Curves

- **Planar curve (point set)**

$$\{(x, y) | f(x, y) = 0\}$$

- **The implicit function is not unique**

$$\{(x, y) | + \alpha f(x, y) = 0\}$$

$$\{(x, y) | - \alpha f(x, y) = 0\}$$

- **Comparison with parametric representation**

$$\mathbf{p}(u) = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix}$$

- **Implicit function is a level-set**

$$\begin{cases} z &= f(x, y) \\ z &= 0 \end{cases}$$

Straight Line

- **Mathematics**

$$ax + by + c = 0$$

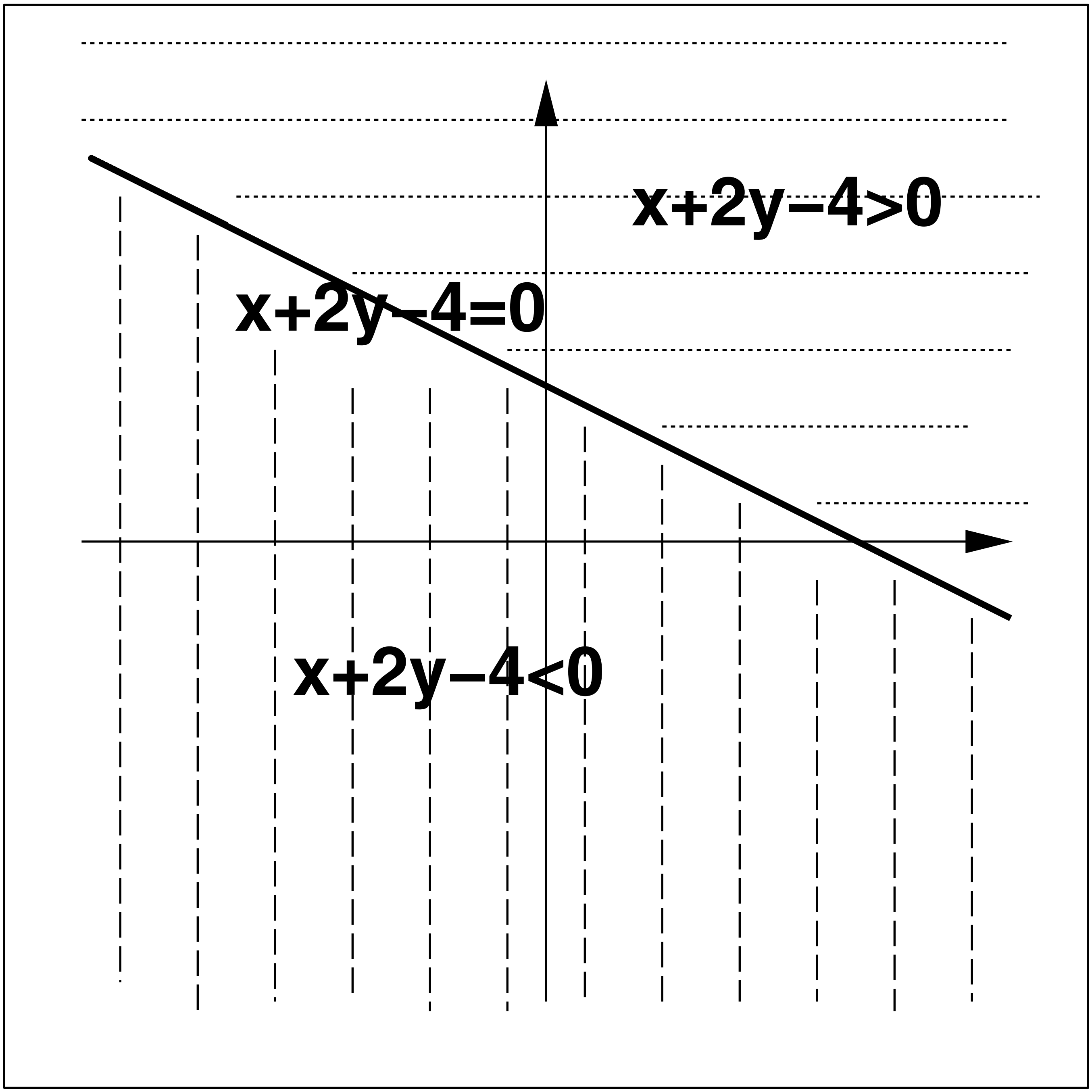
$$+\alpha(ax + by + c) = 0$$

$$-\alpha(ax + by + c) = 0$$

- **Example**

$$x + 2y - 4 = 0$$

Straight Line



Conic Sections

- **Mathematics**

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

- **Examples**
  - **ellipse**

$$2x^2 + 3y^2 - 5 = 0$$

  - **hyperbola**

$$2x^2 - 3y^2 - 5 = 0$$

  - **parabola**

$$2x^2 + 3y = 0$$

  - **empty set**

$$2x^2 + 3y^2 + 1 = 0$$

  - **point**

$$2x^2 + 3y^2 = 0$$

- **pair of lines**

$$2x^2 - 3y^2 = 0$$

- **parallel lines**

$$2x^2 - 7 = 0$$

- **repeated lines**

$$2x^2 = 0$$

- **Parametric equations of conics**

- **Generalization to higher-degree curves**

- **How about non-planar (spatial) curves**

Circle

- **Implicit equation**

$$x^2 + y^2 - 1 = 0$$

- **Parametric function**

$$\mathbf{c}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

where

$$0 <= \theta <= 2\pi$$

- **Parametric representation using rational polynomials (the first quadrant)**

$$x(u) = \frac{1 - u^2}{1 + u^2}$$

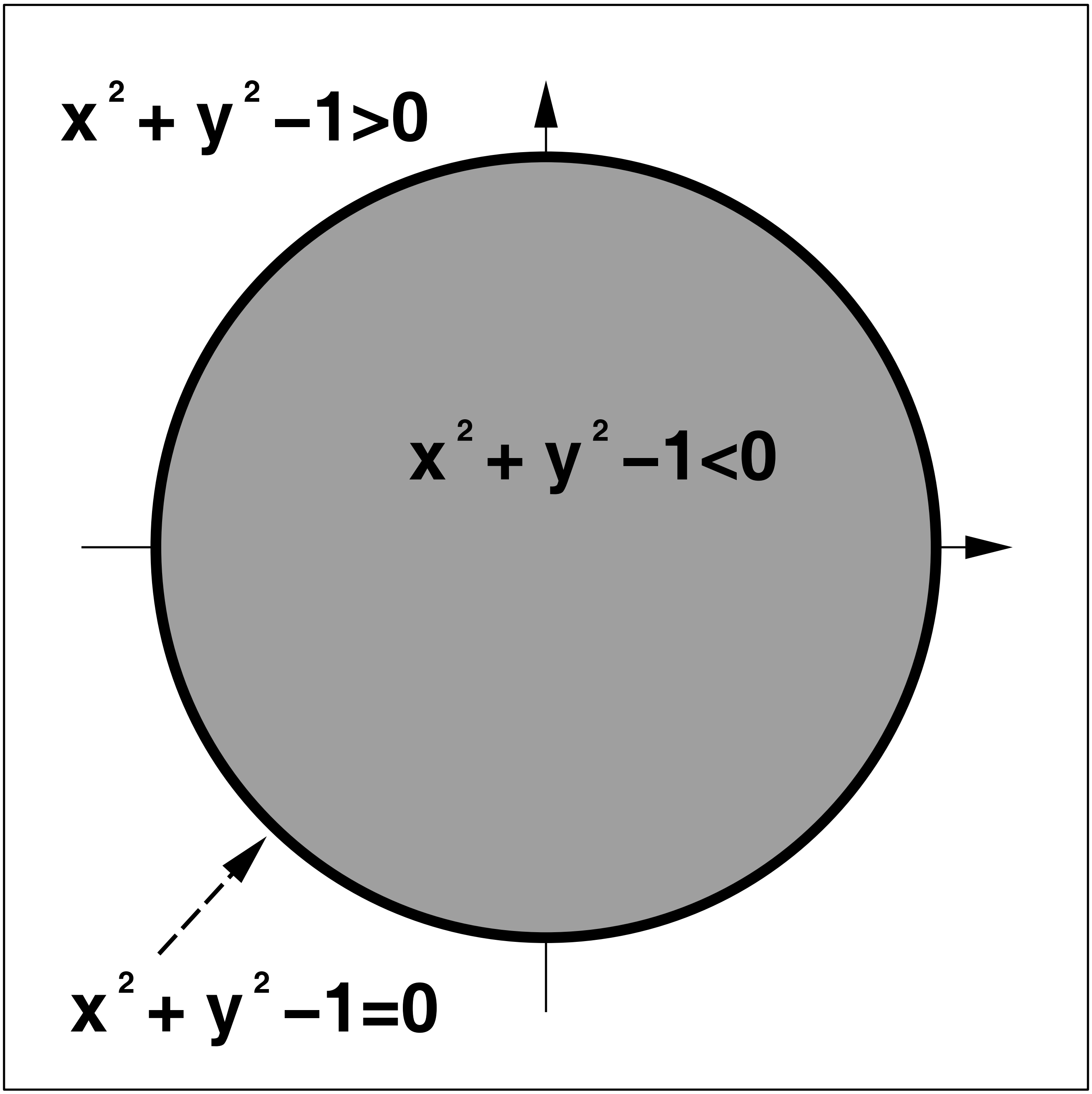
$$y(u) = \frac{2u}{1 + u^2}$$

where

$$0 <= u <= 1$$

- **Parametric representation is not unique**

Circle



Surfaces

- **Mathematics**

$$\{(x,y,z)|f(x,y,z)=0\}$$

- **The implicit function for surfaces is not unique**

$$\{(x,y,z)|+\alpha f(x,y,z)=0\}$$

$$\{(x,y,z)|-\alpha f(x,y,z)=0\}$$

- **Comparison with parametric representation**

$$\mathrm{s}(u,v)=\begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

- **Surface defined by implicit function is a level-set**

$$\begin{cases} w &= f(x,y,z) \\ w &= 0 \end{cases}$$

Plane

- General plane equation

$$ax + by + cz + d = 0$$

- parametric representation

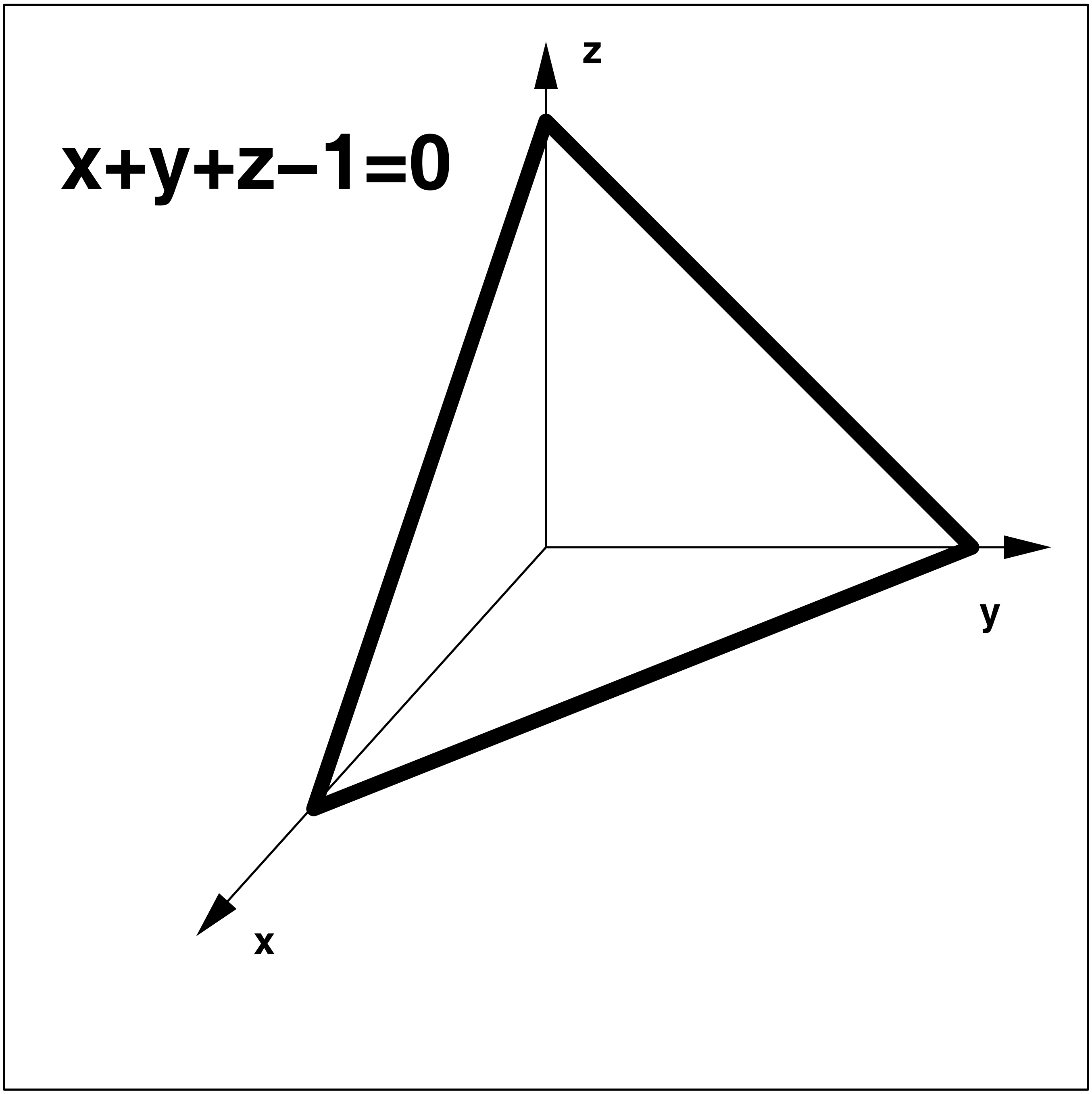
$$\mathbf{p}(u, v) = \mathbf{p}_a + (\mathbf{p}_b - \mathbf{p}_a)u + (\mathbf{p}_c - \mathbf{p}_a)v$$

  where $\mathbf{p}_a$, $\mathbf{p}_b$, and $\mathbf{p}_c$ are non-colinear!
  All three points are on the plane.

- Example

$$x + y + z - 1 = 0$$

Plane



Quadric Surfaces

- **Implicit functions**

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

$$+gx + hy + jz + k = 0$$

- **Examples**
  - **sphere**

$$x^2 + y^2 + z^2 - 1 = 0$$

  - **cylinder**

$$x^2 + y^2 - 1 = 0$$

  - **cone**

$$x^2 + y^2 - z^2 = 0$$

  - **paraboloid**

$$x^2 + y^2 + z = 0$$

  - **ellipsoid**

$$2x^2 + 3y^2 + 4z^2 - 5 = 0$$

- **hyperboloid**

$$x^2 + y^2 - z^2 + 4 = 0$$

- **two parallel planes**
- **two intersecting planes**
- **single plane**
- **line**
- **point**

• **parametric representation of sphere**

$$x = r \cos\alpha \cos\beta$$

$$y = r \cos\alpha \sin\beta$$

$$z = r \sin\alpha$$

where $-\frac{\pi}{2} <= \alpha <= \frac{\pi}{2}$, **and** $-\pi <= \beta <= \pi$

• **Ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

• **Parametric representation of ellipsoid**

$$x = a \cos\alpha \cos\beta$$

$$y = b \cos\alpha \sin\beta$$

$$z = c \sin\alpha$$

where $-\frac{\pi}{2} <= \alpha <= \frac{\pi}{2}$**, and** $-\pi <= \beta <= \pi$

- **Parametric representation of quadric surfaces**
- **Generalization to higher-degree surfaces**

$$\sum_i \sum_j \sum_k a_{ijk} x^i y^j z^k = 0$$

where

$$0 <= i + j + k <= n$$

- **Torus**
- **Superquadrics**
- **Blobby objects**

Spatial Curves

- **Intersection of two surfaces**

$$f(x, y, z) = 0$$

$$g(x, y, z) = 0$$

Superquadrics

- **Geometry (Generalization of quadrics)**
- **Superellipse**

$$(x/a_1)^{2/s} + (y/a_2)^{2/s} - 1 = 0$$

- **Superellipsoid**

$$((x/a_1)^{2/s_2} + (y/a_2)^{2/s_2})^{s_2/s_1} + (z/a_3)^{2/s_1} - 1 = 0$$

- **Parametric representation**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \cos^{s_1}(u) \cos^{s_2}(v) \\ a_2 \cos^{s_1}(u) \sin^{s_2}(v) \\ a_3 \sin^{s_1}(v) \end{bmatrix}$$

where

$$-\pi/2 \leq u \leq \pi/2$$

$$-\pi \leq v < \pi$$

- s_1 **and** s_2 **control the shape**

Algebraic Solid

- **Half space**

  $$\{(x,y,z)|f(x,y,z) <= 0\}$$

  Or

  $$\{(x,y,z)|f(x,y,z) => 0\}$$

- **Useful for complex objects**
  (refer to notes on solid modeling)

$$\mathbf{f}(x,y,z) = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \\ \cdots \end{bmatrix} = \mathbf{0}$$