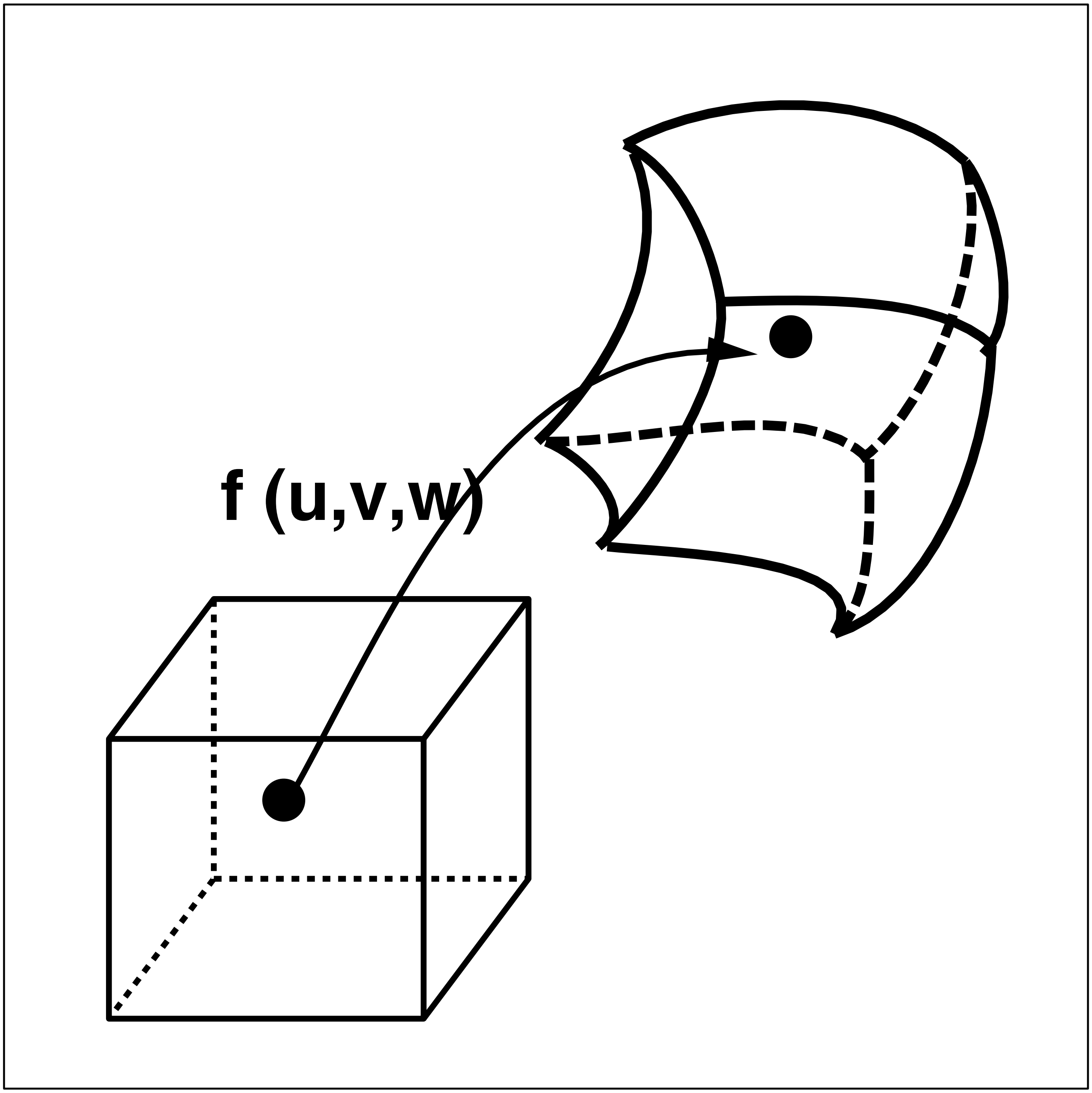
Solid



Parametric Solids

- **Three-parameter functions**

$$x = x(u, v, w)$$

$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

$$\mathbf{p}(u, v, w) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $u, v, w \in [0, 1]$

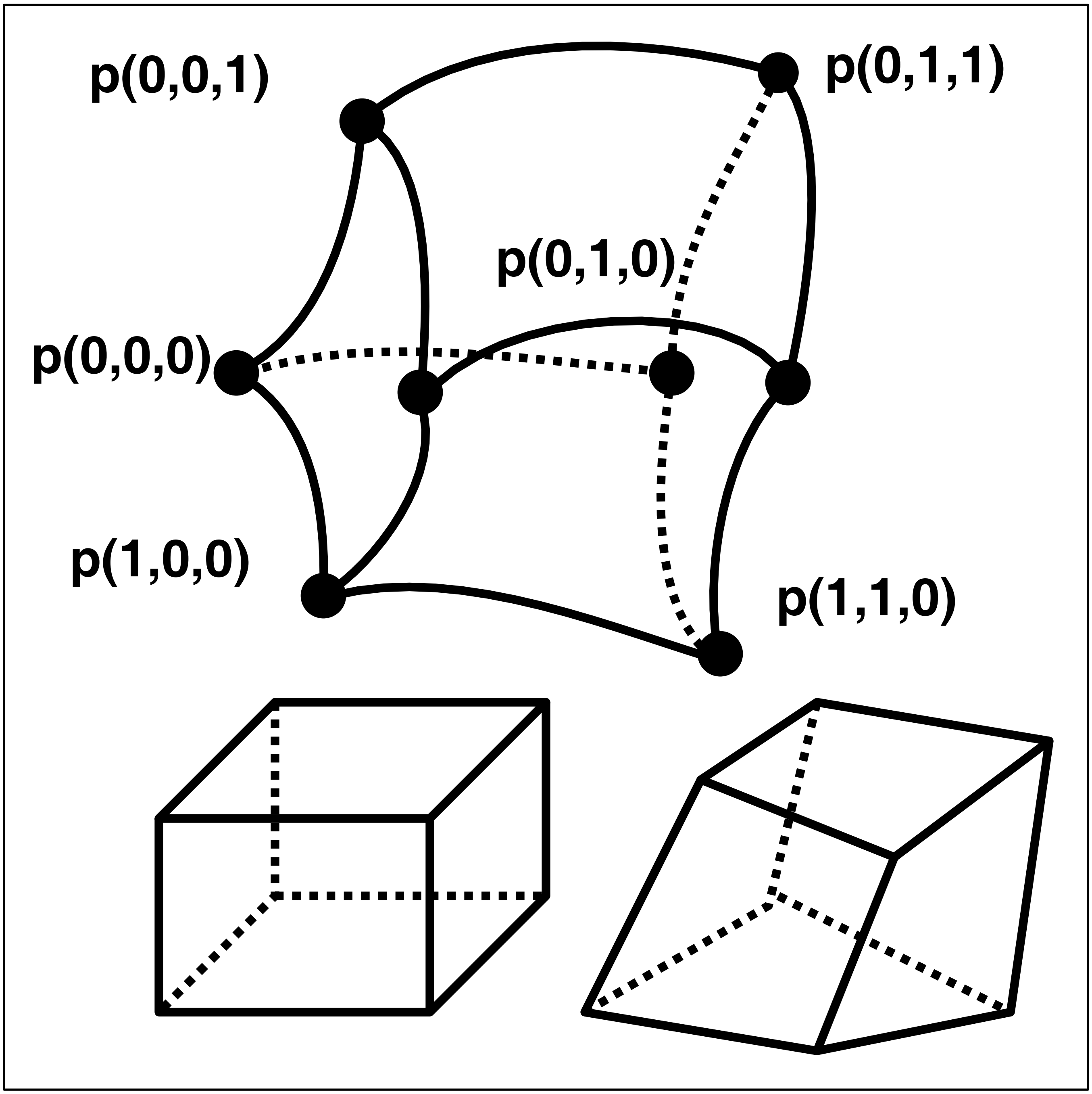
- **Also known as "hyperpatch"**

- **Parametric solids represent both exterior and interior**

- **Examples**
  - **a rectangular solid**
  - **a trilinear solid**

- **Isoparametric surfaces**
- **Boundary elements**
  - **8 corner points**
  - **12 curved edges**
  - **6 curved faces**

Solid



Parametric Solids

- **Tricubic solid**

$$\mathbf{p}(u,v,w) = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} \mathbf{a}_{ijk} u^i v^j w^k$$

where $u, v, w \in [0, 1]$

- **Bezier solid**

$$\mathbf{p}(u,v,w) = \sum_{i}\sum_{j}\sum_{k} \mathbf{p}_{ijk} B_i(u) B_j(v) B_k(w)$$

- **B-spline solid**

$$\mathbf{p}(u,v,w) = \sum_{i}\sum_{j}\sum_{k} \mathbf{p}_{ijk} B_{i,I}(u) B_{j,J}(v) B_{k,K}(w)$$

- **NURBS solid**

$$\mathbf{p}(u,v,w) = \frac{\sum_i \sum_j \sum_k \mathbf{p}_{ijk} q_{ijk} B_{i,I}(u) B_{j,J}(v) B_{k,K}(w)}{\sum_i \sum_j \sum_k q_{ijk} B_{i,I}(u) B_{j,J}(v) B_{k,K}(w)}$$

- **Tricubic Hermite solid**

- **Continuity for composite solids**
- **Matrix representation**
- **Higher dimension elements**
- **Trimmed solid and boundary**
- **Subdivision-based solid**
- **Procedure-based solid**
- **Properties**
  - **tangent, normal, curvature, derivatives, etc.**

Curves, Surfaces, Solids

- **Isoparametric curves for surfaces**

$$\mathrm{s}(u, v)$$

$$\mathrm{s}(u_i, v)$$

$$\mathrm{s}(u, v_j)$$

where u_i, v_j **are constant**

- **Isoparametric curves for solids**

$$\mathrm{s}(u, v, w)$$

$$\mathrm{s}(u_i, v_j, w)$$

$$\mathrm{s}(u_i, v, w_k)$$

$$\mathrm{s}(u, v_j, w_k)$$

where u_i, v_j, w_k **are constant**

- **Isoparametric surfaces for solids**

$$\mathrm{s}(u, v, w)$$

$$\mathbf{s}(u_i, v, w)$$

$$\mathbf{s}(u, v_j, w)$$

$$\mathbf{s}(u, v, w_k)$$

- **Non-isoparametric curves for surfaces**

$$\mathbf{s}(u, v)$$

$$\mathbf{c}(t) = \left[\begin{array}{c} u(t) \\ v(t) \end{array} \right]$$

$$\mathbf{s}(u(t), v(t))$$

- **Non-isoparametric curves for solids**

$$\mathbf{s}(u, v, w)$$

$$\mathbf{c}(t) = \left[\begin{array}{c} u(t) \\ v(t) \\ w(t) \end{array} \right]$$

$$\mathbf{s}(u(t), v(t), w(t))$$

- **Non-isoparametric surfaces for solids**

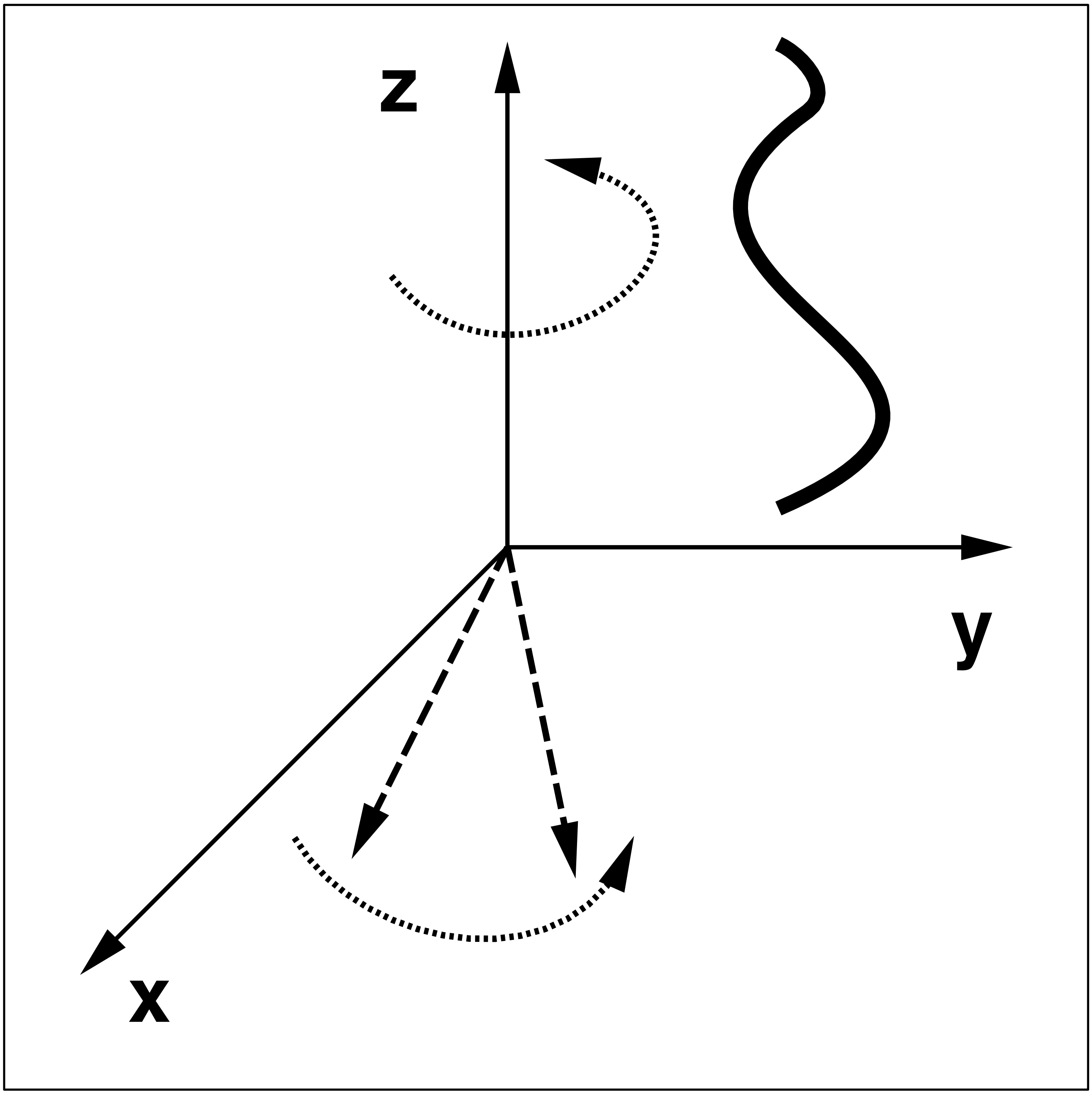
$$\mathbf{s}(u, v, w)$$

$$\mathbf{p}(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \\ w(s,t) \end{bmatrix}$$

$$\mathbf{s}(u(s,t), v(s,t), w(s,t))$$

- **Compare with curvilinear coordinate systems**
- **Irregular shape from trimming operation**
- **Applications in graphics and visualization**
  - **intensity, color, texture, gradient, material, etc.**

Surface of Revolution



Surfaces of Revolution

- Geometric construction
  - specify a planar curve profile on $y - z$ plane
  - rotate this profile with respect to z-axis

- Procedure-based model

- What kinds of shape can we model?

- Review: three-dimensional rotation w.r.t. z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

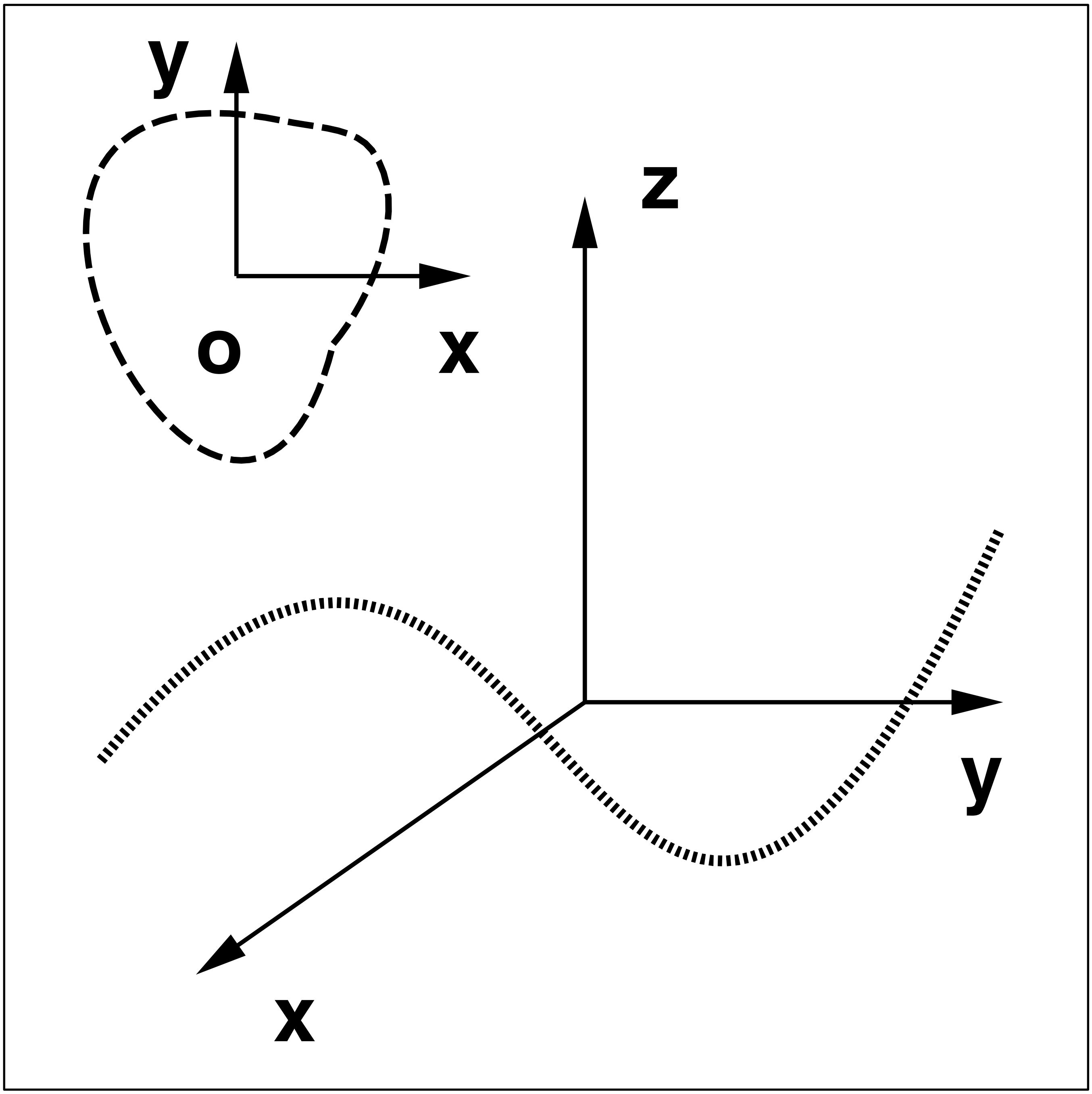
- Mathematics: surfaces of revolution

$$\mathbf{c}(u) = \begin{bmatrix} 0 \\ y(u) \\ z(u) \end{bmatrix}$$

$$\mathbf{s}(u, v) = \begin{bmatrix} -y(u)\sin v \\ y(u)\cos v \\ z(u) \end{bmatrix}$$

- **How do we render it?**

Sweeping Surface



General Sweeping Surfaces

- Surface of revolution is a special case of a sweeping surface
- Idea: a profile curve and a trajectory curve
- Profile: $\mathbf{c}_1(u)$
- Trajectory: $\mathbf{c}_2(v)$
- Move a profile curve along a trajectory curve to generate a sweeping surface
- Question: How to orient $\mathbf{c}_1(u)$ as it moves along $\mathbf{c}_2(v$
- Answer: various options
- Fixed orientation, simple translation of the coordinat system of $\mathbf{c}_1(u)$ along $\mathbf{c}_2(v)$
- Rotation: if the trajectory curve is a circle

- Move using the "Frenet Frame" of $c_2(v)$, smoothly varying orientation

- Example: surface of revolution

- Differential geometry fundamentals: Frenet frame

Frenet Frames

- Motivation: attach a smoothly-varying coordinate system to any location of a curve

- 3 independent direction vectors for a 3D coordinate system

- Tangent

$$\mathbf{t}(u) = normalize(\mathbf{c}_u(u))$$

- Bi-normal

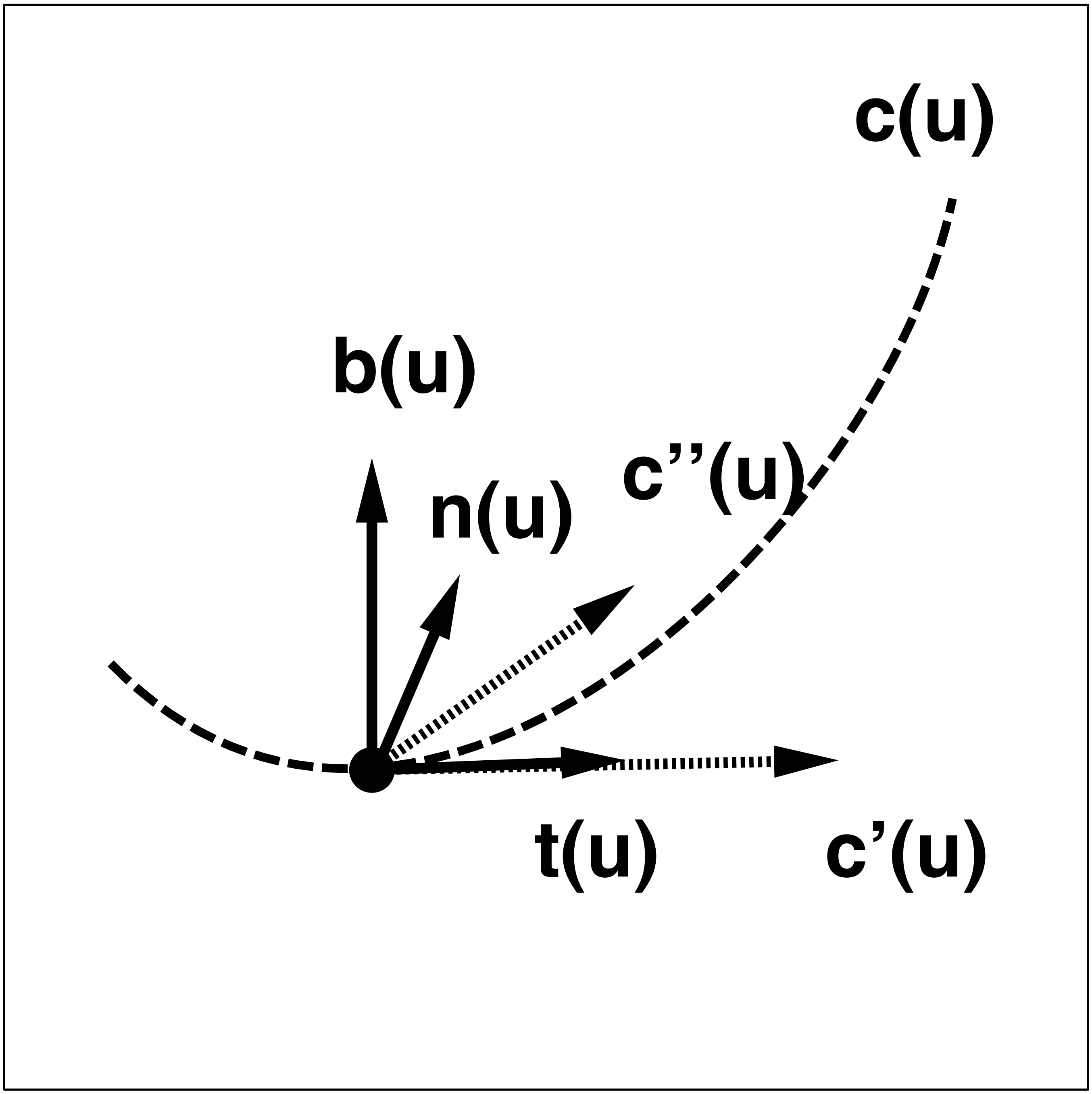
$$\mathbf{b}(u) = normalize(\mathbf{c}_u(u) \times \mathbf{c}_{uu}(u))$$

- Normal

$$\mathbf{n}(u) = normalize(\mathbf{b}(u) \times \mathbf{t}(u))$$

- Frenet coordinate system (frame) $(\mathbf{t}, \mathbf{b}, \mathbf{n})$ varies smoothly, as we move along the curve $\mathbf{c}(u)$

Frenet Coordinate System



Frenet Swept Surfaces

- Orient the profile curve $\mathbf{c}_1(u)$ using the Frenet frame $\mathbf{c}_2(v)$
  - put $c_1(u)$ on the normal plane $(\mathbf{n}, \mathbf{b})$.
  - place O_c of $\mathbf{c}_1(u)$ on $\mathbf{c}_2(v)$
  - align x_c of $\mathbf{c}_1(u)$ with $\mathbf{n}$
  - align y_c of $\mathbf{c}_1(u)$ with $\mathbf{b}$
- Example: if $\mathbf{c}_2(v)$ is a circle
- Variation (generalization)
- Scale $c_1(u)$ as it moves
- Morph $c_1(u)$ into $\mathbf{c}_3(u)$ as it moves
- Use your own imagination!

Ruled Surfaces

- **Move one straight line along a curve**
- **Examples: plane, cone, cylinder**
- **Cylindrical surface**
- **Surface equation**

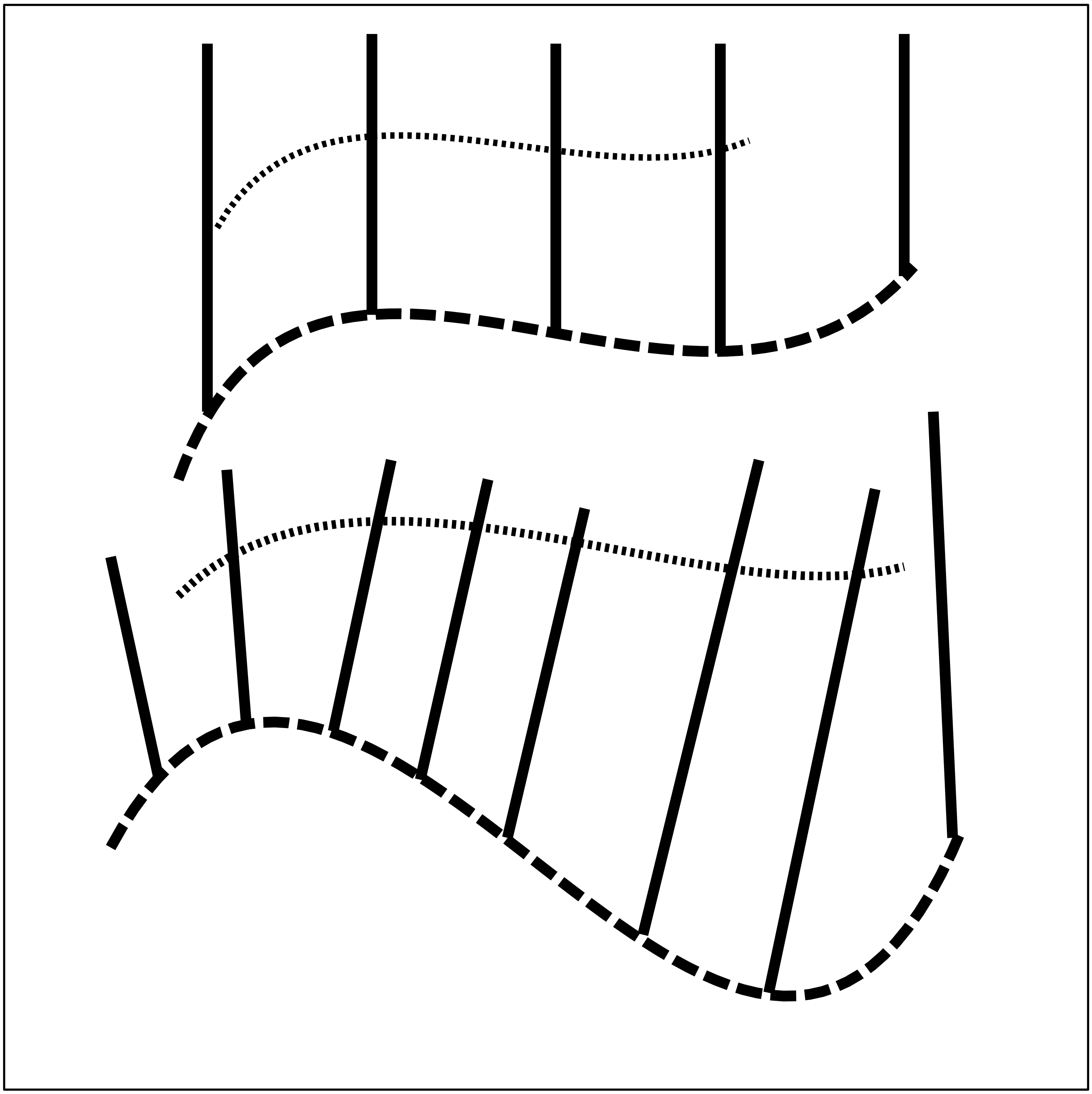
$$\mathrm{s}(u,v) = (1-v)\mathbf{a}(u) + v\mathbf{b}(u)$$

$$\mathrm{s}(u,v) = (1-v)\mathrm{s}(u,0) + v\mathrm{s}(u,1)$$

$$\mathrm{s}(u,v) = \mathbf{p}(u) + v\mathbf{q}(u)$$

- **Isoparametric lines**
- **More examples**

Ruled Surfaces



Developable Surfaces

- Deform a surface to planar shape without length/area changes
- Unroll a surface to a plane without stretching/distorting
- Example: cone, cylinder
- Developable surfaces $\subset$ Ruled surfaces
- More examples???

Implicit Equations For Curves

- **Describe an implicit relationship**
- **Planar curve**

$$f(x, y) = 0$$

- **The implicit function is not unique**
- **Comparison with parametric representation**

$$\mathbf{p}(u) = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix}$$

- **Straight line**

$$ax + by + c = 0$$

- **Conic curve (section)**

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

- **Examples**

  - **parabola**
  - **two parallel lines**
  - **ellipse**
  - **hyperbola**
  - **two intersection lines**

- **Parametric equations of conics**

- **Generalization to higher-degree curves**

- **How about non-planar (spatial) curves**

Implicit Functions

- Long history: classical algebraic geometry
- Implicit and parametric forms
  - advantages
  - disadvantages
- (-) Curves, surfaces, solids in higher-dimension
- (+) Intersection Computation
- (+) Point classification
- (+) Larger than parameter-based modeling
- (+) Unbounded geometry
- (-) Object traversal
- (-) Evaluation
- (-) Efficient algorithms, toolkits, software

- (-) Computer-based shape modeling and design
- (+) Geometric degeneracy and anomaly
- (+) Algebraic and geometric operations are often clo
- (+) Mathematics: algebraic geometry
- (+,-) Symbolic computation
- (-) Deformation and transformation
- (-) Shape editing, rendering, and control
- Conversion between parametric and implicit forms
- Implicitization vs. Parameterization
- Strategy: integration of both techniques
- Approximation using parametric models

Implicit Equations For Surfaces

- **Surface**

$$f(x, y, z) = 0$$

$$\sum_i \sum_j \sum_k a_{ijk} x^i y^j z^k = 0$$

- **Comparison with parametric representation**

$$\mathrm{s}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$$

- **Plane**

- **Quadric Surfaces**

- **More examples**
  - single plane
  - two planes (parallel or intersecting)
  - cylinder
  - cone

- ellipsoid
- paraboloid
- hyperboloid

- **Parametric representation of quadric surfaces**

- **Generalization to higher-degree surfaces**

- **Example: Superquadrics**

Superquadrics

- **Geometry (Generalization of quadrics)**
- **Superellipse**

$$(x/a_1)^{2/s} + (y/a_2)^{2/s} - 1 = 0$$

- **Superellipsoid**

$$((x/a_1)^{2/s_2} + (y/a_2)^{2/s_2})^{s_2/s_1} + (z/a_3)^{2/s_1} - 1 = 0$$

- **Parametric representation**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \cos^{s_1}(u) \cos^{s_2}(v) \\ a_2 \cos^{s_1}(u) \sin^{s_2}(v) \\ a_3 \sin^{s_1}(v) \end{bmatrix}$$

where

$$-\pi/2 \leq u \leq \pi/2$$

$$-\pi \leq v < \pi$$

- s_1 **and** s_2 **control the shape**

Spatial Curves

- **Intersection of two surfaces**

$$f(x, y, z) = 0$$

$$g(x, y, z) = 0$$

Research Areas

- Implicit-function-based free-form modeling
- Algebraic splines
- Parametric/implicit conversion
- Discretization (polygonization) of implicit primitives
- Topological and geometric properties
- Efficient algorithms
- Interpolation/approximation using implicit functions
- Optimization
- Deformable models
- Intersections
- Solid (volume) modeling

- **Applications**
  - **Human body and animated character (blobby mode**
  - **graphics, visualization, etc.**
- **Much more!!!**