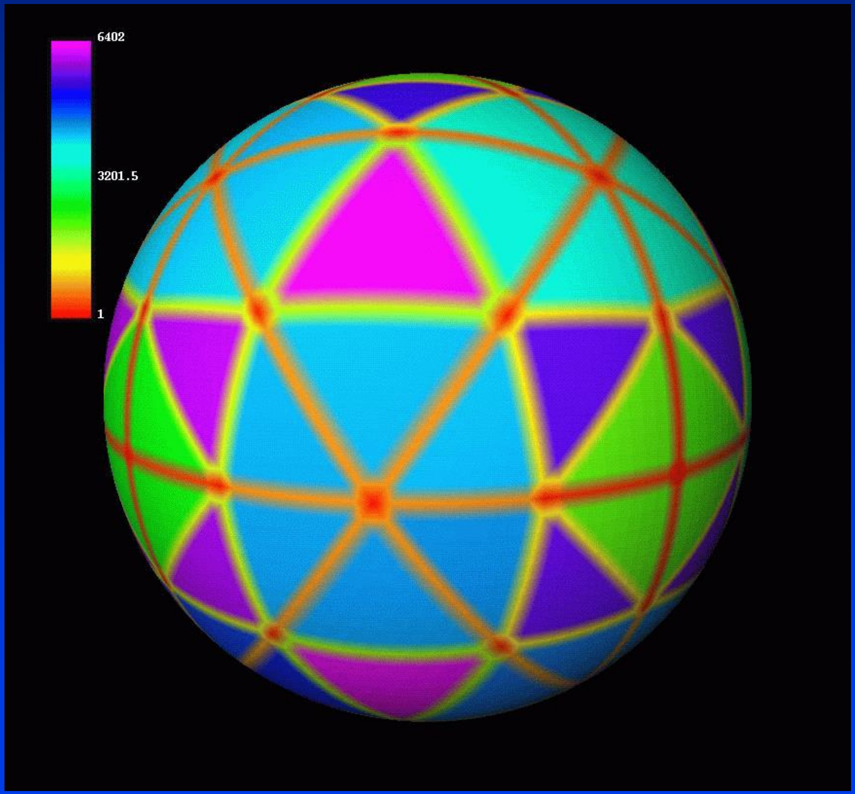


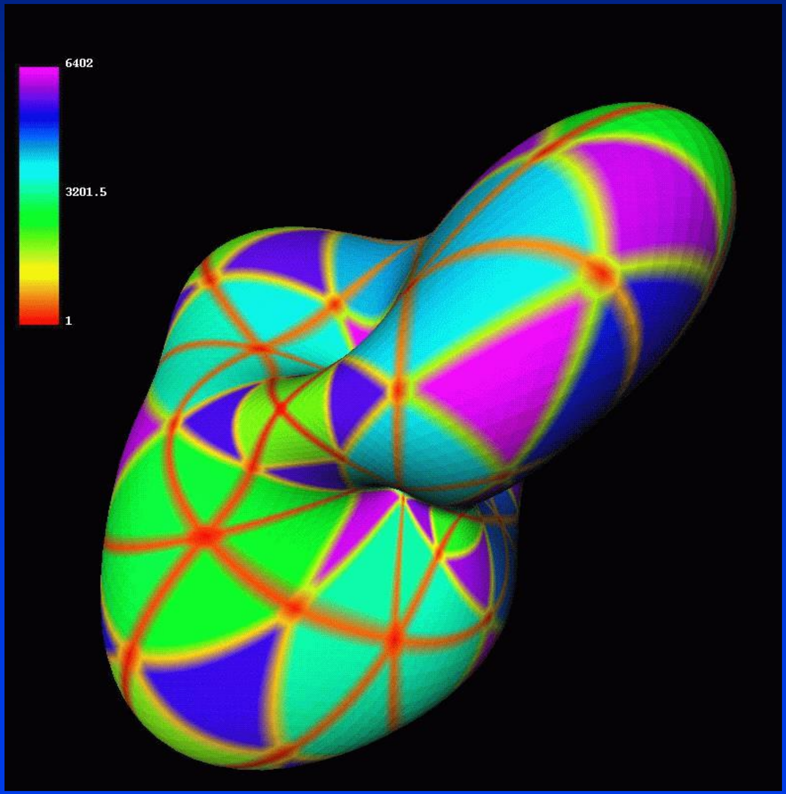
Bivariate Splines on Triangulations

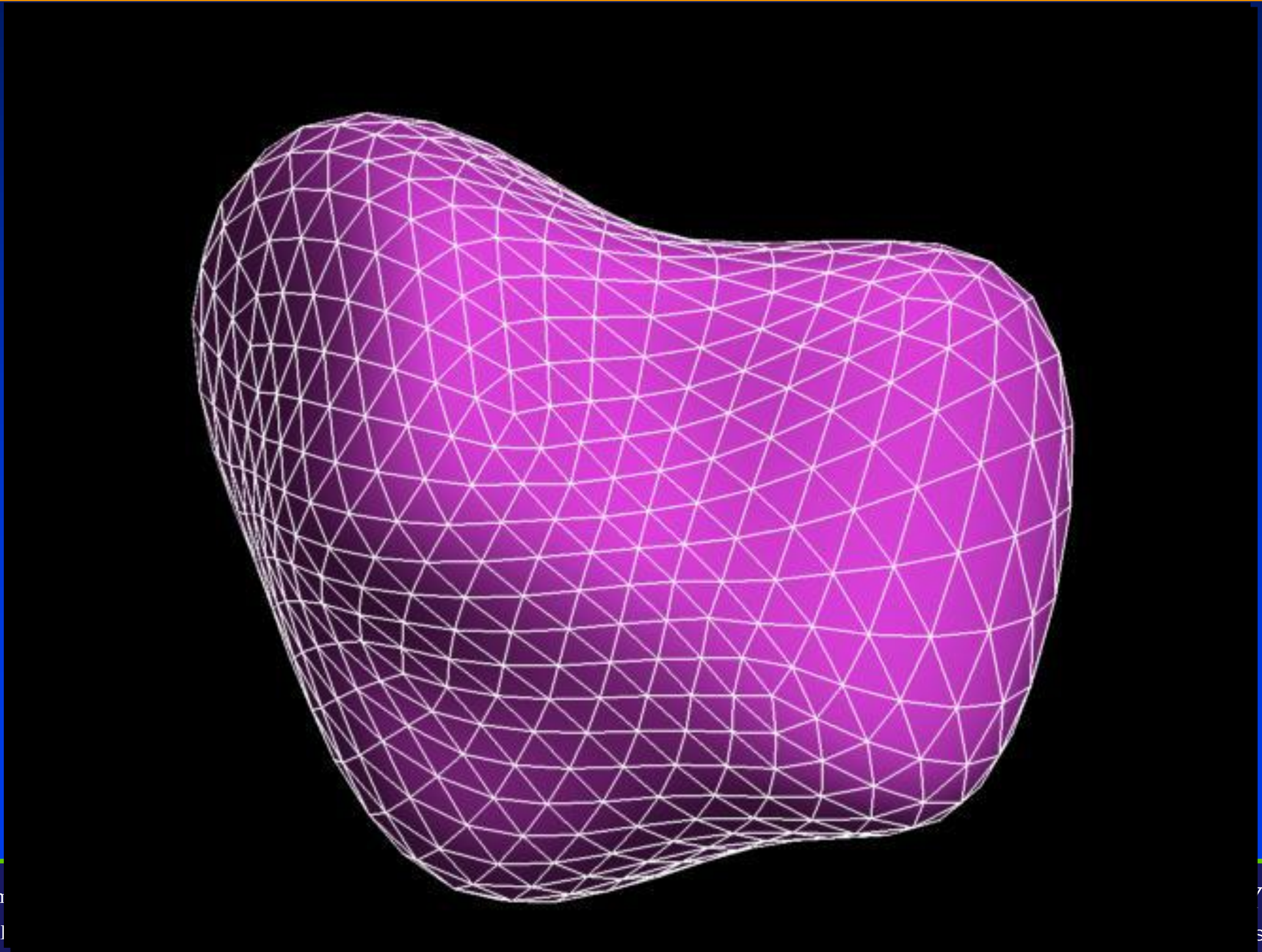


$$S_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathbb{P}_d, \text{ for all } T \in \triangle\},$$

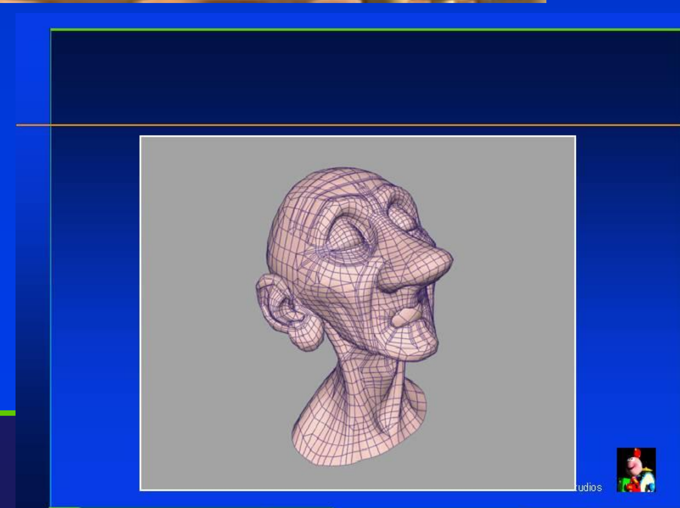
Splines defined on the Sphere and Other Manifolds



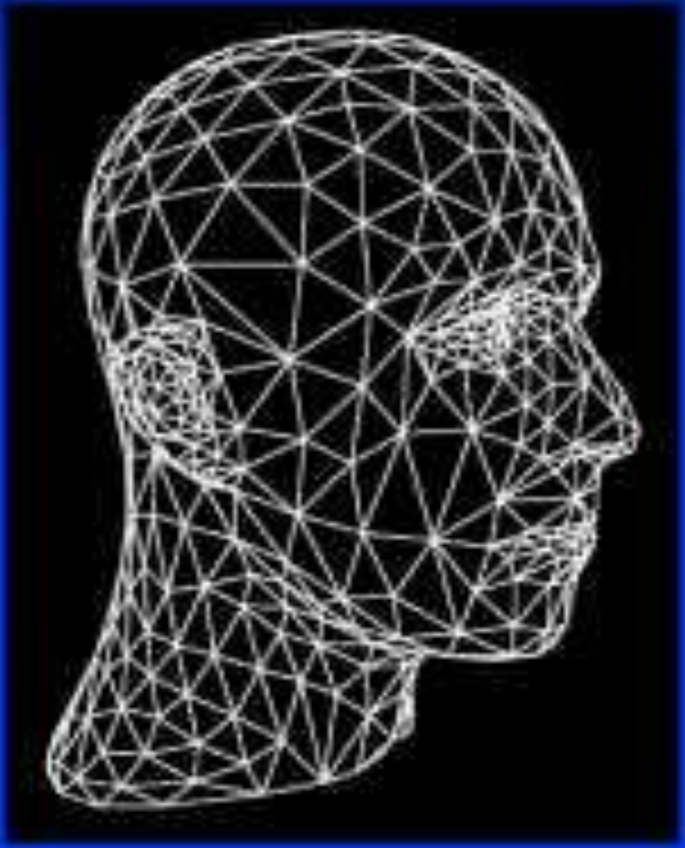


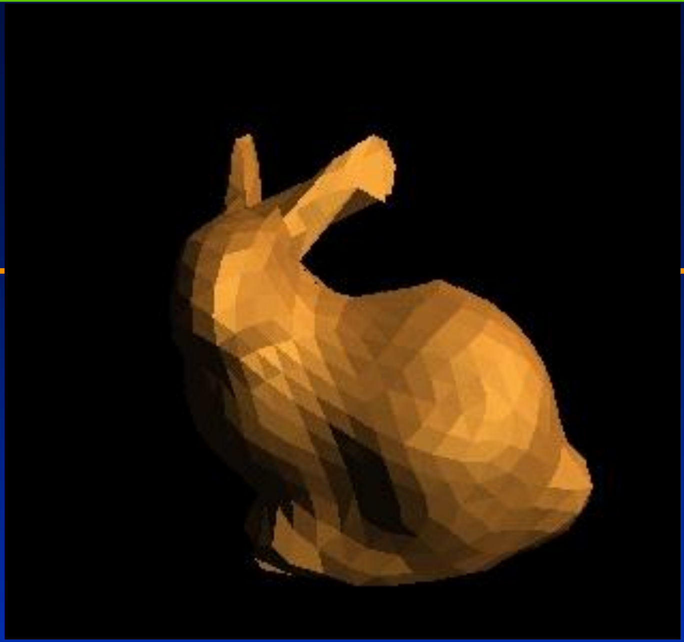


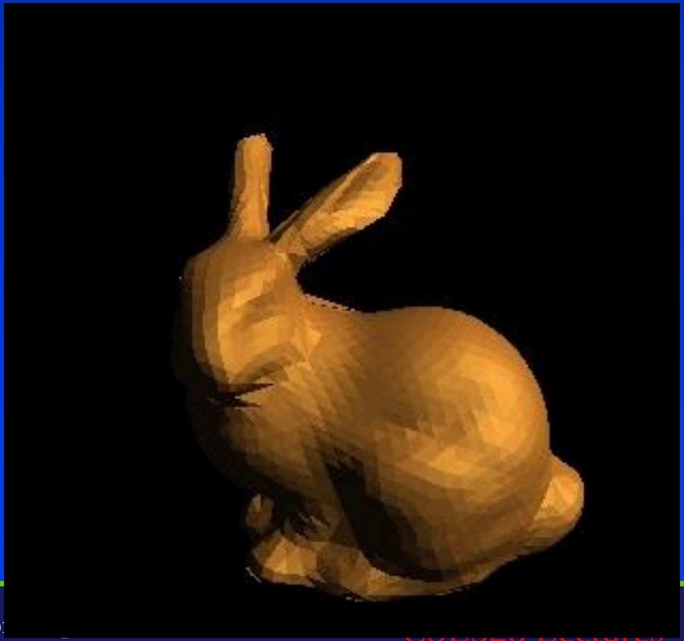

PIXAR SHORTS



Arbitrary Control Meshes







Interactive Multi–Res. Modeling

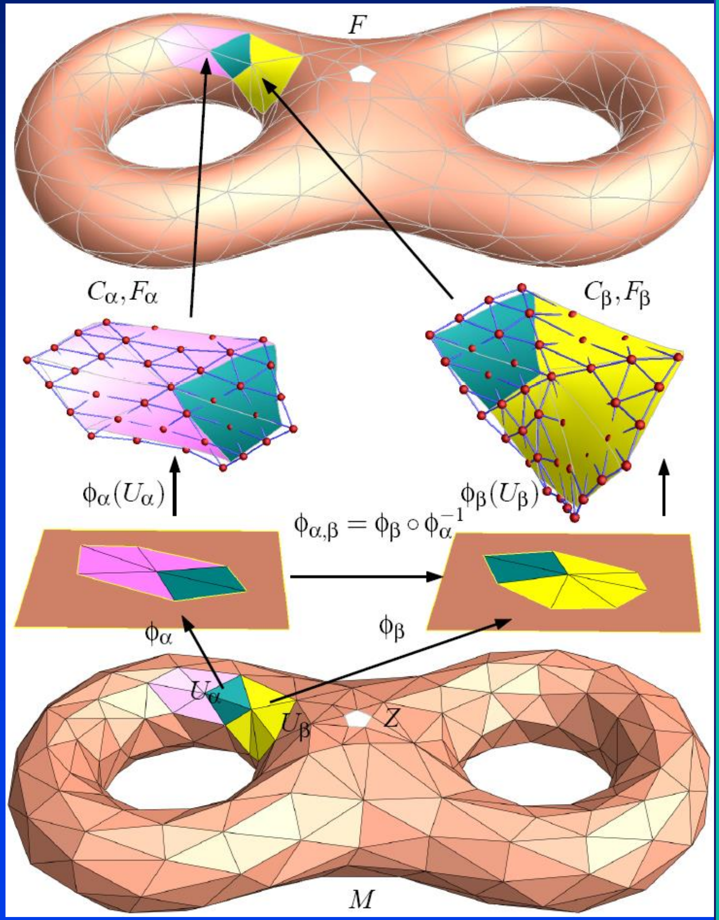
- Triangular Manifold Splines

  Xianfeng David Gu, Ying He, Hong Qin

- SMI 2005, “Manifold Splines”
  GMP 2006, “Manifold T-Splines” (Kexiang Wang, Hongyu Wang)

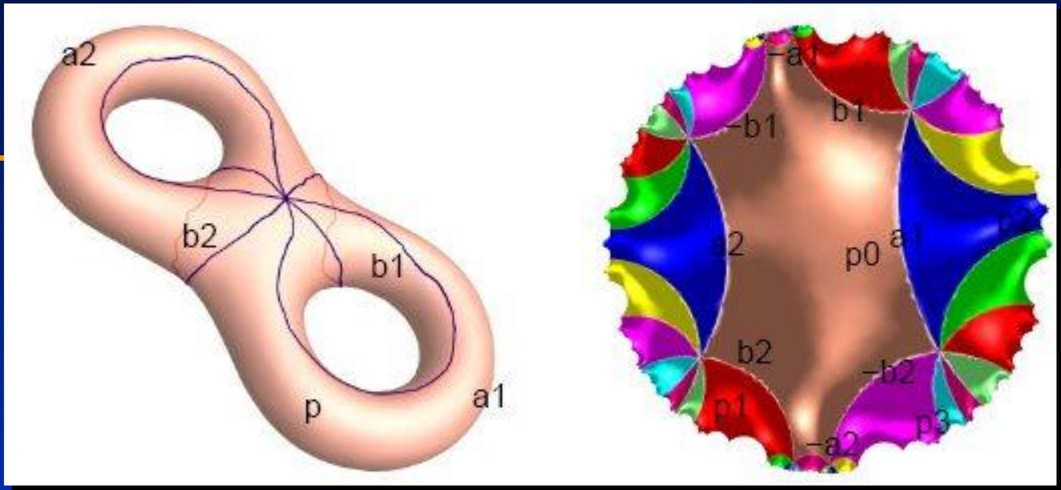
General Ideas and Pipeline

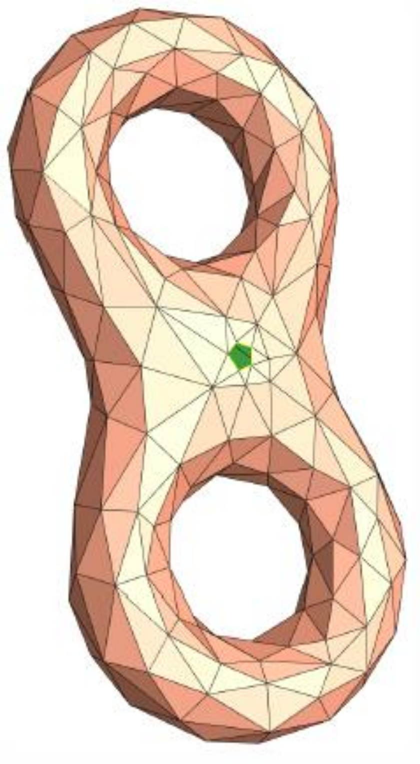
- Input: reasonably dense, manifold triangular mesh
- Construct a global affine parameterization
  - Affine transformations between groups of triangles
  - *(Except for a few points)*
- Construct charts on groups of triangles
  - Transition functions trivially affine
- Embedding functions are triangular splines
  - Splines with triangular mesh as "knot vector"
  - "Fix" holes by patching

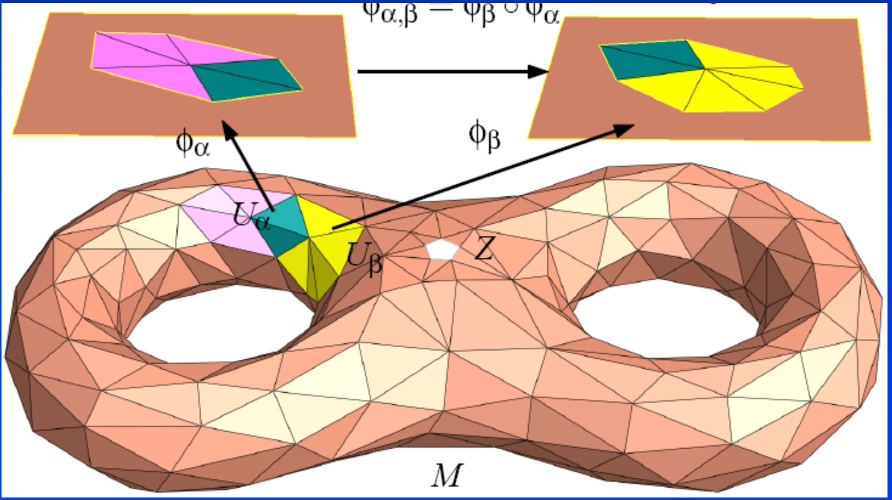


Global Parameterization

- Conformal parameterization
- Checkerboard, except for octagon
- Cut by removing center point
- Can now "unfold" locally into plane

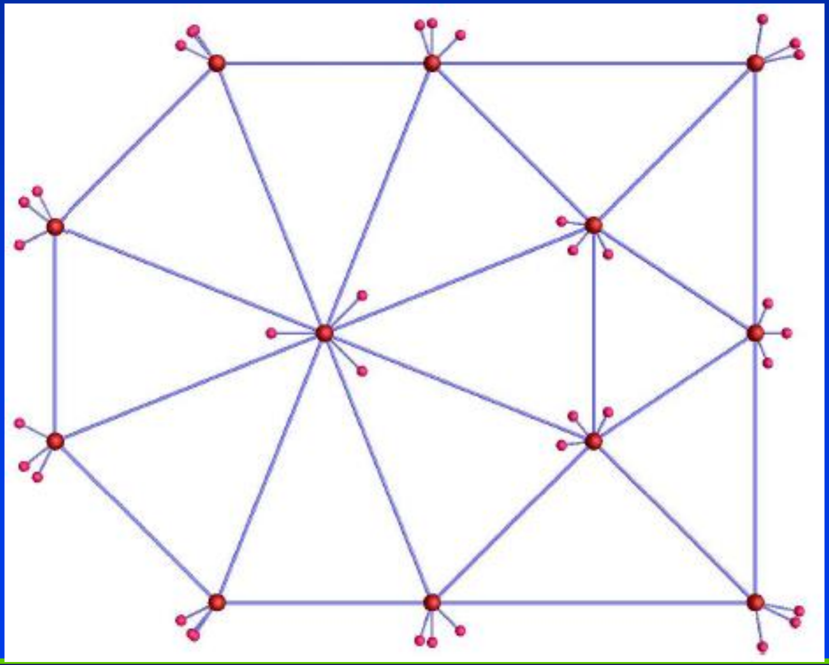

a2
b2
b1
p
a1
-a1
b1
-b1
a2
p0
a1
b2
-b2
p1
p3
-a2

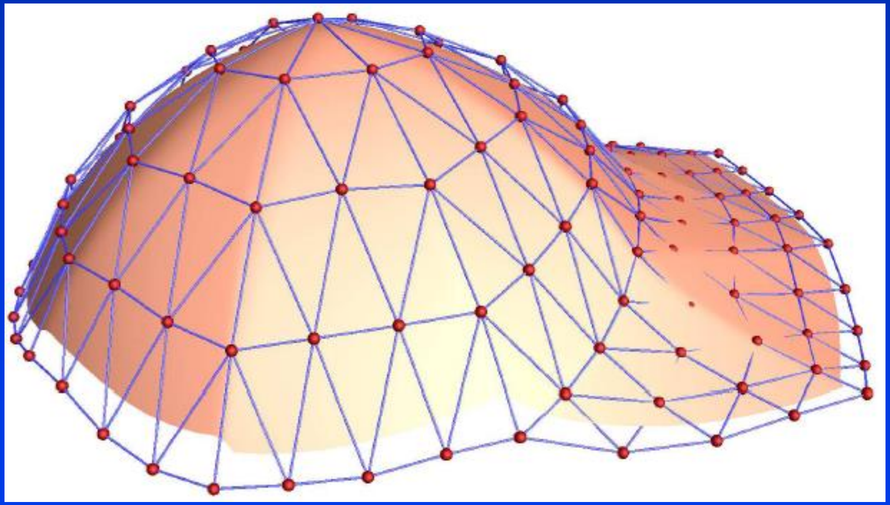



$\phi_{\alpha,\beta} = \phi_\beta \circ \phi_\alpha$
ϕ_α
ϕ_β
U_α
U_β
Z
M

Embedding Functions

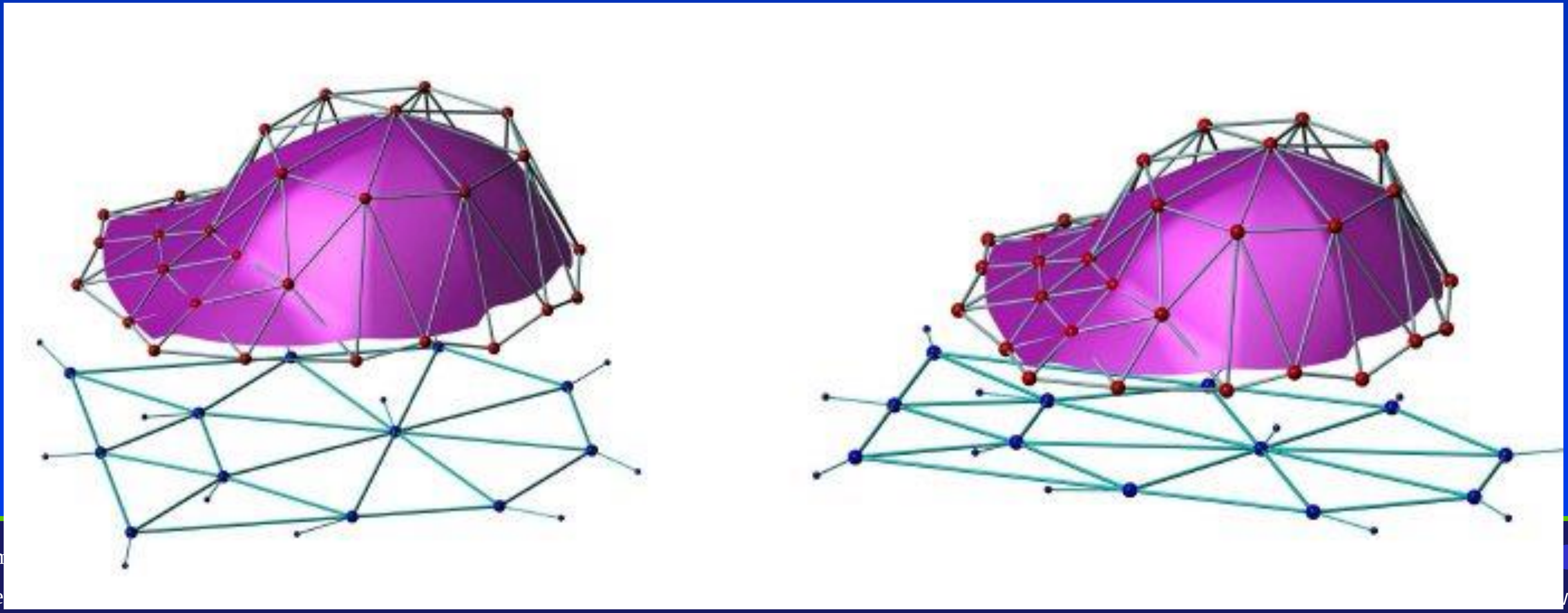
- Triangular splines
  - Built on 2D triangular mesh
  - Affine invariant





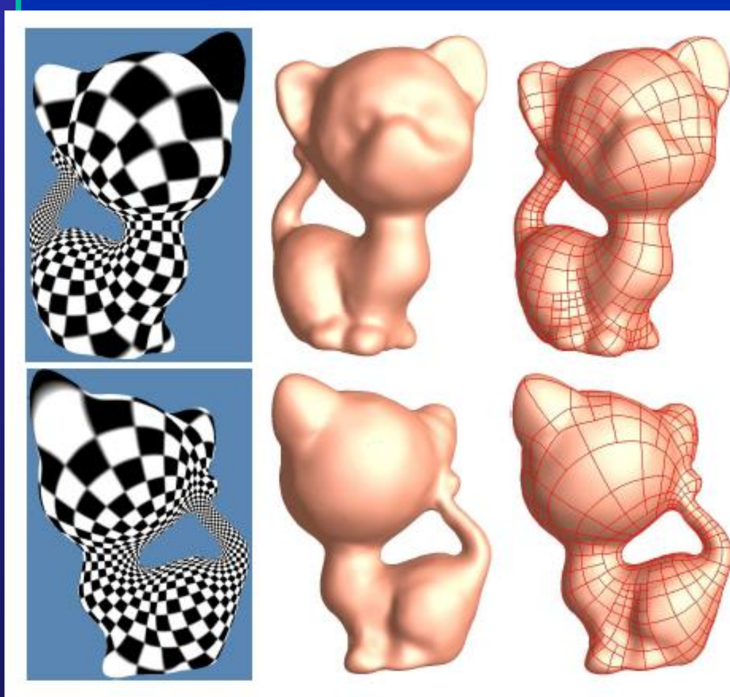
Avoid Functions' Blending

- Affine invariant embedding functions + affine transition functions means chart functions agree where they overlap
  - Use same control points



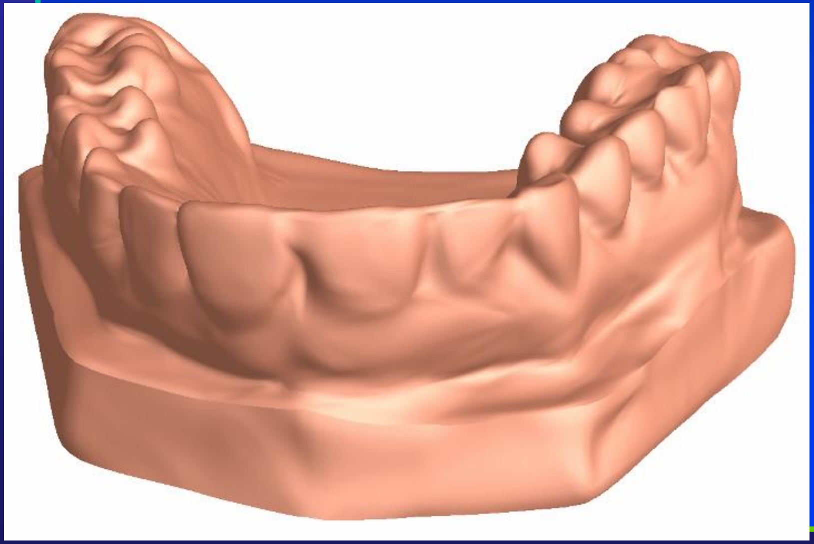
Other Types of Spline Functions

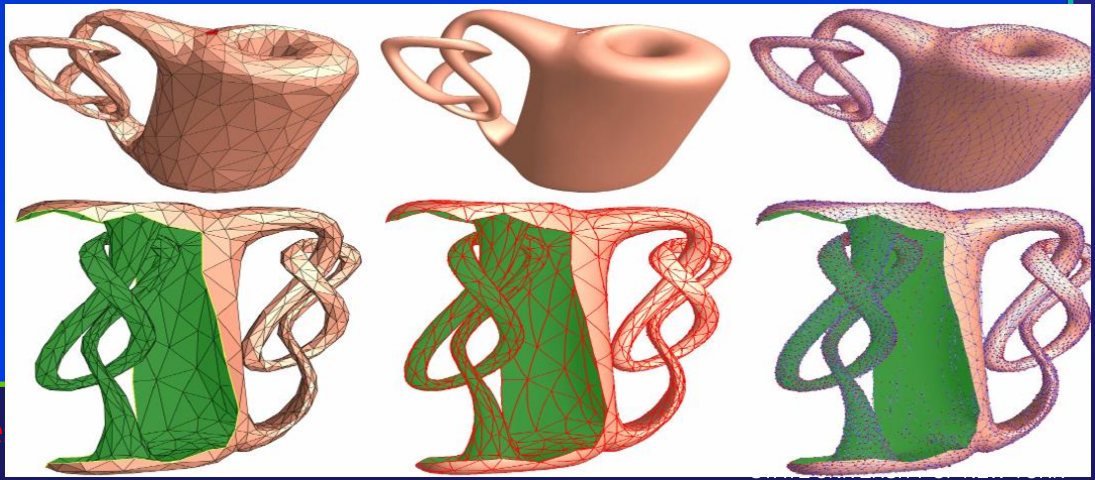
- Use T-splines, Powell-Sabin splines
- Can also use T-splines for embedding function
  - Globally parameterize to a square



Advantages

- Simple, affine transformations for transition functions
- C^k
- Triangular splines can handle sharp features





Disadvantages

- Need to fix holes in the parameterization
- Triangular splines require optimization
  - Also expensive to compute
- Limited control over parameterization