CSE528 Computer Graphics: Theory, Algorithms, and Applications

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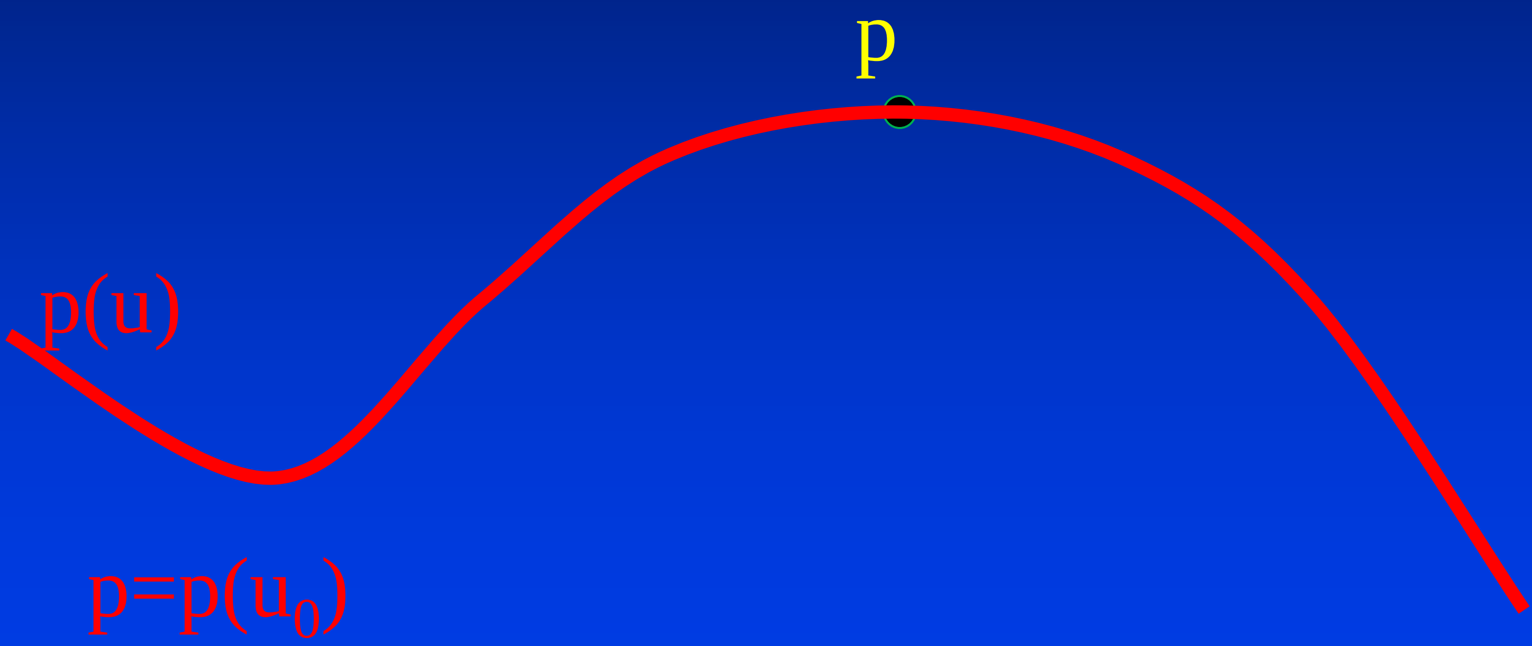
Differential Geometry for Curves and Surfaces (A Very Short Introduction)

Differential Geometry of a Curve

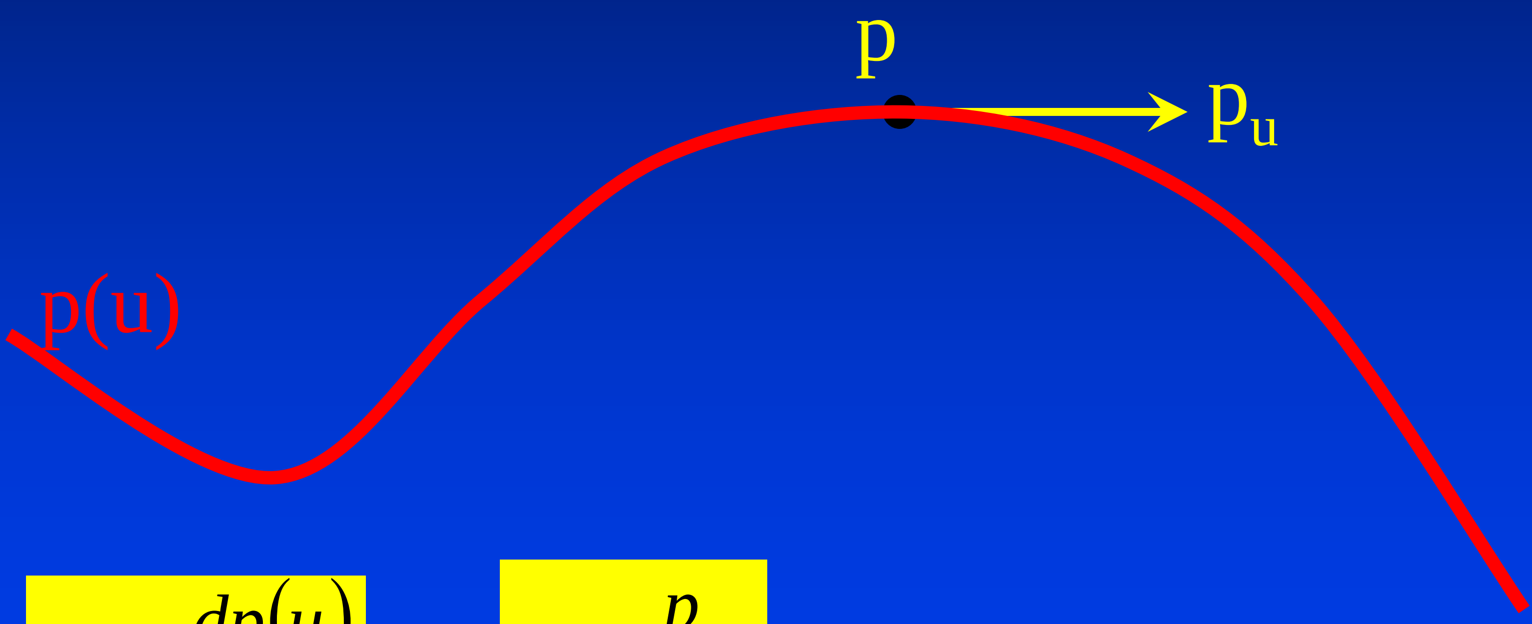
Differential Geometry of a Curve

Point p on the curve at u_0



Differential Geometry of a Curve

Tangent T to the curve at u_0



$$p_u = \frac{dp(u)}{du}$$

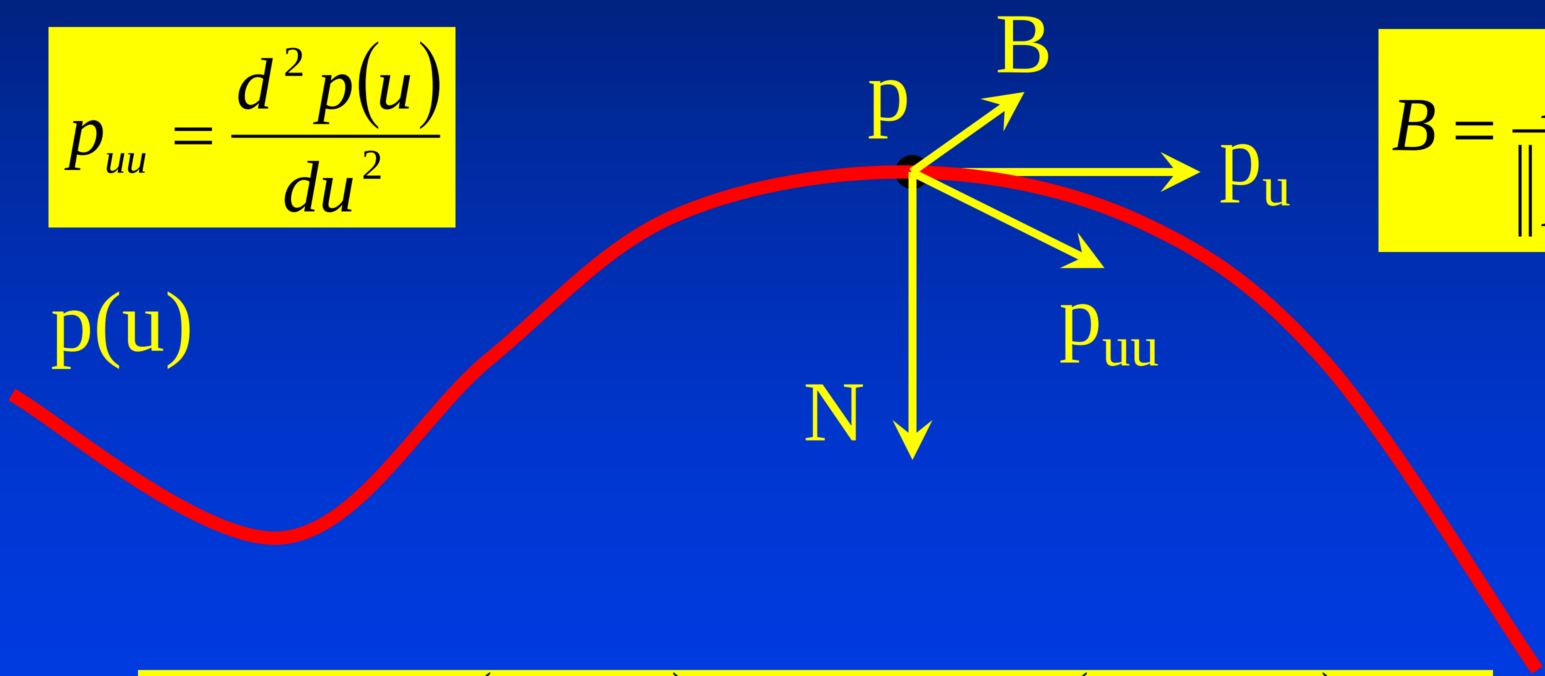
$$T = \frac{p_u}{\|p_u\|}$$

Differential Geometry of a Curve

Normal N and Binormal B to the curve at u_0

$$p_{uu} = \frac{d^2 p(u)}{du^2}$$

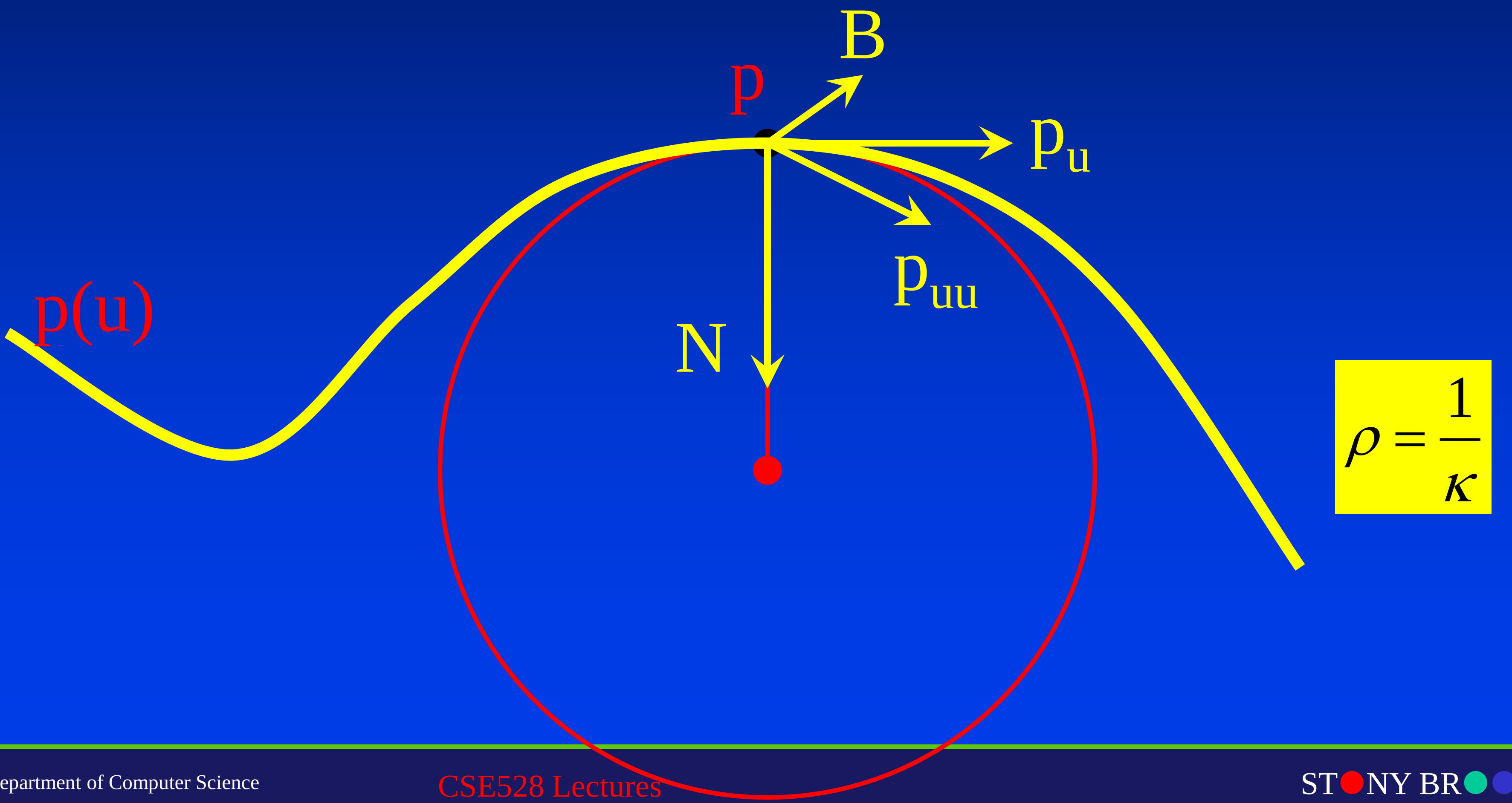
$$B = \frac{p_u \times p_{uu}}{\left\| p_u \times p_{uu} \right\|}$$



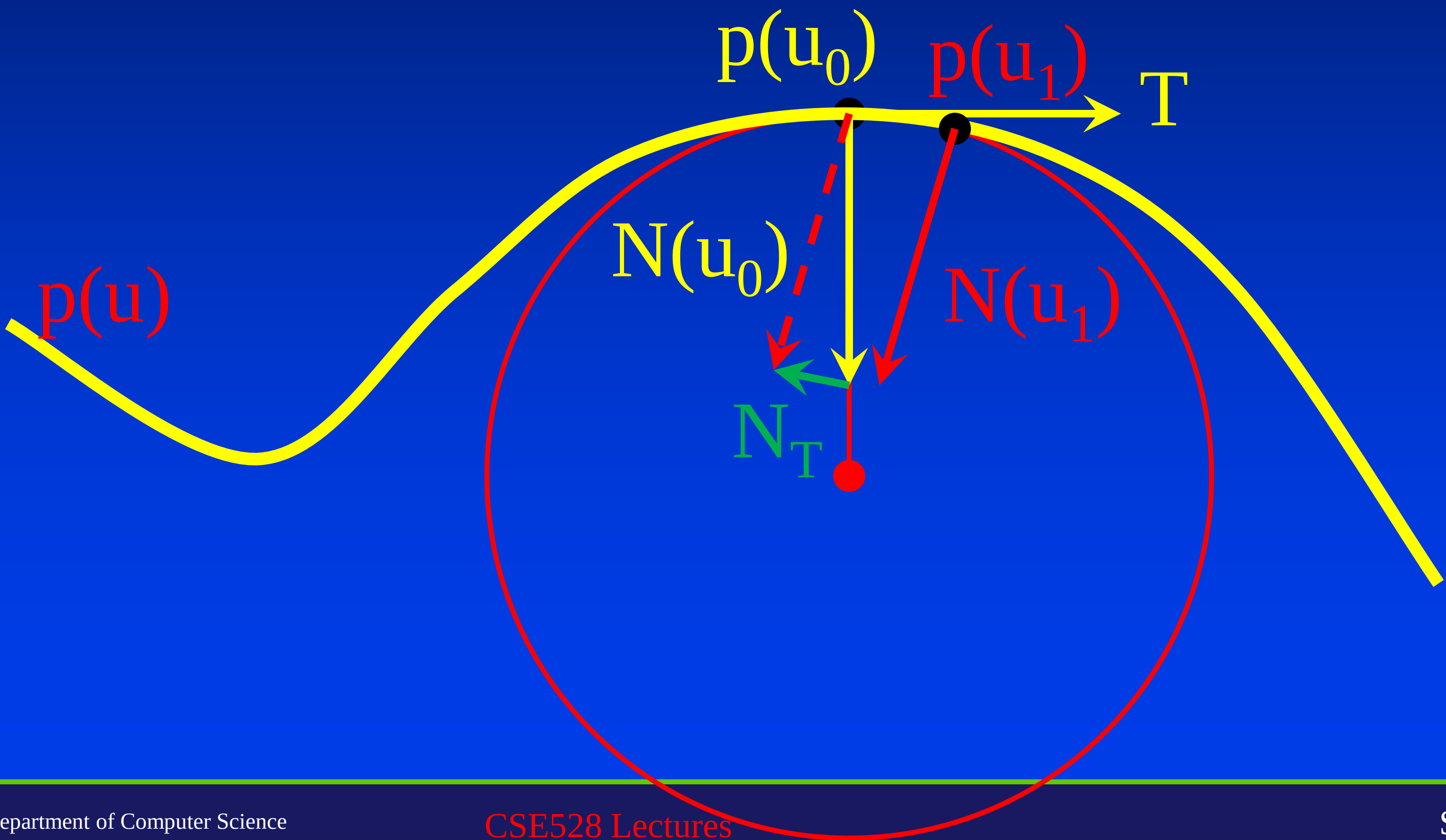
$$N = \frac{p_{uu} - (T \cdot p_{uu})T}{\left\| p_{uu} - (T \cdot p_{uu})T \right\|} = B \times T = \frac{(p_u \times p_{uu}) \times p_u}{\left\| p_u \times p_{uu} \right\| \left\| p_u \right\|}$$

Differential Geometry of a Curve

Curvature κ at u_0 and the radius ρ osculating circle



Differential Geometry of a Curve



Intrinsic Properties of Curves

- Different representations for the SAME curve

$$p(t) = (\cos(t), \sin(t))$$

$$q(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t^2}{1+t^2} \right)$$

Intrinsic Properties of Curves

$$p(t) = (\cos(t), \sin(t))$$

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$$p(0) = q(0) = (1,0)$$

Intrinsic Properties of Curves

$$p(t) = (\cos(t), \sin(t))$$

$$q(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t^2}{1+t^2} \right)$$

$$p(0) = q(0) = (1,0)$$

$$p'(0) = (0,1) \neq (0,2) = q'(0)$$

Identical curves but different derivatives!!!

Arc Length: The Basic Concept

$$s(t) = \int_a^t \| p'(t) \| dt$$

- $s(t)=t$ implies arc-length parameterization
- Independent under any parameterization!!!

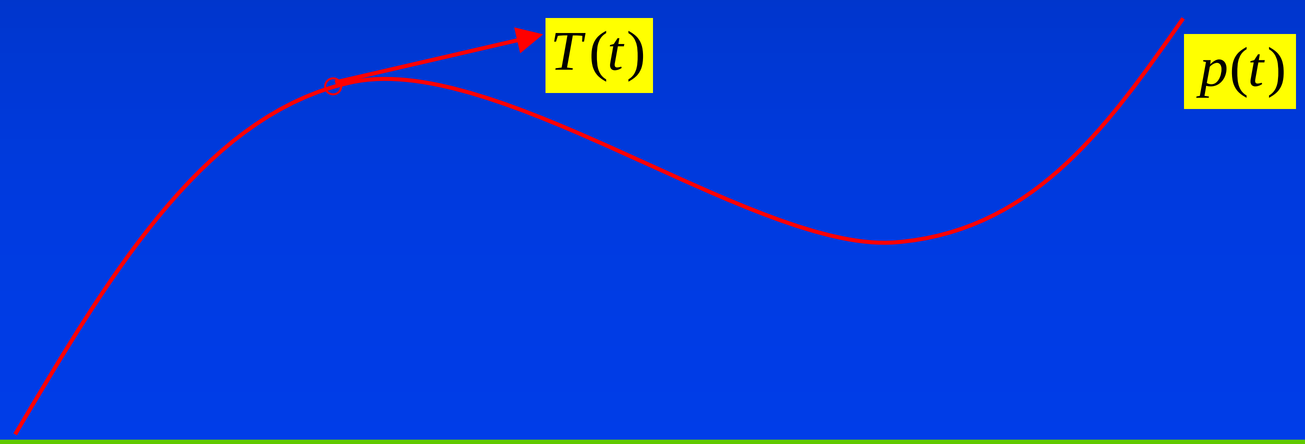
Parametric Curves

- A curve:
  - A set of points moving (along a curve) with one degree of freedom
- Torsion:
  - How much a spatial curve deviates from a plane – how much it attempts to “escape” the osculating plane
- Arc length:
  - The real length that is measured along a curve
- Characterization of all planar curves:
  - torsion = 0
- Characterization of all straight lines:
  - curvature = 0

Frenet Frame

- Unit-length tangent

$$T(t) = \frac{p'(t)}{\|p'(t)\|}$$



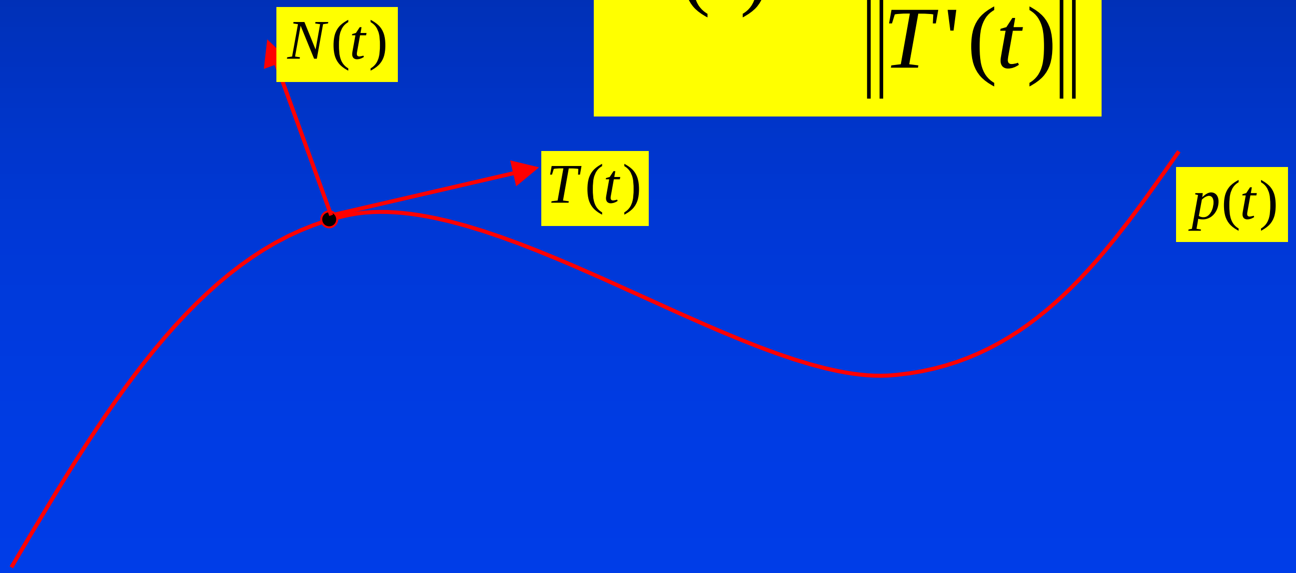
Frenet Frame

- Unit-length tangent

$$T(t) = \frac{p'(t)}{\|p'(t)\|}$$

- Unit-length normal

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$



Frenet Frame

- Unit-length tangent

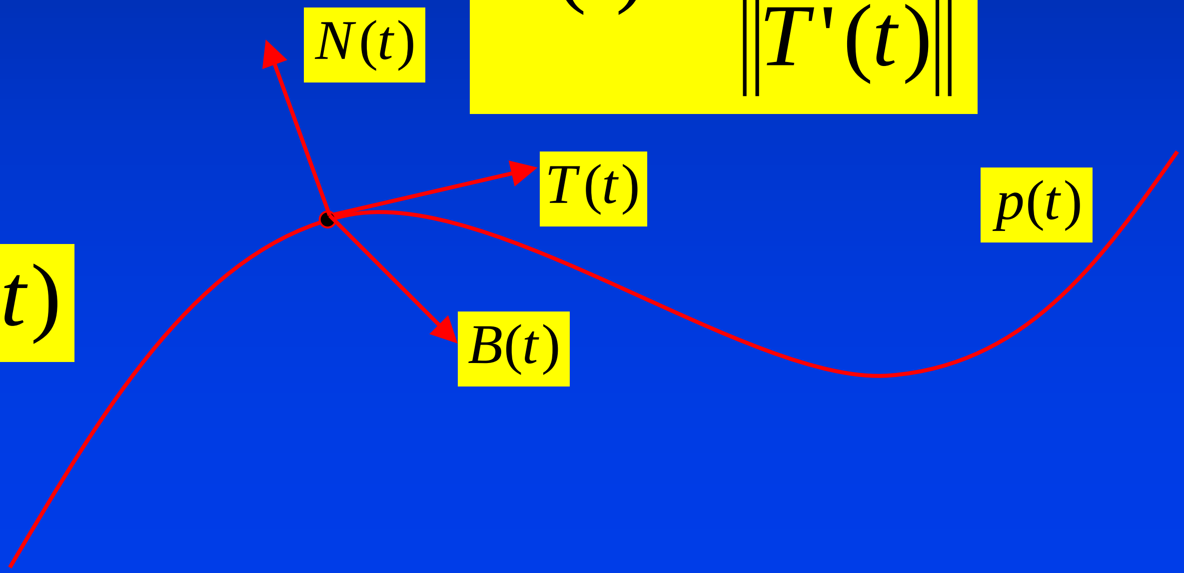
$$T(t) = \frac{p'(t)}{\|p'(t)\|}$$

- Unit-length normal

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

- Binormal

$$B(t) = T(t) \times N(t)$$



Frenet Frame

$$T(t) = \frac{p'(t)}{\left\| p'(t) \right\|}$$

$$N(t) = \frac{T'(t)}{\left\| T'(t) \right\|}$$

$$B(t) = T(t) \times N(t)$$

- Provides an orthogonal frame anywhere on curve

$$B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0$$

Frenet Frame

$$T(t)=\frac{p'(t)}{\|p'(t)\|}$$

$$N(t)=\frac{T'(t)}{\|T'(t)\|}$$

$$B(t)=T(t)\times N(t)$$

- Provides an orthogonal frame anywhere on curve

$$B(t)\cdot T(t)=B(t)\cdot N(t)=T(t)\cdot N(t)=0$$

Trivial due to the cross-product computation

Frenet Frame

$$T(t) = \frac{p'(t)}{\|p'(t)\|}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$B(t) = T(t) \times N(t)$$

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$$T(t) \cdot T(t) = 1$$

Frenet Frame

$$T(t) = \frac{p'(t)}{\|p'(t)\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|} \qquad B(t) = T(t) \times N(t)$$

- Provides an orthogonal frame anywhere on curve

$$B(t) \cdot T(t) = B(t) \cdot N(t) = T(t) \cdot N(t) = 0$$

$$T(t) \cdot T(t) = 1$$

$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$$

Frenet Frame

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$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$$

$$T(t) \cdot N(t) = 0$$

Frenet Frames: Applications

- Camera motion animation
- Extruding a cylinder along a path (generalized cylinders)

Frenet Frames

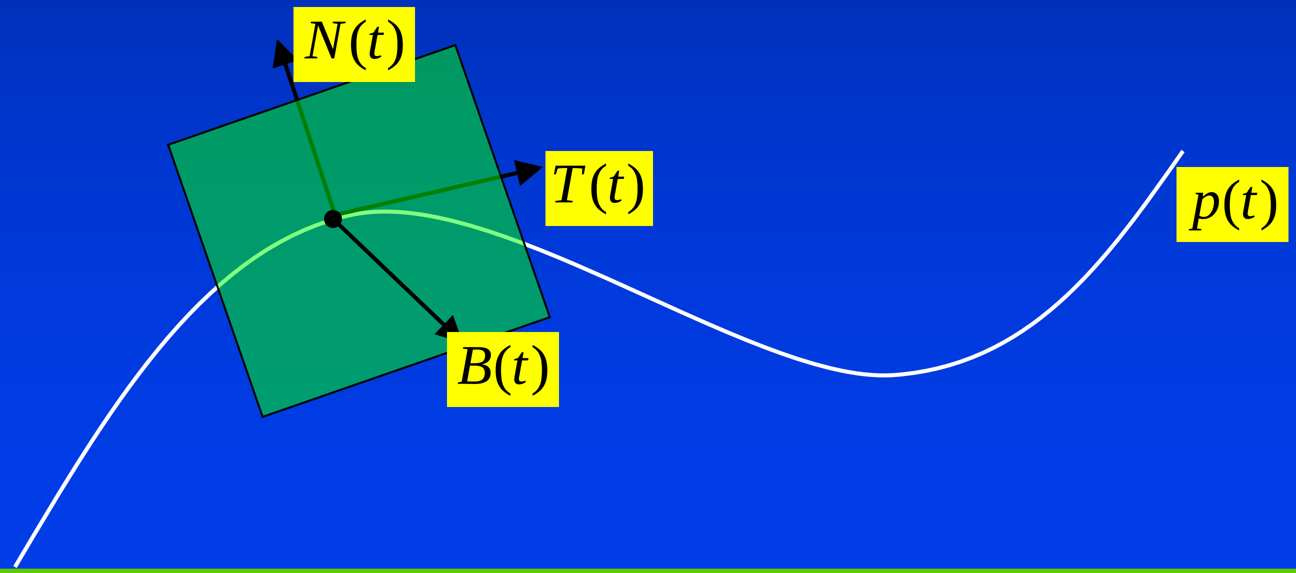
- Problems: The Frenet frame becomes unstable at inflection points or even undefined when

$$T'(t) = 0$$

$$T(t) = \frac{p'(t)}{\|p'(t)\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|} \qquad B(t) = T(t) \times N(t)$$

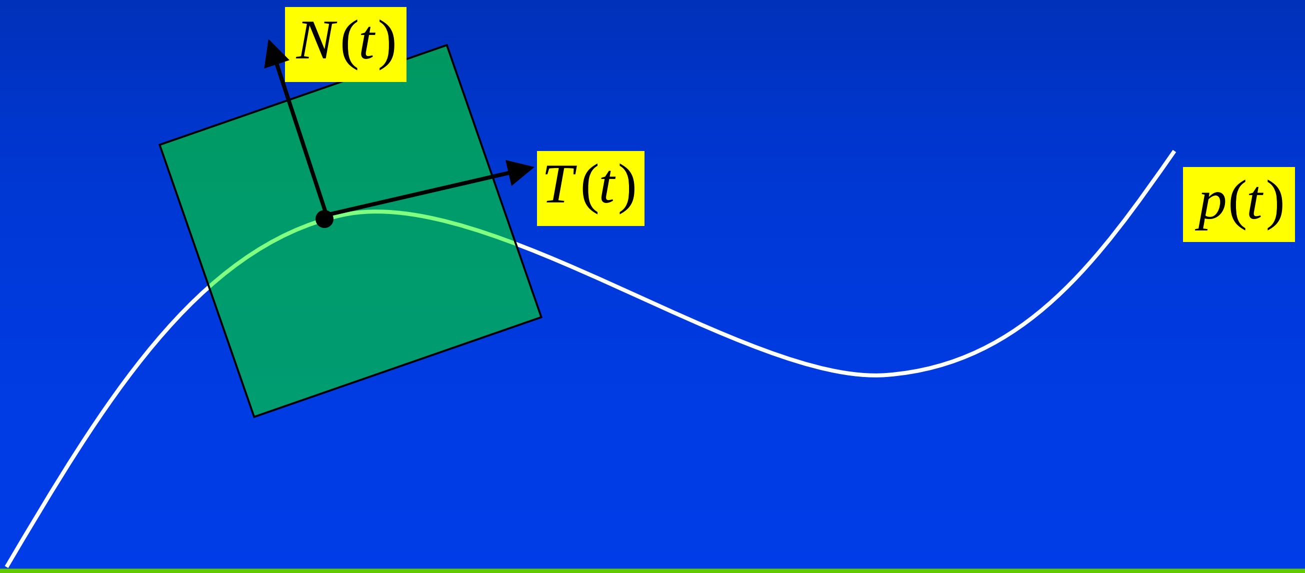
Osculating Plane

- Plane defined by the point $p(t)$ and the vectors $T(t)$ and $N(t)$
- Locally the curve resides in this plane



Curvature

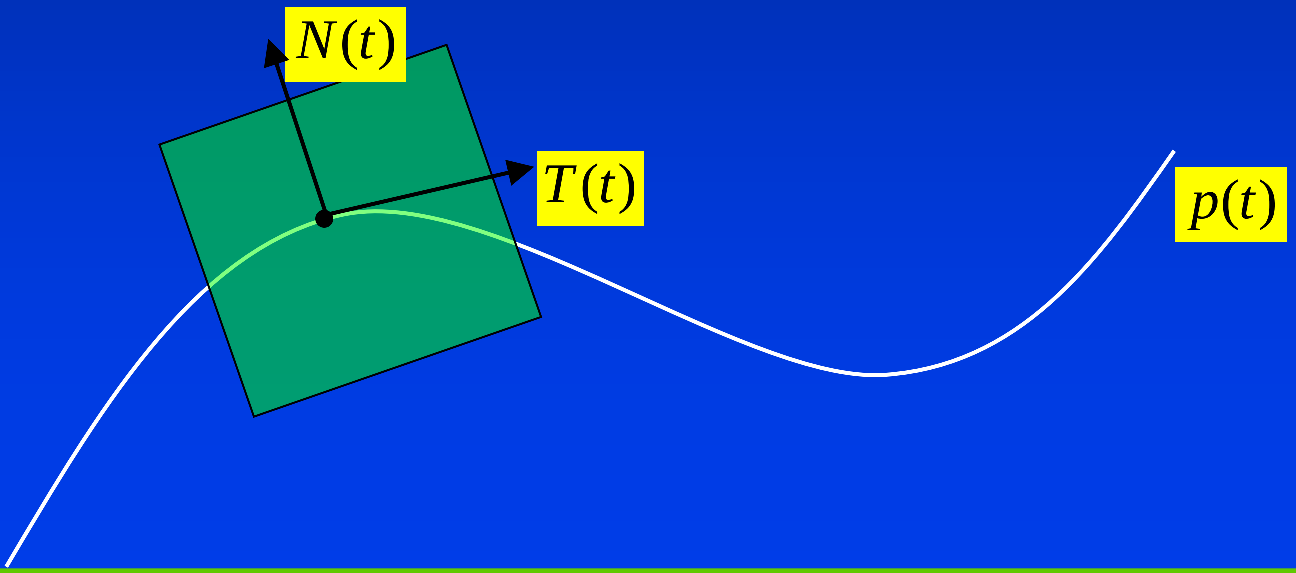
- Measure how much the curve bends



Curvature

- Measure how much the curve bends

$$\kappa = \left\| \frac{\partial T}{\partial s} \right\|$$

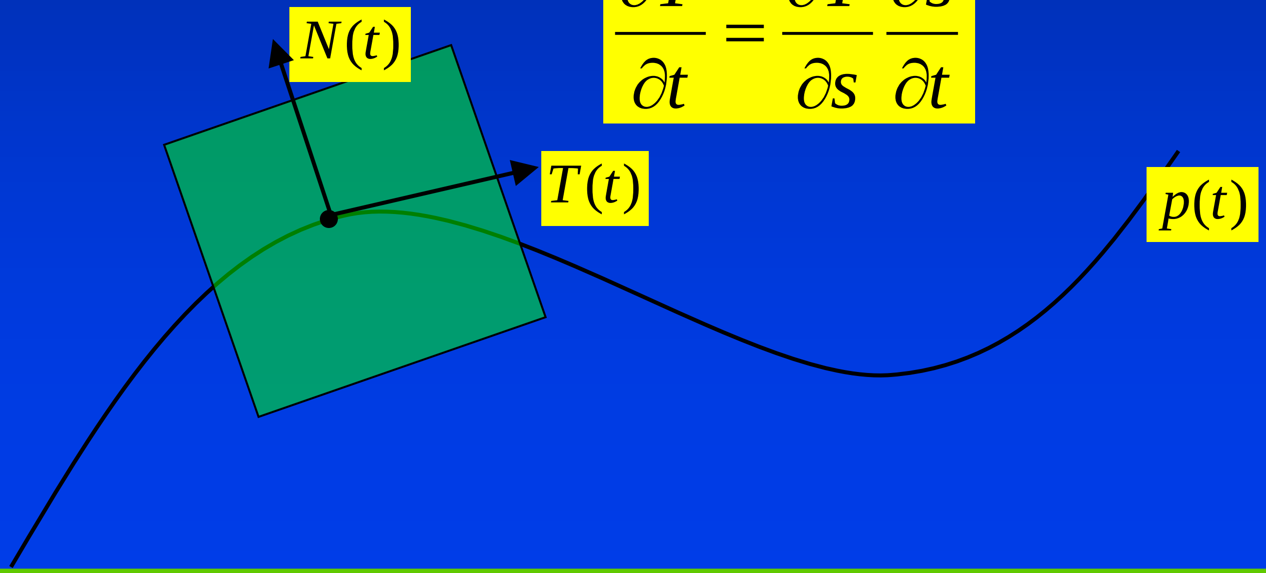


Curvature

- Measure of how much the curve bends

$$\kappa = \left\| \frac{\partial T}{\partial s} \right\|$$

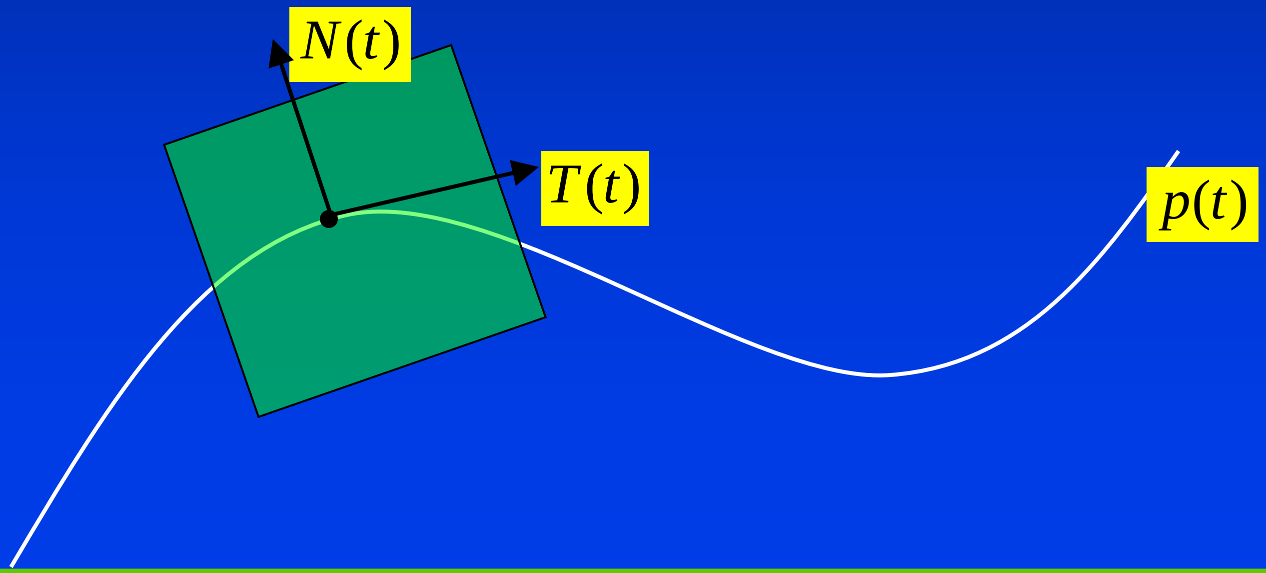
$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial s} \frac{\partial s}{\partial t}$$



Curvature

- Measure how much the curve bends

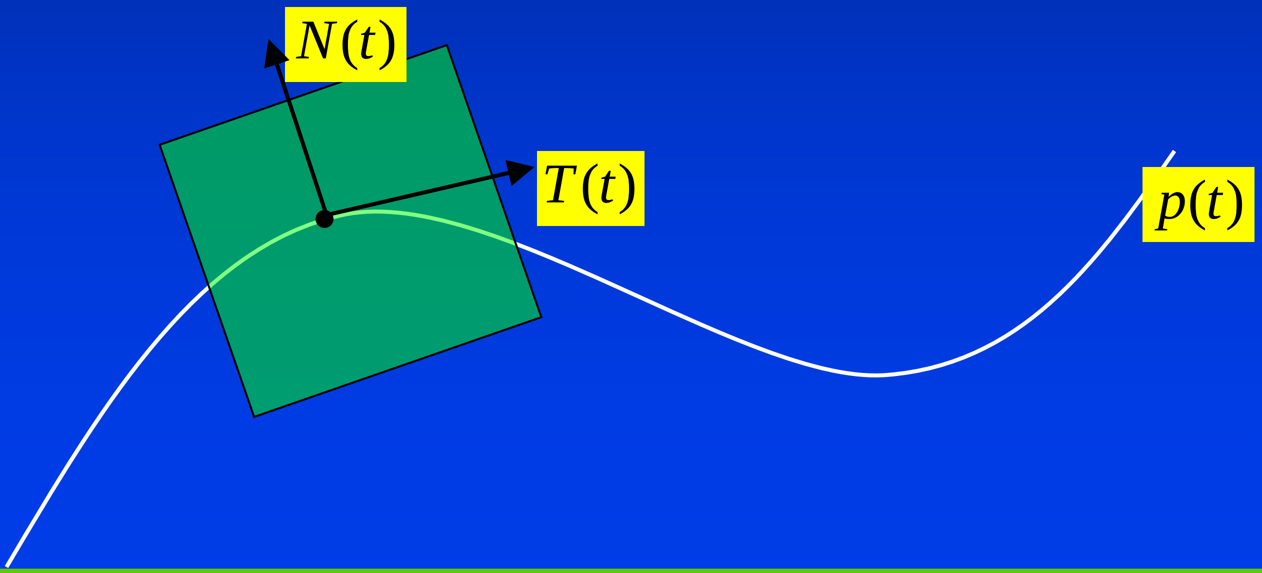
$$\kappa(t) = \frac{\|T'(t)\|}{\|p'(t)\|}$$



Curvature

- Measure how much the curve bends

$$\kappa(t) = \frac{\|T'(t)\|}{\|p'(t)\|} = \frac{\|p'(t) \times p''(t)\|}{\|p'(t)\|^3}$$



Curvature

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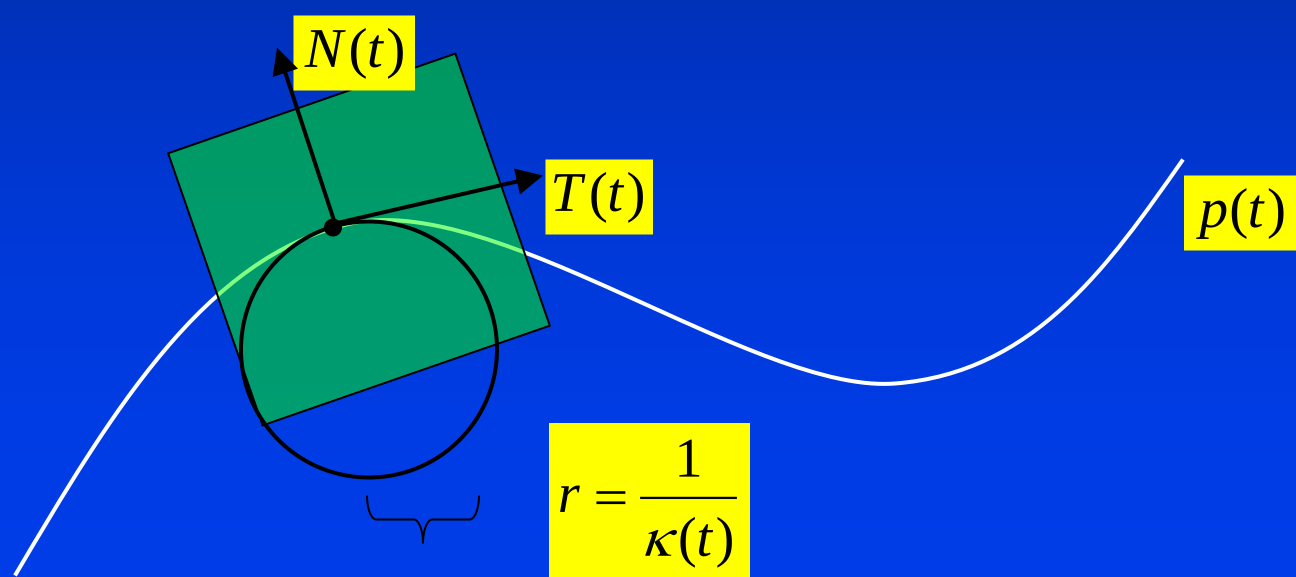
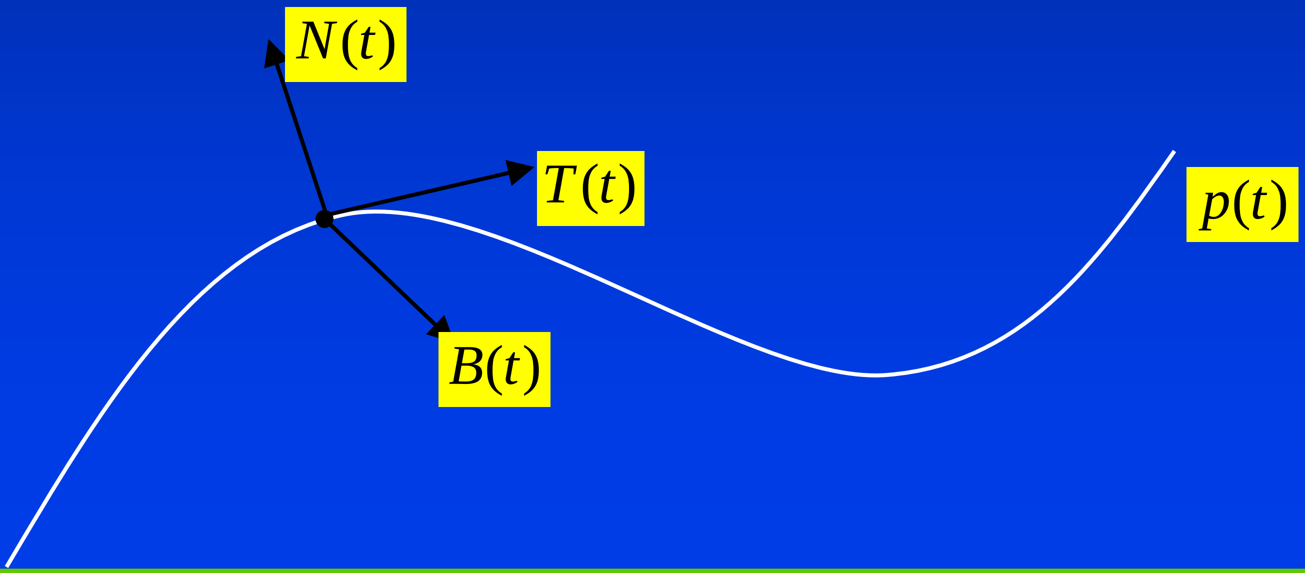


Torsion

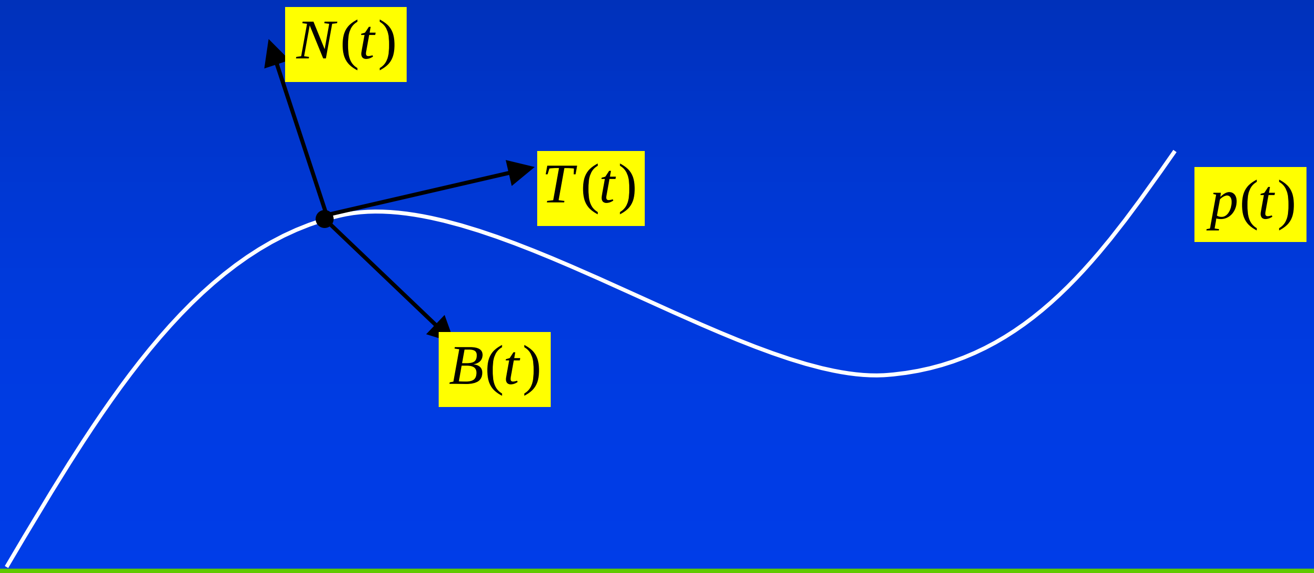
- Measure how much the curve twists or how quickly the curve leaves the osculating plane

$$\tau(s) = \left\| B'(s) \right\|$$



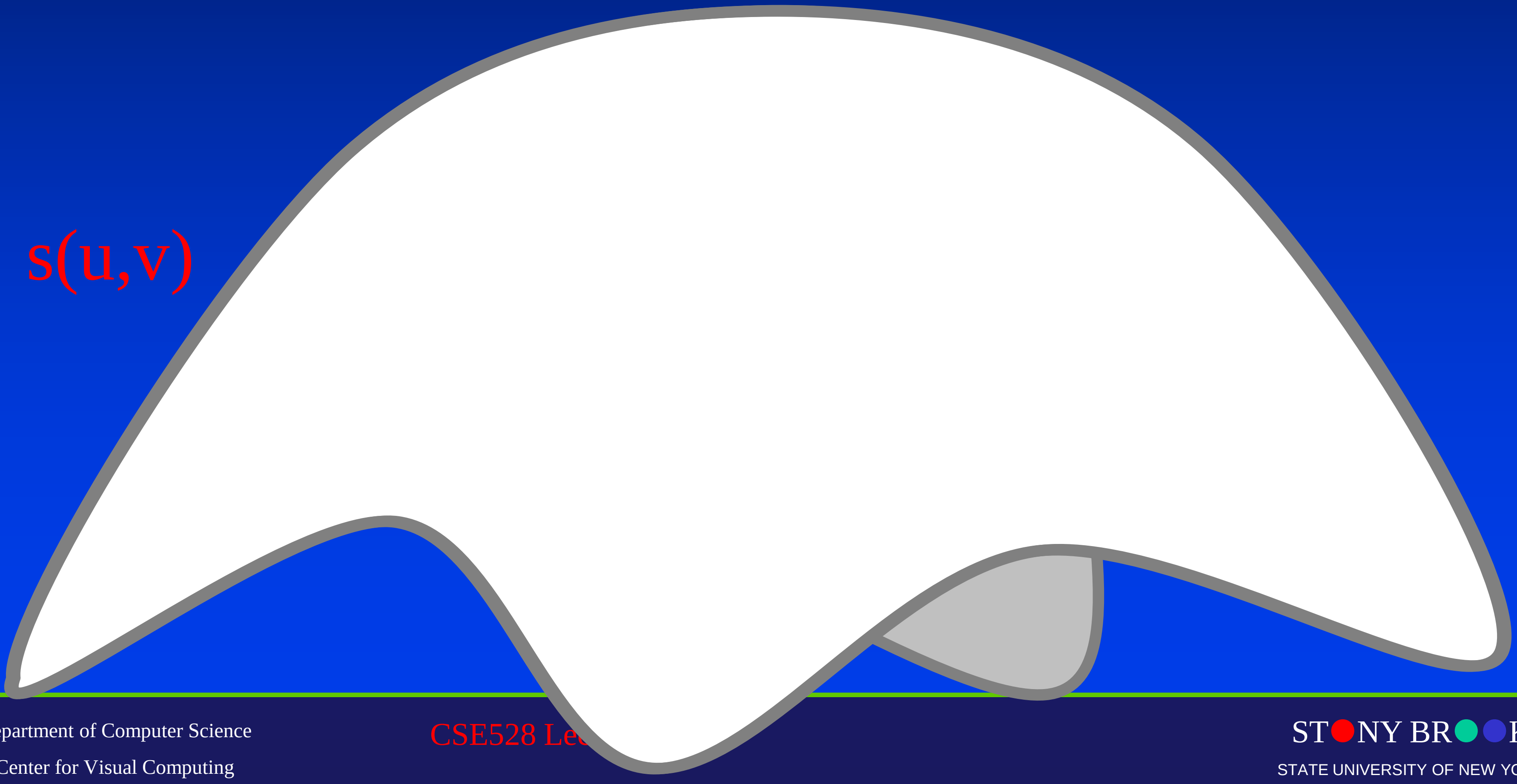
Frenet Equations

- $T'(s) = \kappa(s)N(s)$
- $N'(s) = \tau(s)B(s) - \kappa(s)T(s)$
- $B'(s) = -\tau(s)N(s)$



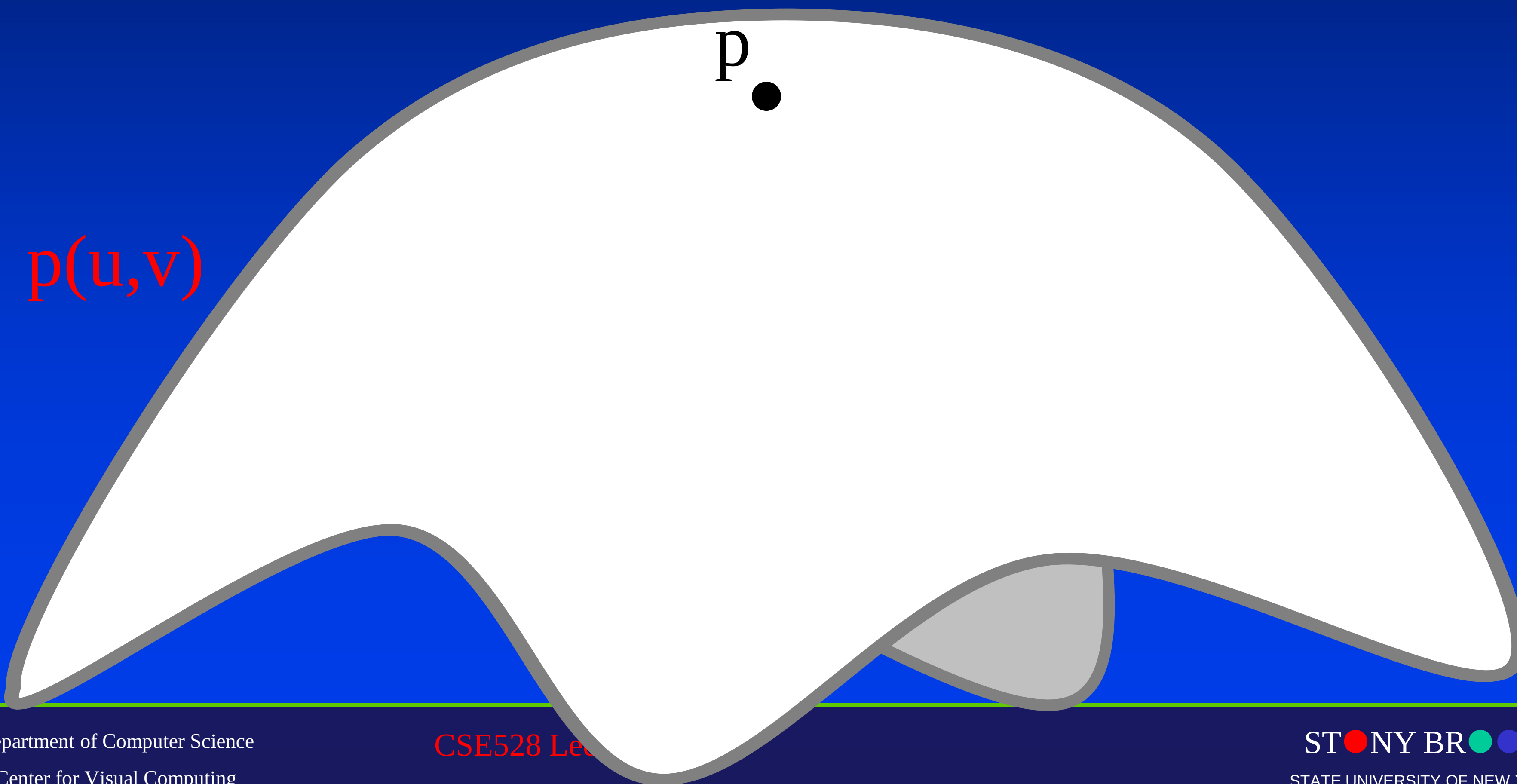
Differential Geometry of Surfaces

Differential Geometry of a Surface



Differential Geometry of a Surface

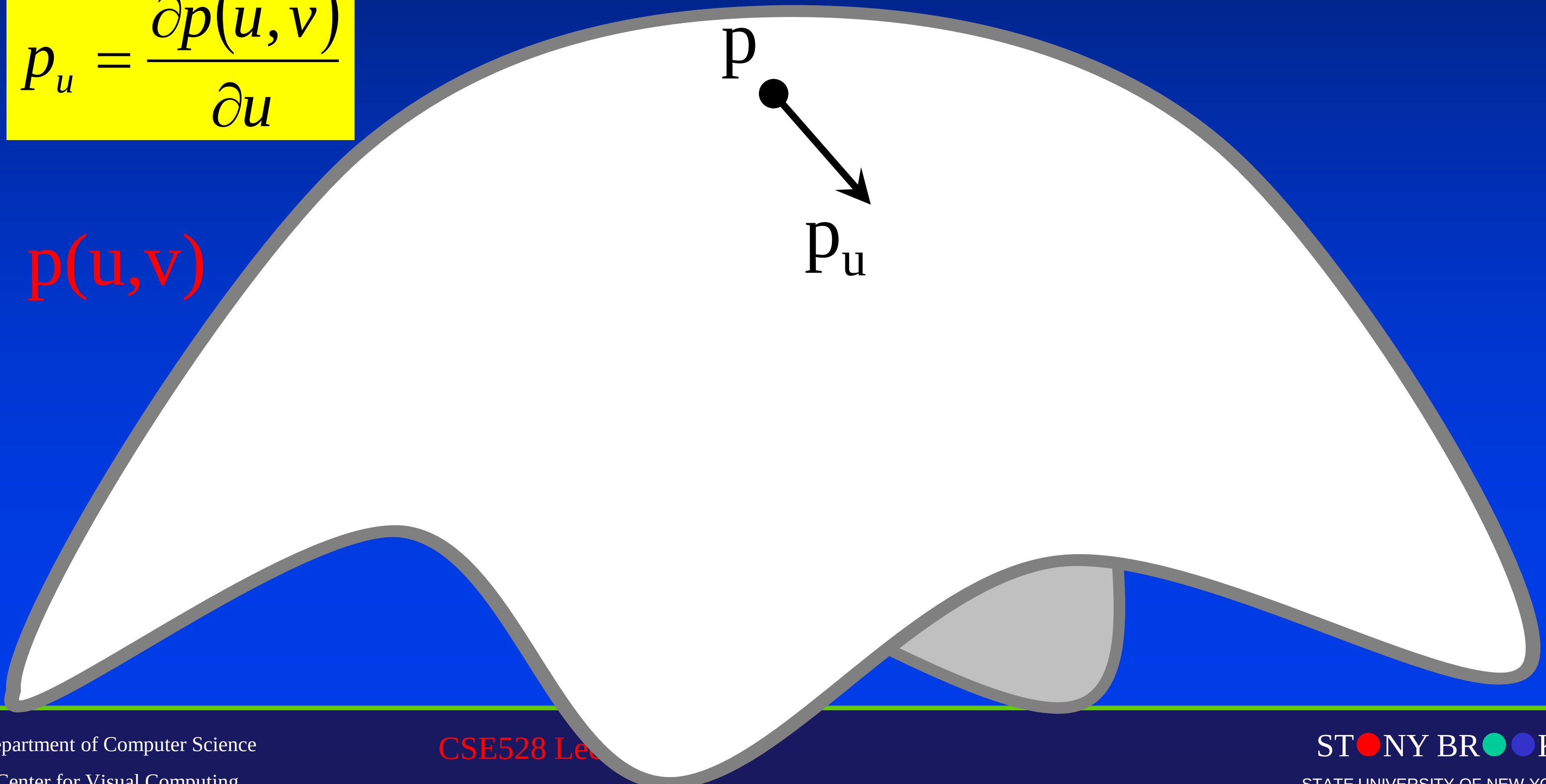
Point p on the surface at $p(u_0,v_0)$



Differential Geometry of a Surface

Tangent p_u in the u direction

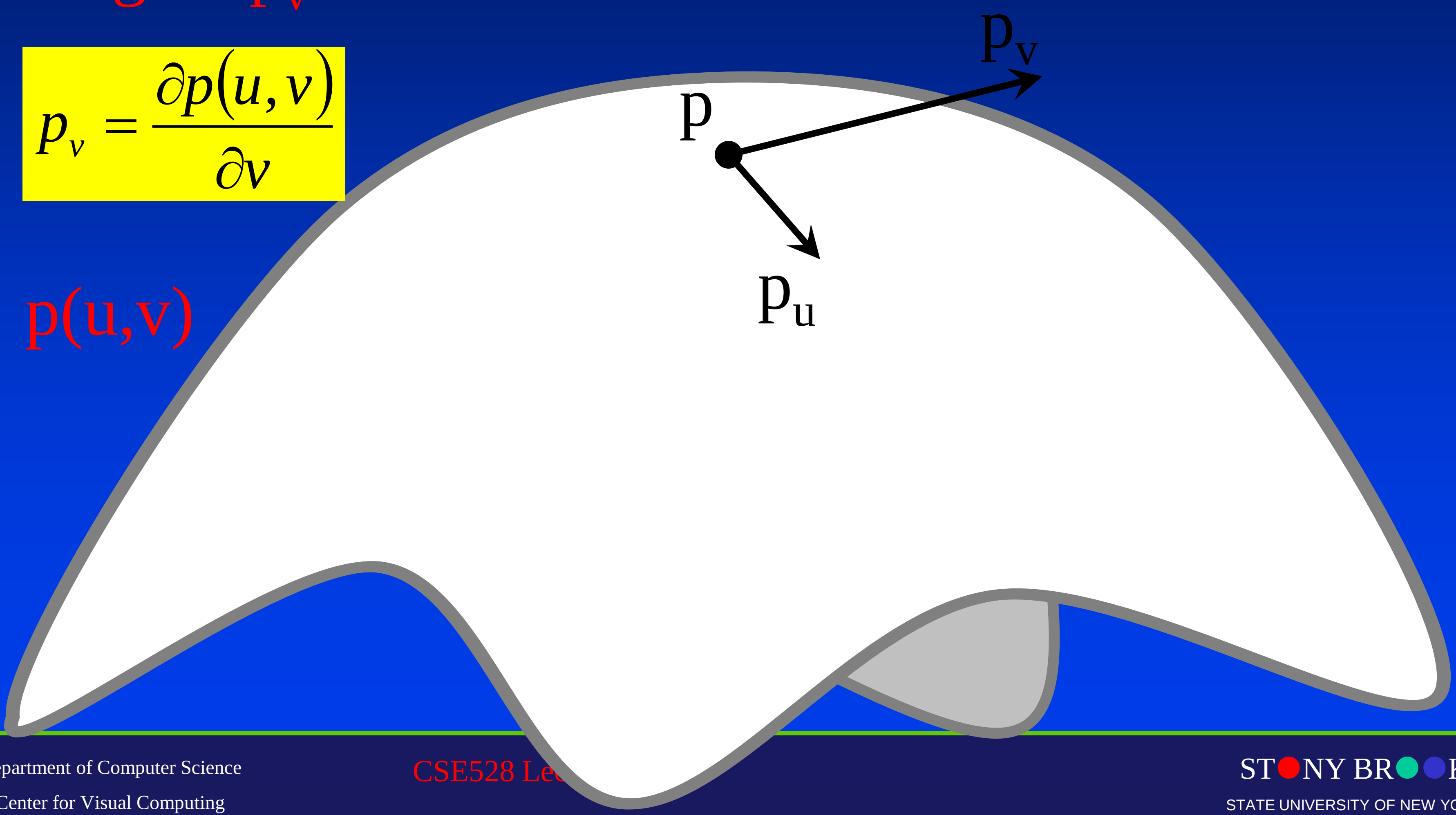
$$p_u = \frac{\partial p(u,v)}{\partial u}$$



Differential Geometry of a Surface

Tangent p_v in the v direction

$$p_v = \frac{\partial p(u,v)}{\partial v}$$

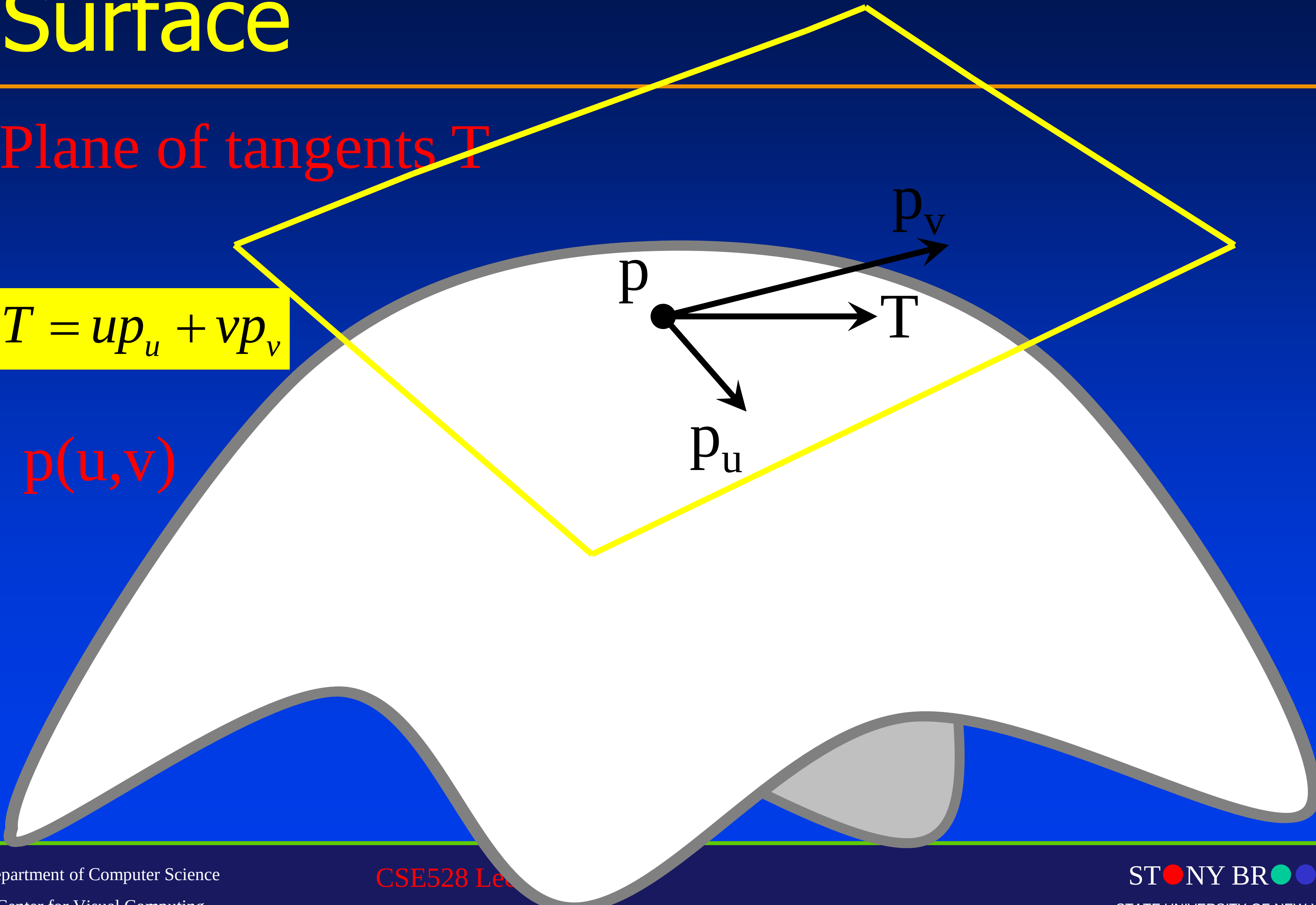


Differential Geometry of a Surface

Plane of tangents T

$$T = up_u + vp_v$$

p(u,v)

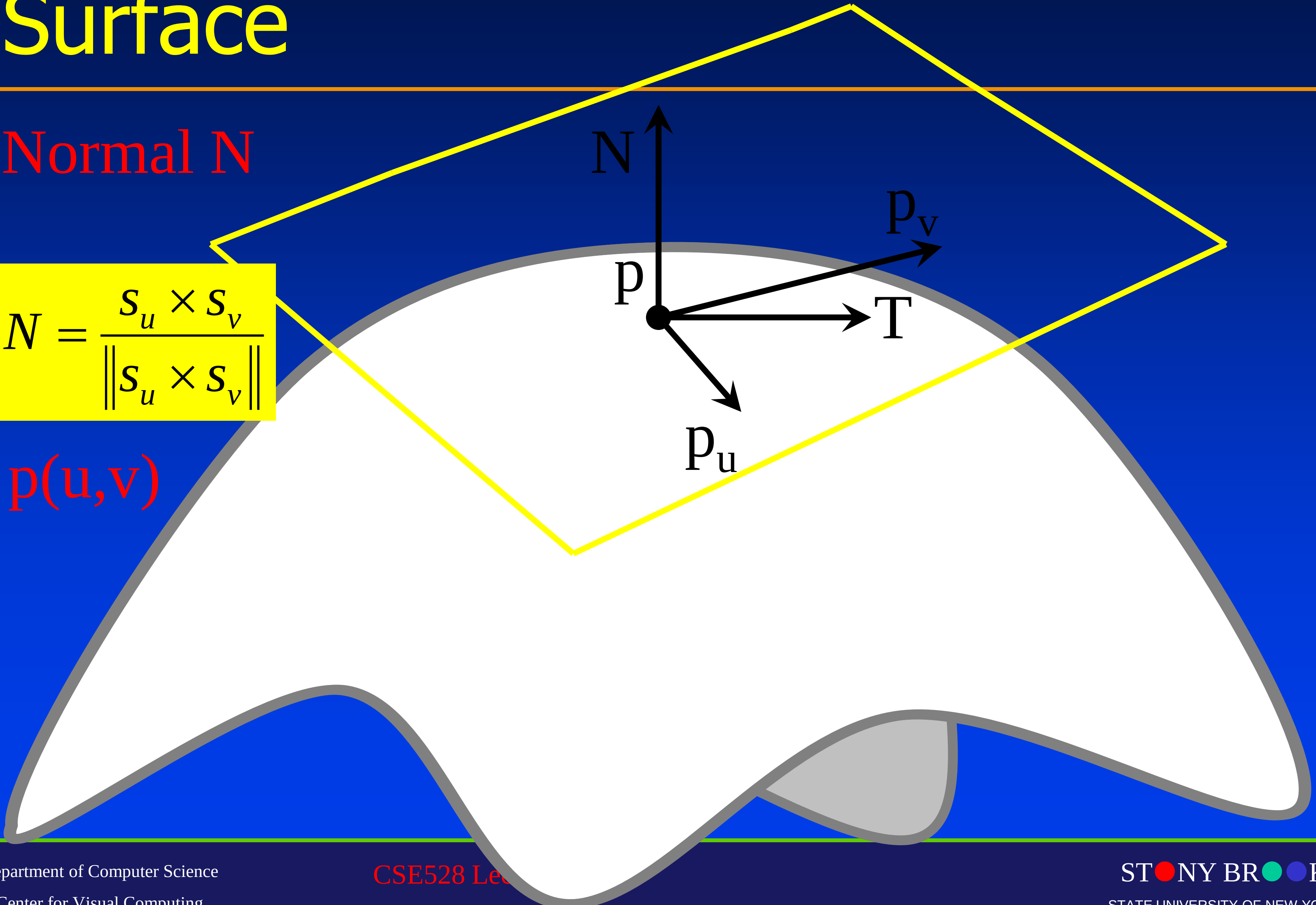


Differential Geometry of a Surface

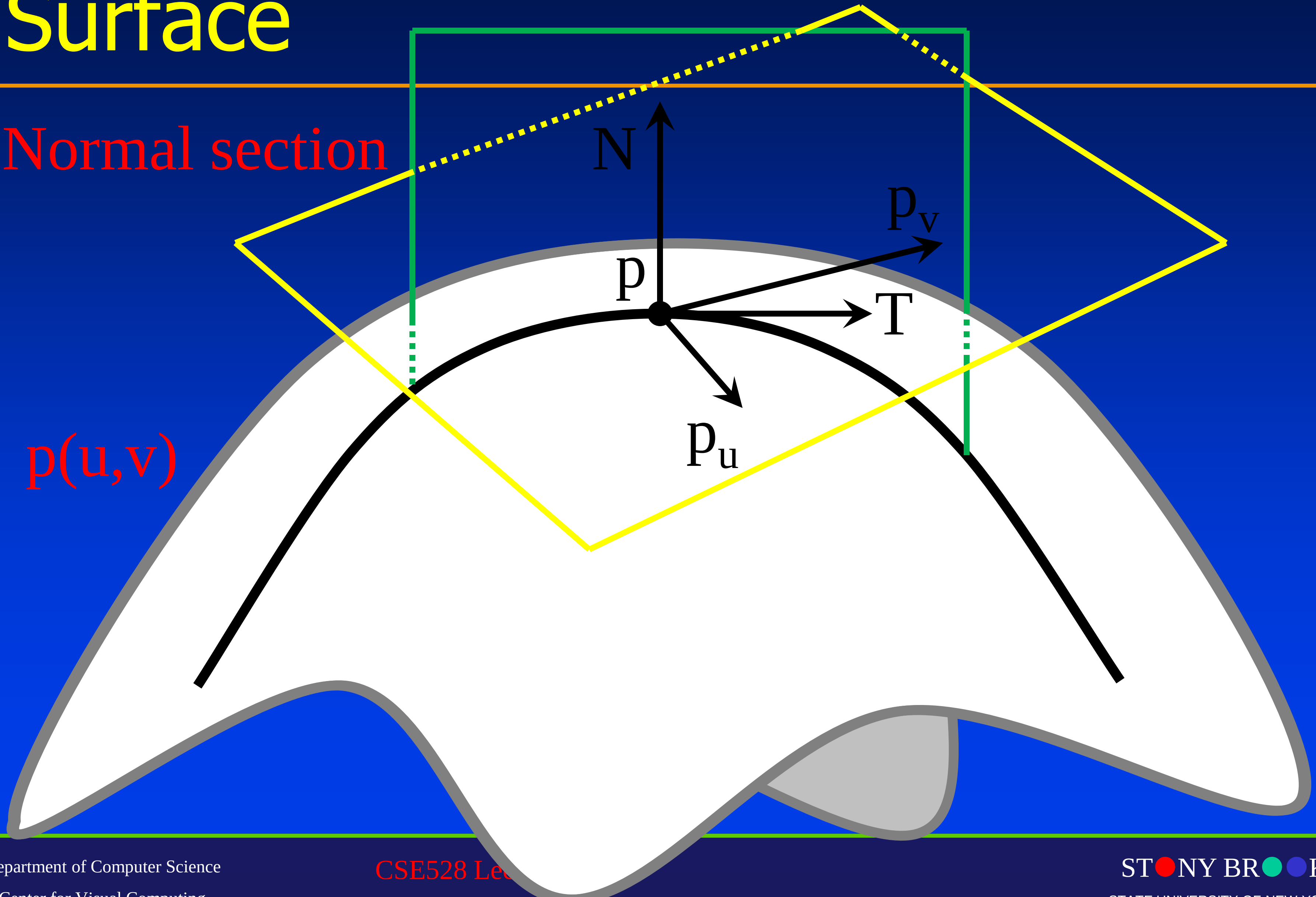
Normal N

$$N = \frac{s_u \times s_v}{\left\| s_u \times s_v \right\|}$$

p(u,v)



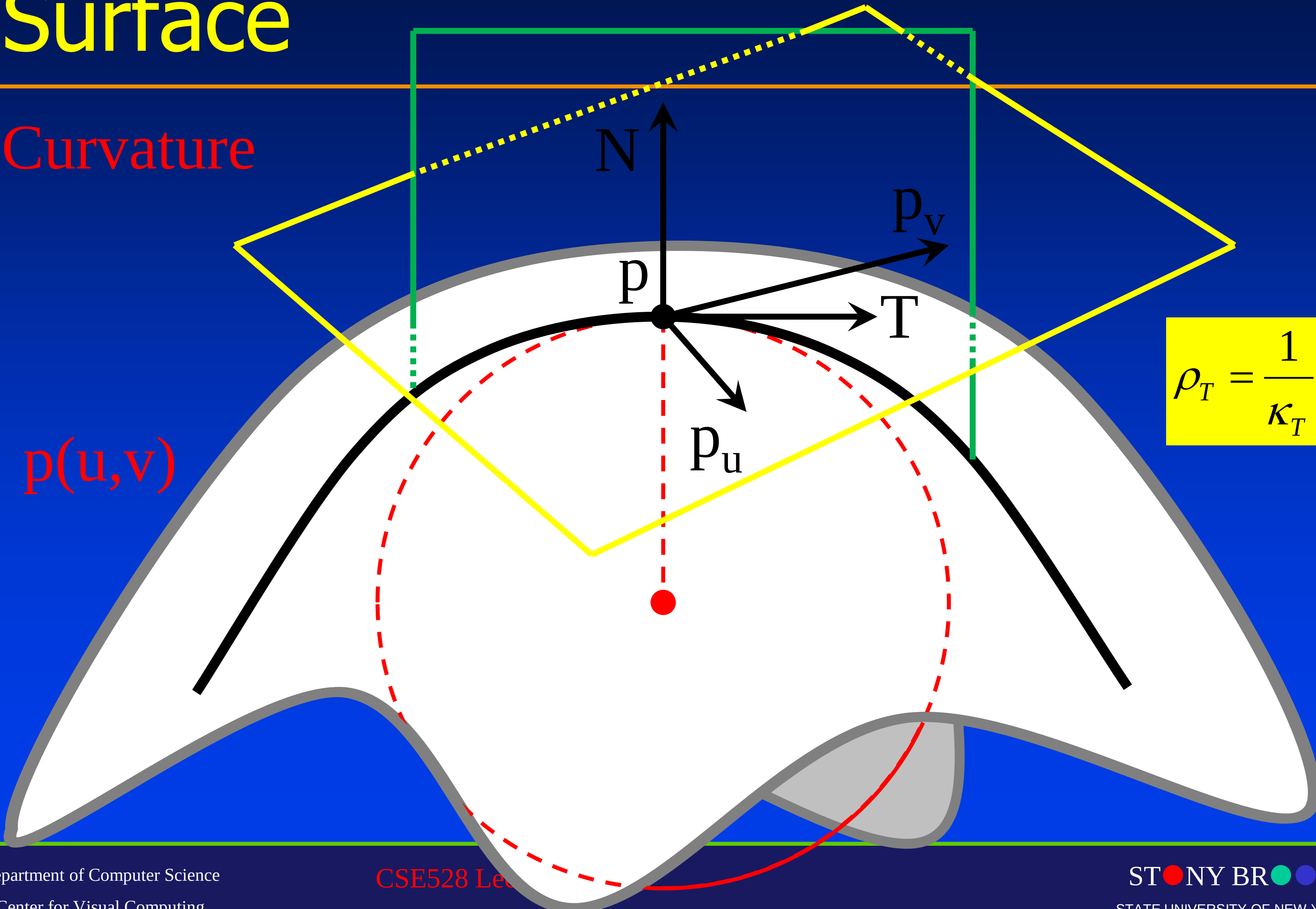
Differential Geometry of a Surface



Differential Geometry of a Surface

Curvature

p(u,v)



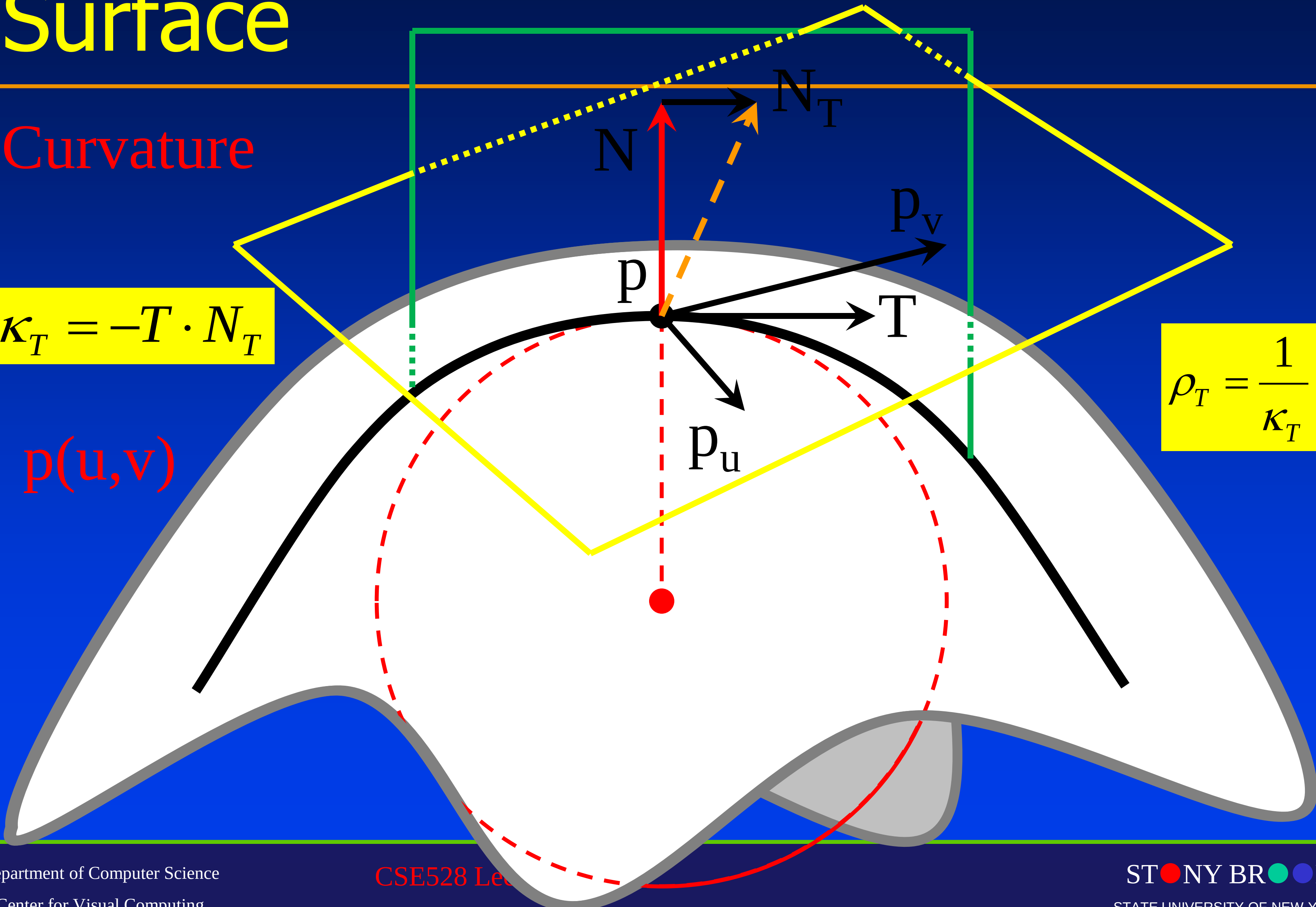
$$\rho_T = \frac{1}{\kappa_T}$$

Differential Geometry of a Surface

Curvature

$$\kappa_T = -T \cdot N_T$$

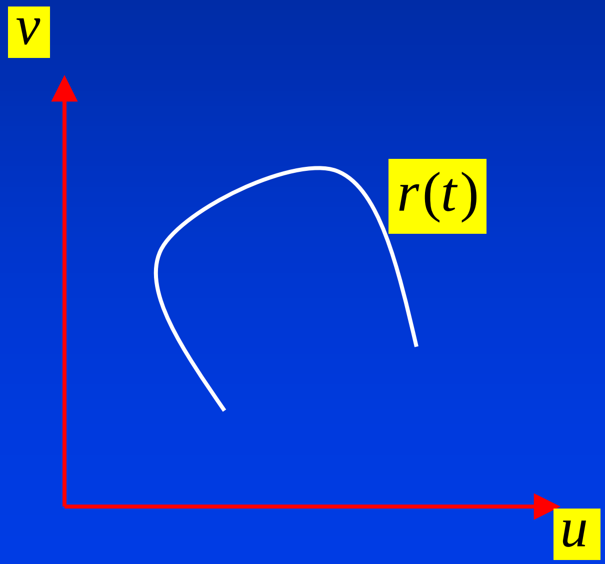
p(u,v)



$$\rho_T = \frac{1}{\kappa_T}$$

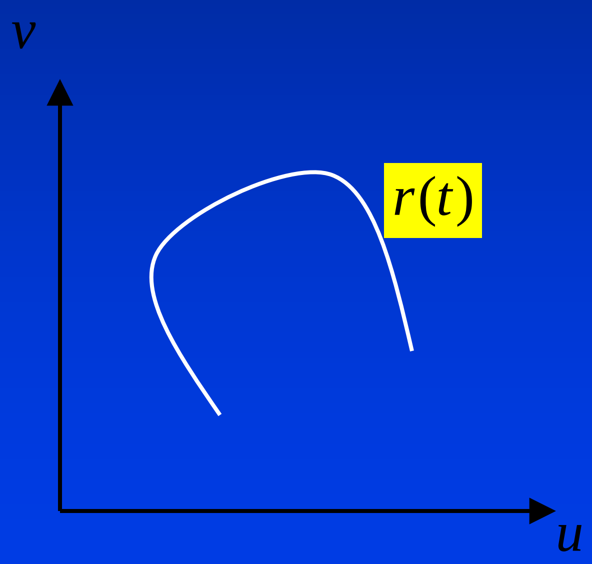
Bi-variate Parametric Surfaces

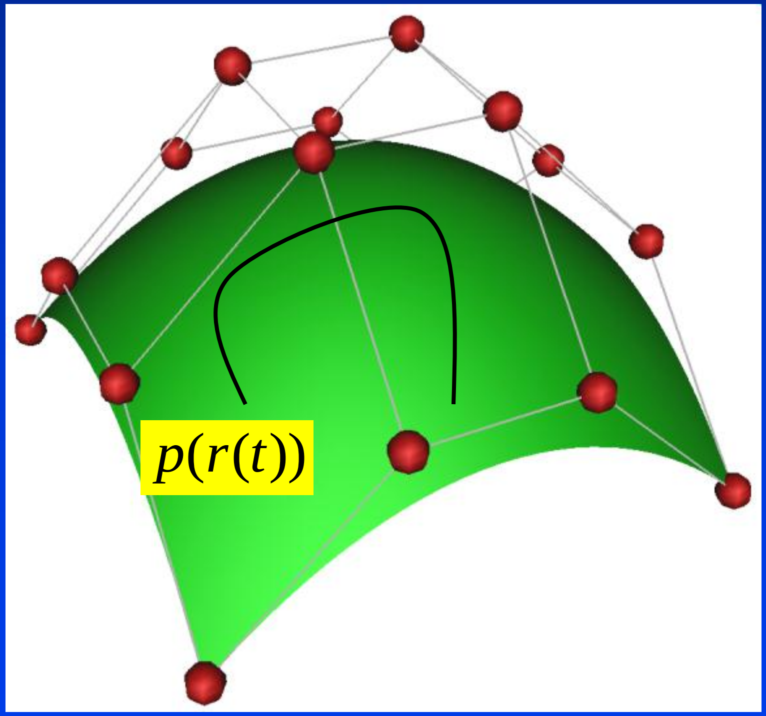
- Consider a curve $r(t)=(u(t),v(t))$



Bi-variate Surfaces

- Consider a curve $r(t)=(u(t),v(t))$
- $p(r(t))$ is a curve on the surface

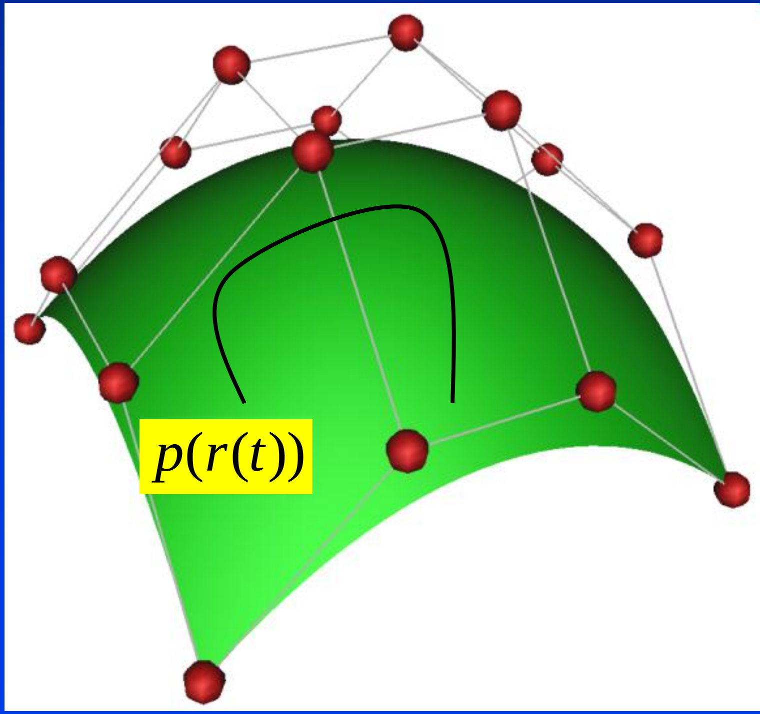




Parametric Surfaces

- Consider a curve $r(t)=(u(t),v(t))$
- $p(r(t))$ is a curve on the surface

$$s(t)=\int_{t_0}^{t}\left\|p'(r(t))\right\|dt$$

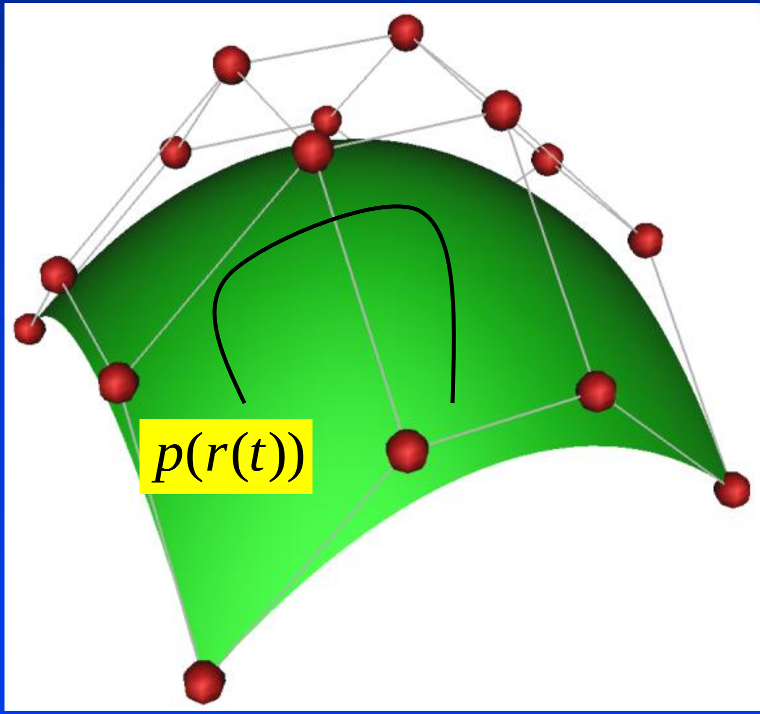


Parametric Surfaces

- Consider a curve $r(t)=(u(t),v(t))$
- $p(r(t))$ is a curve on the surface

$$s(t)=\int_{t_0}^{t}\left\|p'(r(t))\right\|dt$$

$$\left\|p'(r(t))\right\|=\left\|\frac{\partial p}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial p}{\partial v}\frac{\partial v}{\partial t}\right\|$$

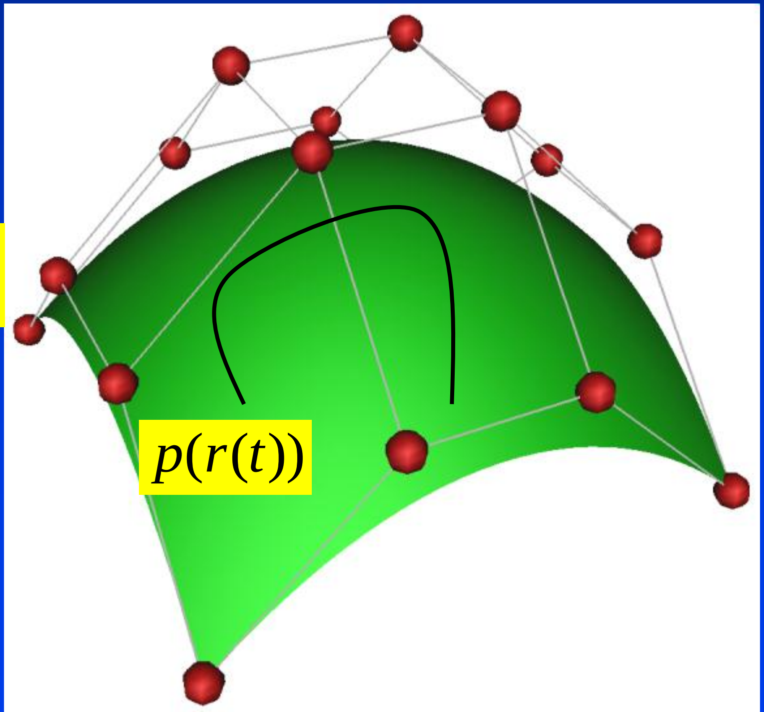


Surfaces

- Consider a curve $r(t)=(u(t),v(t))$
- $p(r(t))$ is a curve on the surface

$$s(t)=\int_{t_0}^{t}\left\|p'(r(t))\right\|dt$$

$$\left\|p'(r(t))\right\|=\sqrt{p_u\cdot p_u\left(\tfrac{\partial u}{\partial t}\right)^2+2p_u\cdot p_v\tfrac{\partial u}{\partial t}\tfrac{\partial v}{\partial t}+p_v\cdot p_v\left(\tfrac{\partial v}{\partial t}\right)^2}$$



Surfaces

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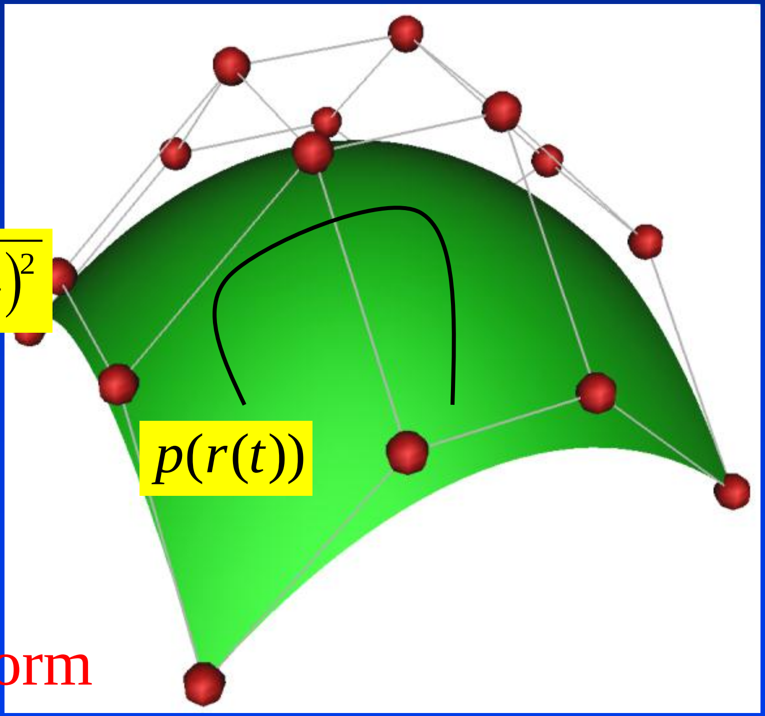
$$\left\|p'(r(t))\right\|=\sqrt{p_u\cdot p_u\left(\tfrac{\partial u}{\partial t}\right)^2+2p_u\cdot p_v\tfrac{\partial u}{\partial t}\tfrac{\partial v}{\partial t}+p_v\cdot p_v\left(\tfrac{\partial v}{\partial t}\right)^2}$$

$$E=p_u\cdot p_u$$
$$F=p_u\cdot p_v$$
$$G=p_v\cdot p_v$$

First fundamental form

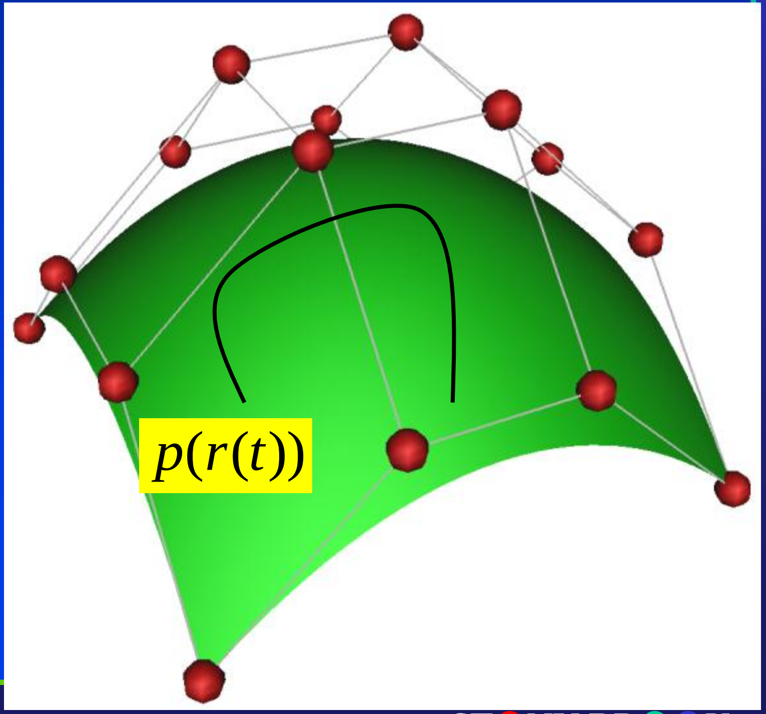


First Fundamental Form

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- Given *any* curve in parameter space *r*(*t*)=(*u*(*t*),*v*(*t*)), arc length of curve on surface is

$$s(t) = \int_{t_0}^{t} \sqrt{E \frac{\partial u}{\partial t}^2 + 2F \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + G \frac{\partial v}{\partial t}^2}\, dt$$

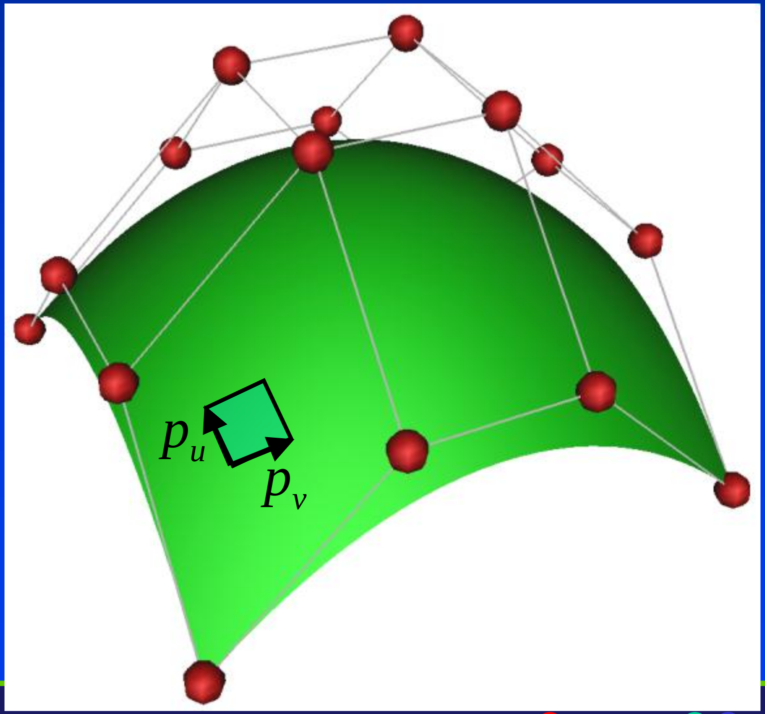


First Fundamental Form

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- The infinitesimal surface area at *u*, *v* is given by

$$\left\| p_u \times p_v \right\|$$



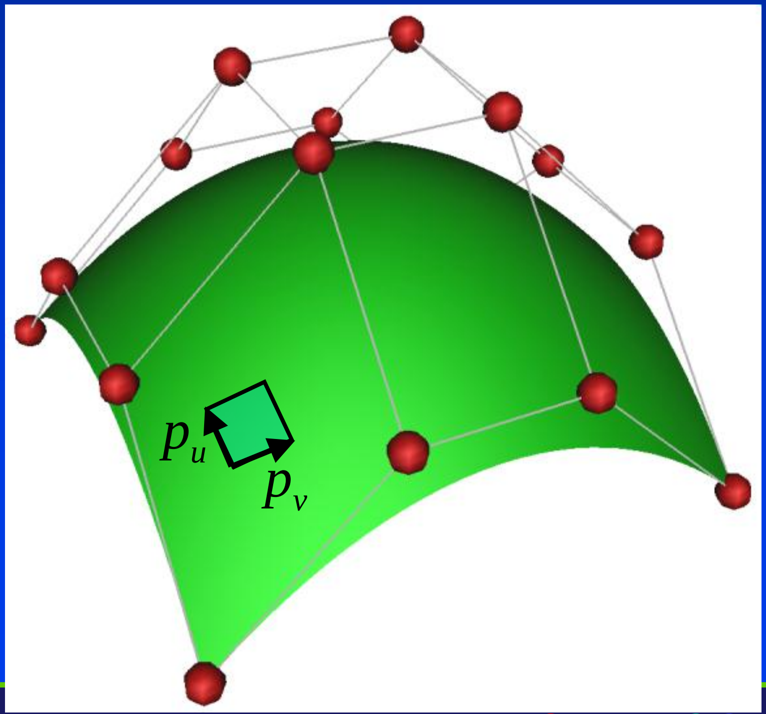
First Fundamental Form

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- The infinitesimal surface area at *u*, *v* is given by

$$\|p_u \times p_v\|$$

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 \sin(\theta)^2$$



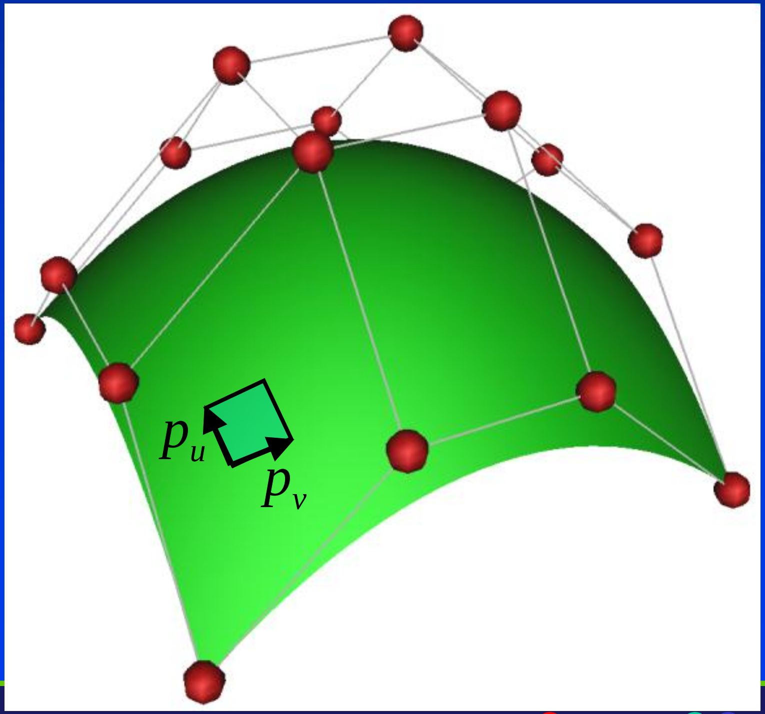
First Fundamental Form

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- The infinitesimal surface area at *u*, *v* is given by

$$\left\| p_u \times p_v \right\|$$

$$\left\| a \times b \right\|^2 = \left\| a \right\|^2 \left\| b \right\|^2 \left(1 - \cos(\theta)^2 \right)$$



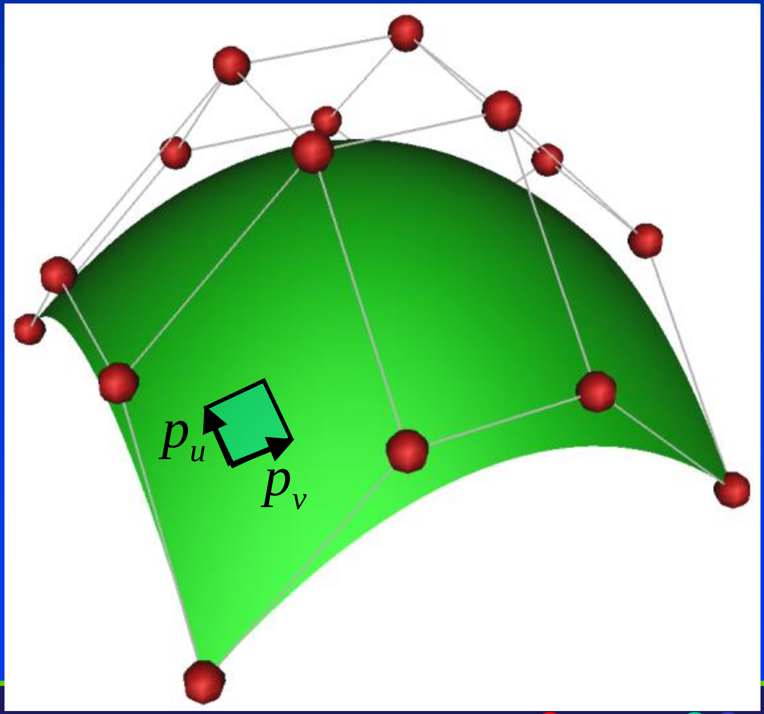
First Fundamental Form

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- The infinitesimal surface area at *u*, *v* is given by

$$\left\| p_u \times p_v \right\|$$

$$\left\| a \times b \right\|^2 = \left\| a \right\|^2 \left\| b \right\|^2 - (a \cdot b)^2$$



First Fundamental Form

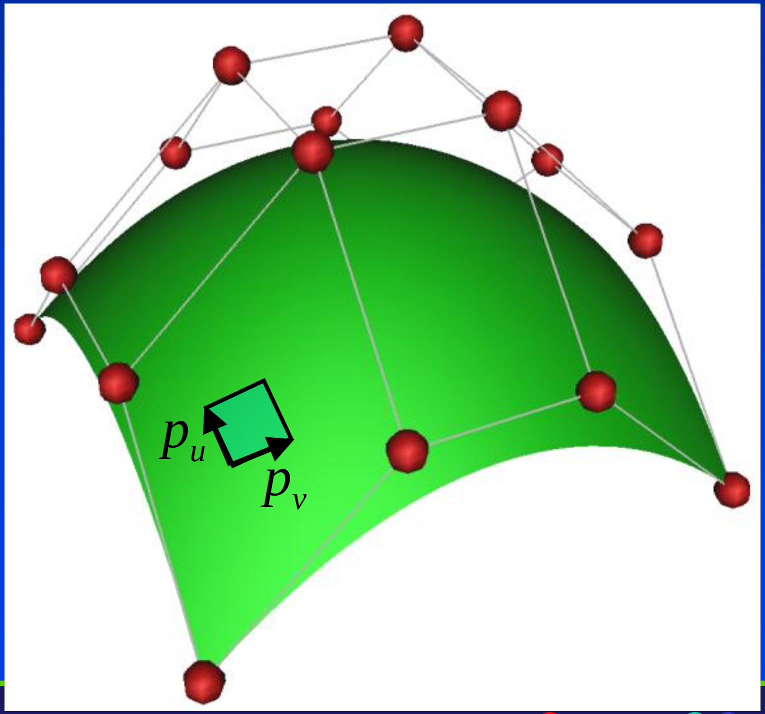
$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- The infinitesimal surface area at *u*, *v* is given by

$$\|p_u \times p_v\|$$

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2$$

$$\|p_u \times p_v\| = \sqrt{\|p_u\|^2 \|p_v\|^2 - (p_u \cdot p_v)^2}$$



First Fundamental Form

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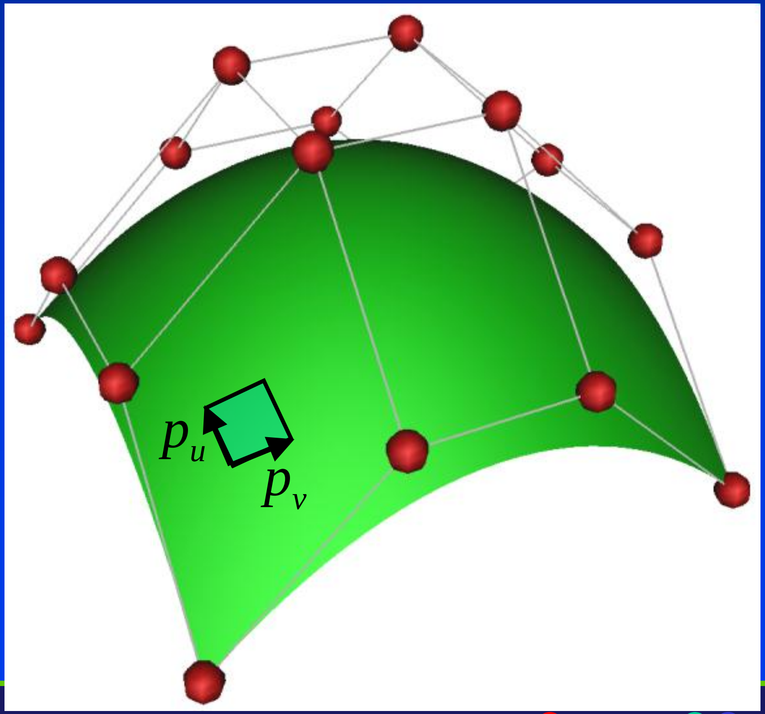
- The infinitesimal surface area at *u*, *v* is given by

$$\|p_u \times p_v\|$$

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2$$

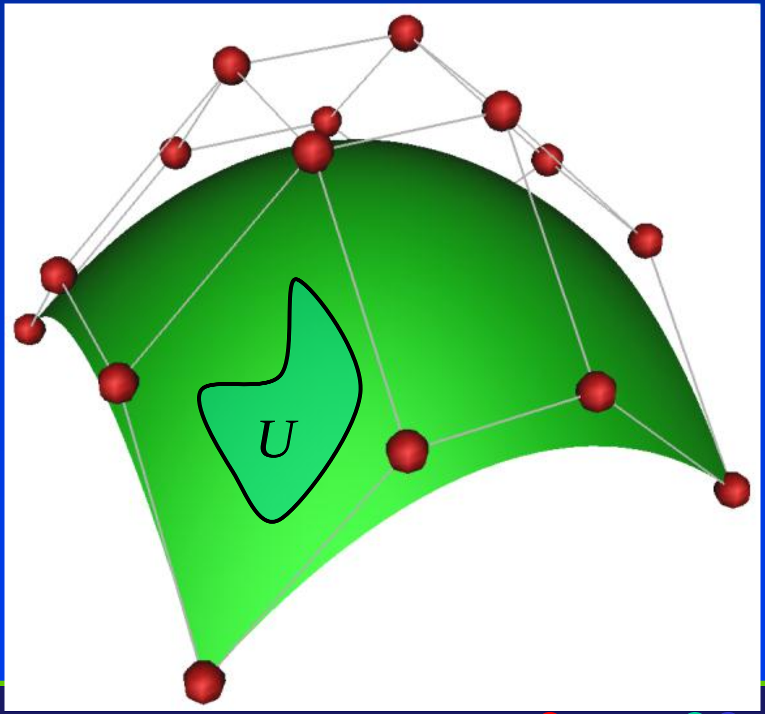
$$\|p_u \times p_v\| = \sqrt{EG - F^2}$$



First Fundamental Form

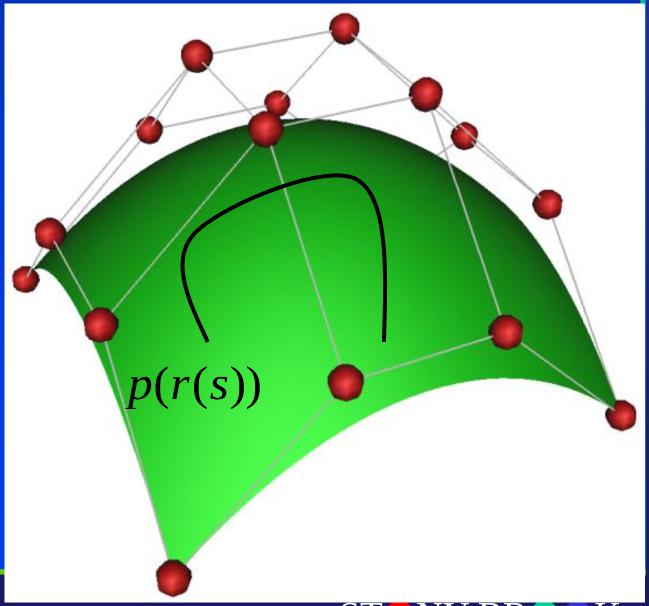
$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

- Surface area at *u*, *v* is given by $\iint_U \sqrt{EG - F^2}$



Second Fundamental Form

- Consider a curve $p(r(s))$ parameterized with respect to arc-length where $r(s)=(u(s),v(s))$
- Curvature is given by $T'(s) = p''(r(s)) = \kappa(s)N(s)$



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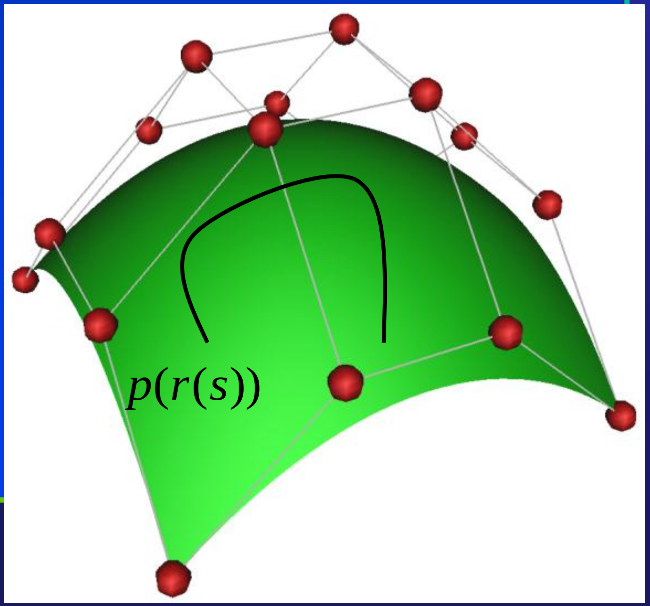
$$T'(s) = \kappa(s)N(s) = p_{uu}\frac{\partial u}{\partial s}^2 + 2p_{uv}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + p_{vv}\frac{\partial v}{\partial s}^2 + p_v\frac{\partial^2 v}{\partial s^2} + p_u\frac{\partial^2 u}{\partial s^2}$$



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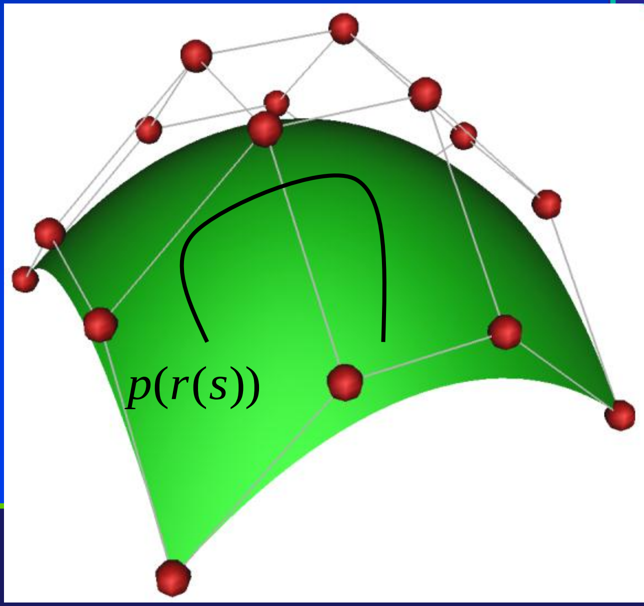
$$T'(s) = \kappa(s)N(s) = p_{uu}\frac{\partial u}{\partial s}^2 + 2p_{uv}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + p_{vv}\frac{\partial v}{\partial s}^2 + p_v\frac{\partial^2 v}{\partial s^2} + p_u\frac{\partial^2 u}{\partial s^2}$$

- Let n be the normal of $p(u,v)$

$$n \cdot N(s) = \cos(\phi)$$



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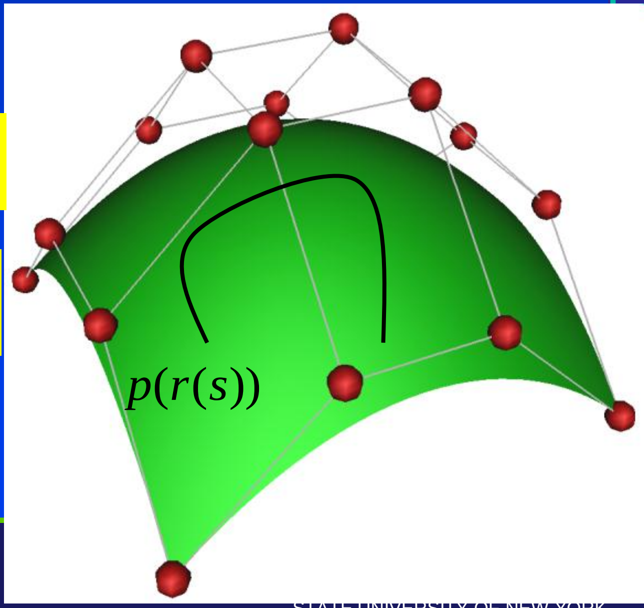
- Let n be the normal of $p(u,v)$

$$n \cdot N(s) = \cos(\phi) \qquad n \cdot T'(s) = \kappa(s)\cos(\phi)$$

$$\kappa(s)\cos(\phi) = n \cdot p_{uu}\frac{\partial u}{\partial s}^2 + 2n \cdot p_{uv}\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + n \cdot p_{vv}\frac{\partial v}{\partial s}^2$$



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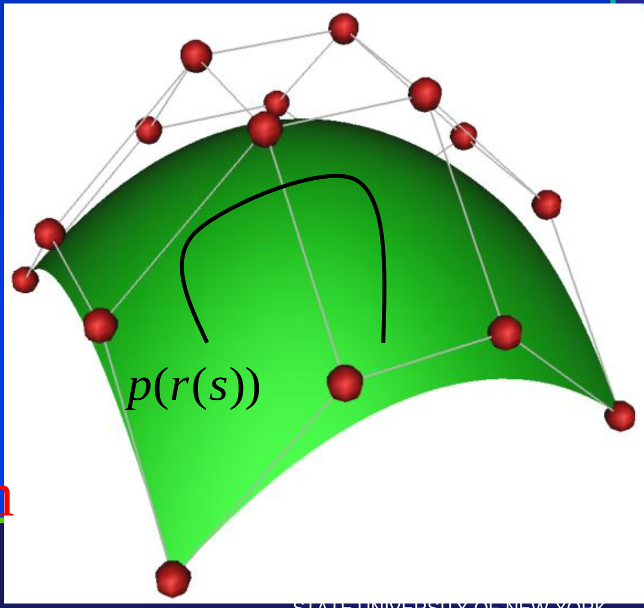
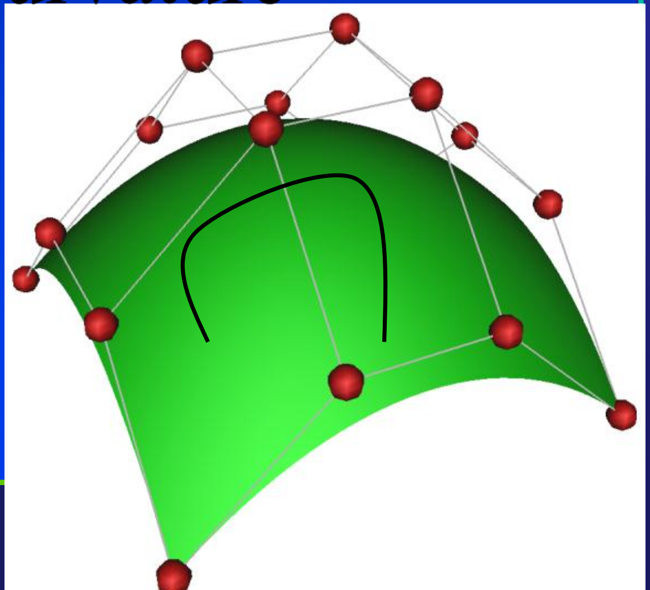
$$L = n \cdot p_{uu}$$
$$M = n \cdot p_{uv}$$
$$N = n \cdot p_{vv}$$

Second Fundamental Form



Meusnier's Theorem

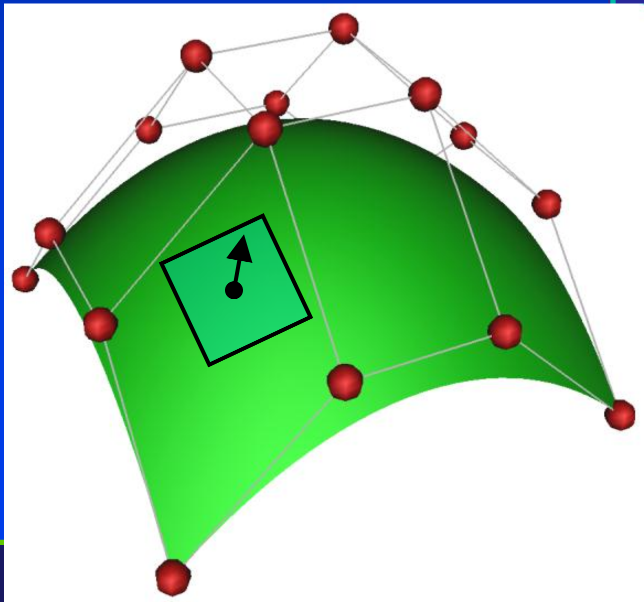
- Assume $n \cdot N(s) = 1$, $\kappa(s)$ is called the *normal curvature*
- Meusnier's Theorem states that all curves on *p*(*u*,*v*) passing through a point *x* having the same tangent, have the same normal curvature



Lines of Curvature

- We can parameterize all tangents through *x* using a single parameter λ

$$\kappa(\lambda) = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$



Principle Curvatures

$$\kappa_1 = \min \kappa(\lambda) \qquad \kappa_2 = \max \kappa(\lambda)$$

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$$EM - FL + (EN - GL)\lambda + (FN - GM)\lambda^2 = 0$$

Gaussian and Mean Curvature

- Gaussian Curvature: $$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$
- Mean Curvature: $$H = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2}$$

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- Mean Curvature: $$H = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2}$$
- $K > 0$: elliptic
- $K < 0$: hyperbolic
- $\kappa_1 = 0 \vee \kappa_2 = 0$: parabolic
- $\kappa_1 = 0 \wedge \kappa_2 = 0$: flat

Energy Formulation for Surfaces

$$\kappa_G = \kappa_1 \kappa_2$$

$$\kappa_M = \frac{\kappa_1 + \kappa_2}{2}$$

$$\begin{aligned}
\kappa_B &= \kappa_1^2 + \kappa_2^2 \\
&= \left(\kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2\right) - 2\kappa_1\kappa_2 \\
&= 4\left(\frac{\kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2}{4}\right) - 2\kappa_1\kappa_2 \\
&= 4\left(\frac{\kappa_1 + \kappa_2}{2}\right)^2 - 2\kappa_1\kappa_2 \\
&= 4\kappa_M^2 - 2\kappa_G
\end{aligned}$$

$$\begin{aligned}
E_B &= \int_S \kappa_1^2 + \kappa_2^2 \partial A \\
&= \int_S 4\kappa_M^2 - 2\kappa_G \partial A \\
&= 4\int_S \kappa_M^2 \partial A - 2\int_S \kappa_G \partial A \\
&= 4\int_S \kappa_M^2 \partial A - 2(2\pi\chi(S)) \\
&= 4\int_S \kappa_M^2 \partial A - 4\pi(2 - 2G)
\end{aligned}$$

Bending Energy

$$\kappa_G = \kappa_1 \kappa_2$$

$$\kappa_M = \frac{\kappa_1 + \kappa_2}{2}$$

$$\begin{aligned}
\kappa_B &= \kappa_1{}^2 + \kappa_2{}^2 \\
&= \left(\kappa_1{}^2 + 2\kappa_1\kappa_2 + \kappa_2{}^2\right) - 2\kappa_1\kappa_2 \\
&= 4\left(\frac{\kappa_1{}^2 + 2\kappa_1\kappa_2 + \kappa_2{}^2}{4}\right) - 2\kappa_1\kappa_2 \\
&= 4\left(\frac{\kappa_1 + \kappa_2}{2}\right)^2 - 2\kappa_1\kappa_2 \\
&= 4\kappa_M{}^2 - 2\kappa_G
\end{aligned}$$

$$\begin{aligned}
E_B &= \int_S \kappa_1{}^2 + \kappa_2{}^2 \partial A \\
&= \int_S 4\kappa_M{}^2 - 2\kappa_G \partial A \\
&= 4\int_S \kappa_M{}^2 \partial A - 2\int_S \kappa_G \partial A \\
&= 4\int_S \kappa_M{}^2 \partial A - \underline{2\left(2\pi\chi(S)\right)} \\
&= 4\int_S \kappa_M{}^2 \partial A - 4\pi\left(2 - 2G\right)
\end{aligned}$$

$$\text{Minimizing} \int_S \kappa_1{}^2 + \kappa_2{}^2 \partial A = \text{Minimizing} \int_S \kappa_M{}^2 \partial A$$

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