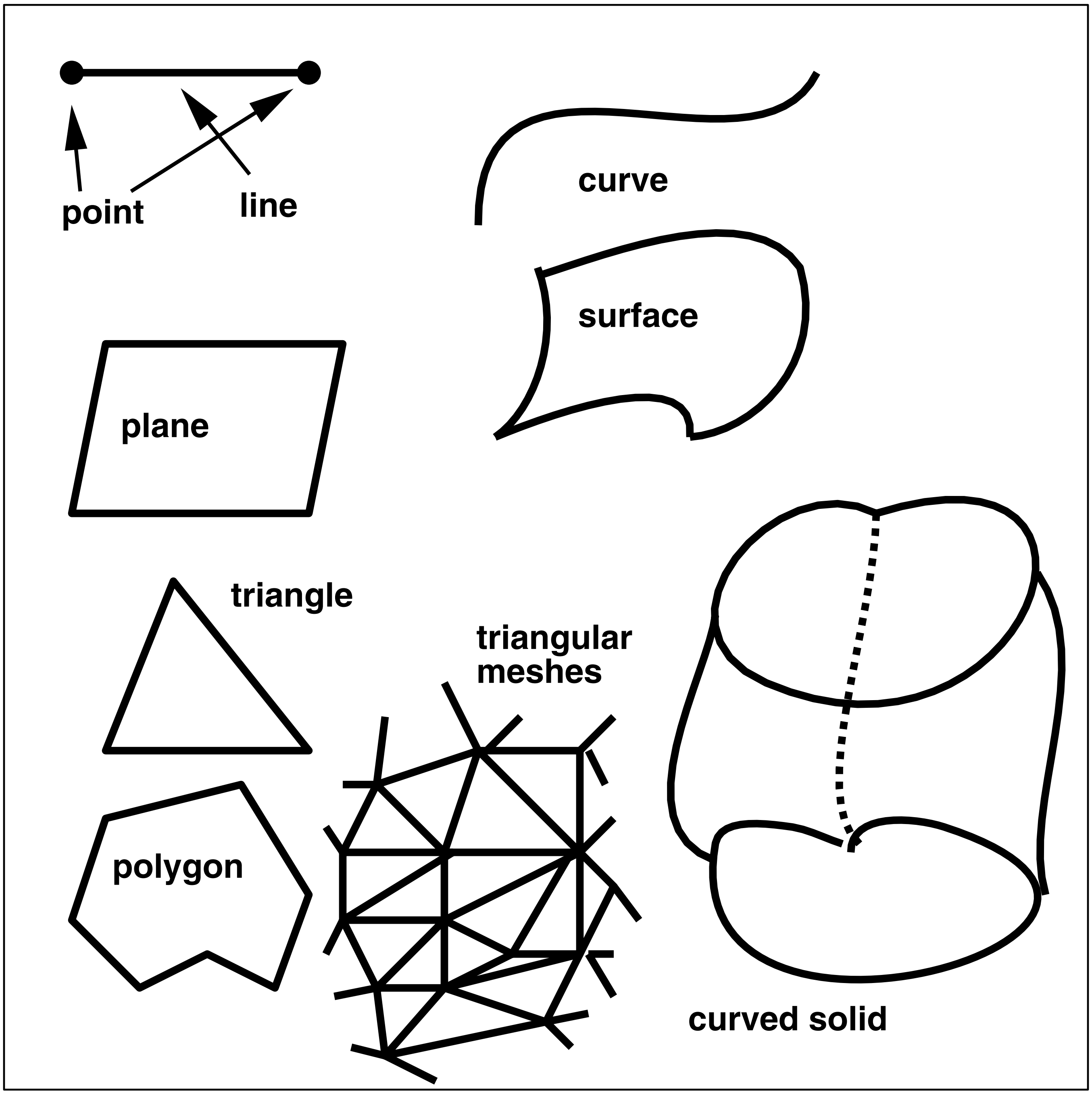
GEOMETRIC MODELING

- Why geometric modeling?
- Fundamental for visual computing
- Critical for virtual engineering
- Interaction
- Geometric information for decision making

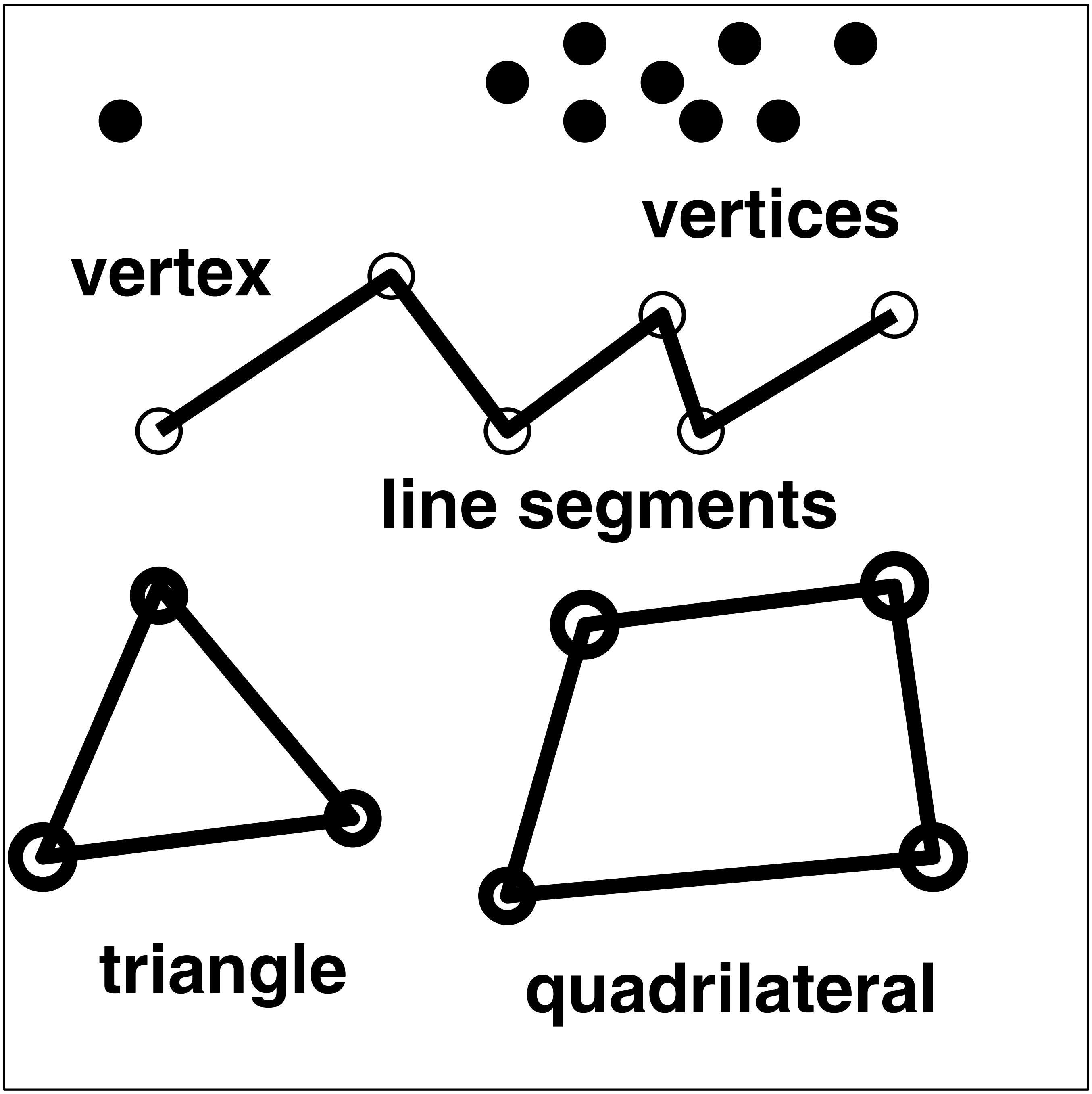
3D Shape Representation

- **Points (vertices), a set of points**
- **Lines, polylines, curves**
- **Triangles, polygons**
- **Triangular meshes, polygonal meshes**
- **Analytic (commonly-used) shape**
- **Quadric surfaces, sphere, ellipsoid, torus**
- **Superquadric surfaces, superellipse, superellipsoid**
- **Blobby models**

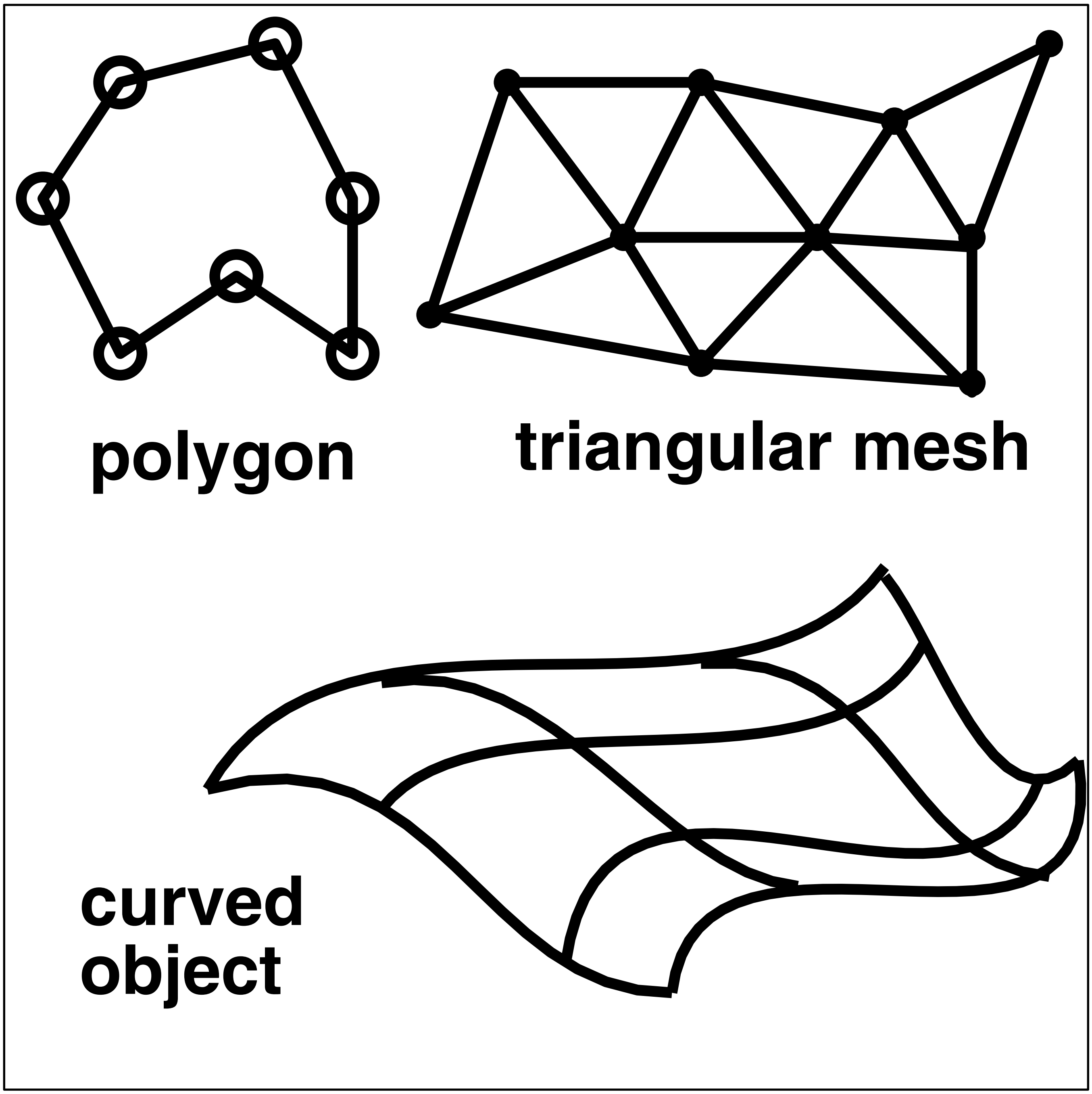
Basic Shapes



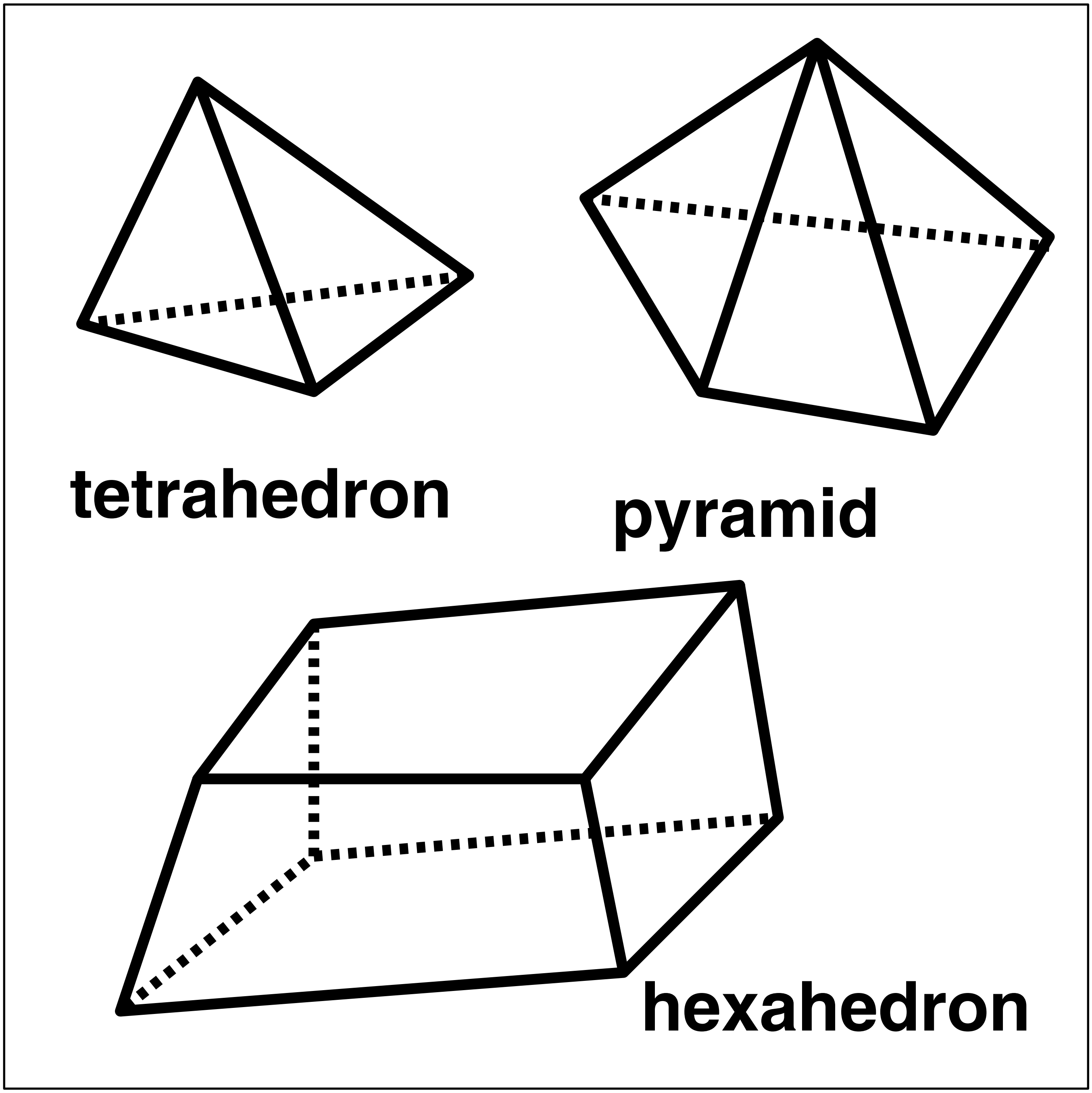
Fundamental Shapes



Fundamental Shapes



Fundamental Shapes



Mathematical Tools

- Parametric curves and surfaces
- Spline-based objects (piecewise polynomials)
- Explicit, implicit and parametric representations
- The integrated way to look at the shape:
- Object can be considered as a set of faces
  each face can be further decomposed into a set of ed
  each edge can be decomposed into vertices
- Subdivision models
- Other procedure-based models
- Sweeping
- Surfaces of revolution
- Volumetric models

Line

- **Parametric representation**

$$\mathbf{l}(\mathbf{p}_0, \mathbf{p}_1) = \mathbf{p}_0 + (\mathbf{p}_1 - \mathbf{p}_0)u$$

  where $u \in [0, 1]$

- **Parametric representation is not unique**

$$\mathbf{l}(\mathbf{p}_0, \mathbf{p}_1) = \frac{\mathbf{p}_1 + \mathbf{p}_0}{2} + \frac{\mathbf{p}_1 - \mathbf{p}_0}{2}v$$

  where $v \in [-1, 1]$

- **In general, if parametric form** $\mathbf{p}(u)$**, where** $u \in [a, b]$**, th use**

$$v = \frac{u - a}{b - a}$$

$$u = (b - a)v + a$$

  We have $\mathbf{q}(v) = \mathbf{p}((b - a)v + a)$**, where** $v \in [0, 1]$

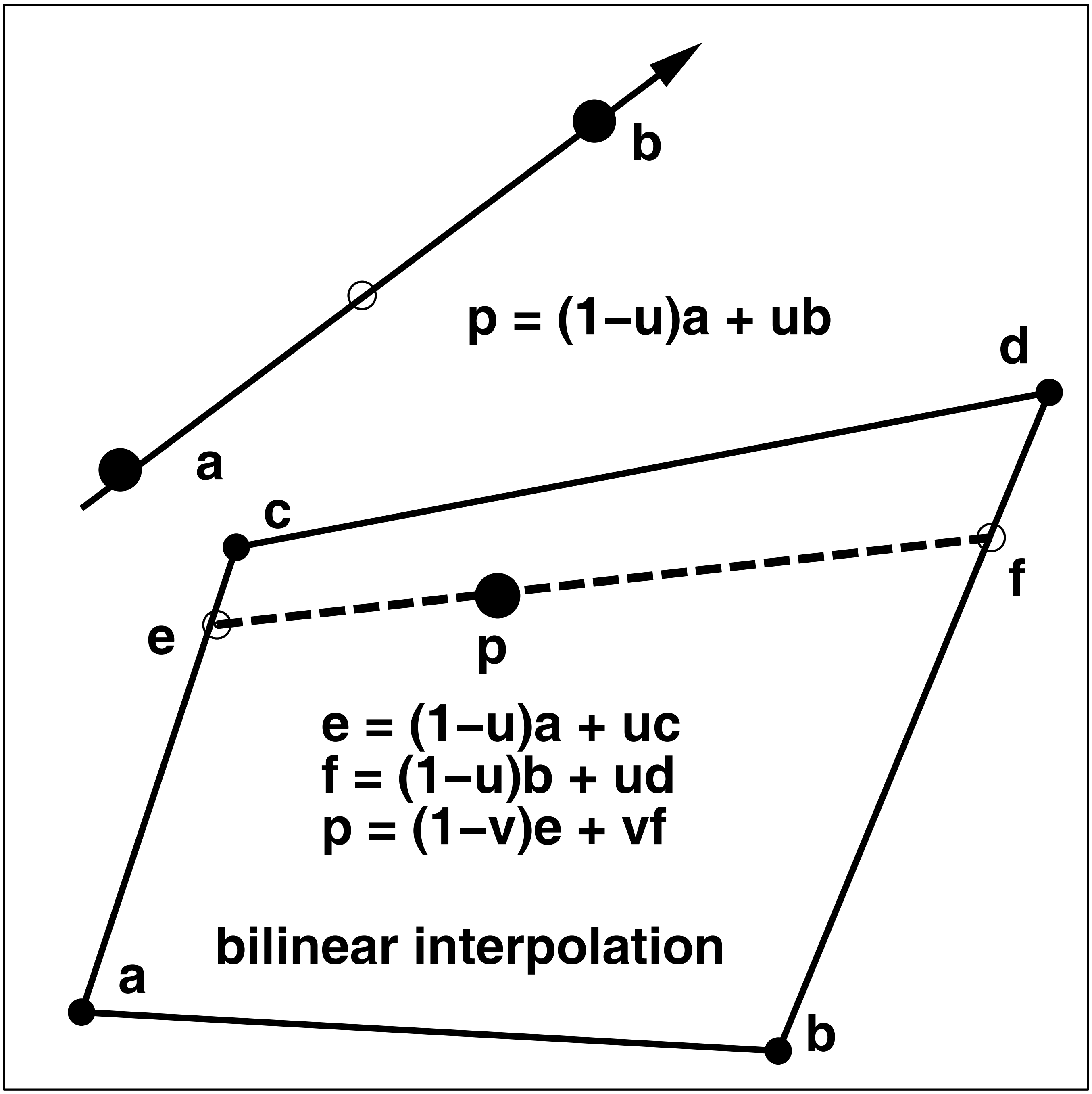
Basic Concepts

- **Linear interpolation:** $\mathbf{v} = \mathbf{v}_0(1-t) + \mathbf{v}_1(t)$
- **Local coordinates:** $\mathbf{v} \in [\mathbf{v}_0, \mathbf{v}_1]$, $t \in [0,1]$
- **Reparameterization:** $f(u)$, $u = g(v)$, $f(g(v)) = h(v)$
- **Affine transformation:**

$$f(ax + by) = af(x) + bf(y), a + b = 1$$

- **Polynomials**
- **Continuity**

Linear Interpolation



Fundamental Features

- Geometry
  - position, direction, length, area
  - normal, tangent, etc.
- Interaction
  - size, continuity
  - collision, intersection
- Topology
- Differential
  - curvature, arc-length
- Physical
- Computer representation & data structure
- Others!

Mathematical Formulations

- **Point:**

$$\mathbf{p} = \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix}$$

- **Line:**

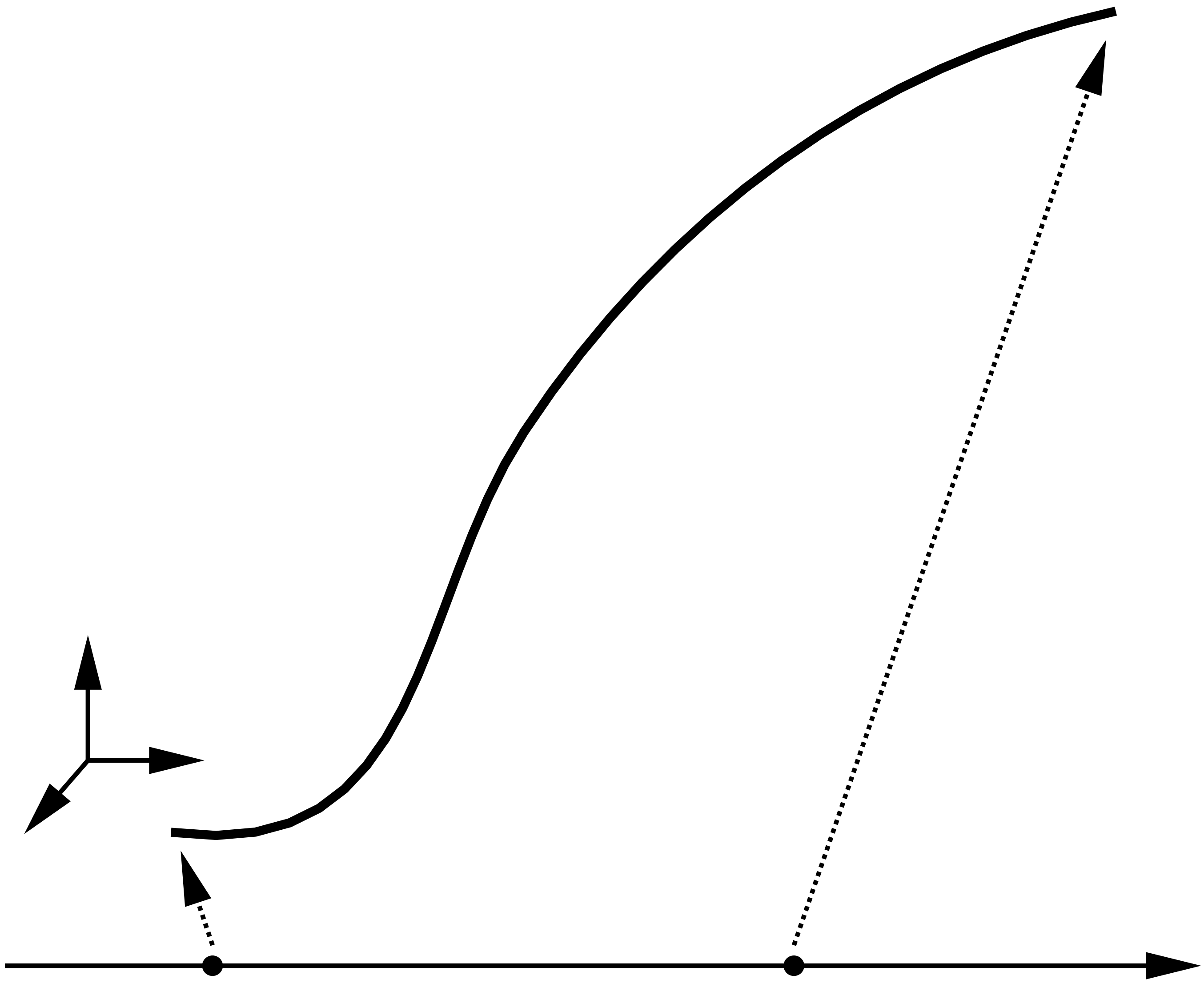
$$\mathbf{l}(u) = \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} u + \begin{bmatrix} \mathbf{b}_x \\ \mathbf{b}_y \\ \mathbf{b}_z \end{bmatrix}$$

- **Quadratic curve:**

$$\mathbf{q}(u) = \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} u^2 + \begin{bmatrix} \mathbf{b}_x \\ \mathbf{b}_y \\ \mathbf{b}_z \end{bmatrix} u + \begin{bmatrix} \mathbf{c}_x \\ \mathbf{c}_y \\ \mathbf{c}_z \end{bmatrix}$$

- **parametric domain:** $u \in [u_s, u_e]$
- **Reparameterization:** $v = (u - u_s)/(u_e - u_s)$**,** $v \in [0, 1]$
- **High-order polynomials**

Parametric Polynomials



Parametric Polynomials

- Polynomial

$$\mathbf{c}(u) = \begin{bmatrix} \mathbf{a}_{0,x} \\ \mathbf{a}_{0,y} \\ \mathbf{a}_{0,z} \end{bmatrix} + \ldots + \begin{bmatrix} \mathbf{a}_{i,x} \\ \mathbf{a}_{i,y} \\ \mathbf{a}_{i,z} \end{bmatrix} u^i + \ldots + \begin{bmatrix} \mathbf{a}_{n,x} \\ \mathbf{a}_{n,y} \\ \mathbf{a}_{n,z} \end{bmatrix} u^n$$

- No intuitive insight for the curve shape

- Difficult for piecewise smooth curves

Cubic Polynomials

- **parametric representation**

$$\begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} u^3 + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} u^2 + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} u + \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

  where $u \in [0, 1]$

- **Each components are treated independently**

- **High-dimension curves can be easily defined**

- **Alternatively**

$$x(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = UA$$

$$y(u) = UB$$

$$z(u) = UC$$

How to Define a Curve

- Specify a set of points for interpolation and/or approximation

$$\begin{bmatrix} x(u_i) \\ y(u_i) \\ z(u_i) \end{bmatrix}$$

  for some u_i

- Specify the derivatives at some locations

$$\begin{bmatrix} x'(u_i) \\ y'(u_i) \\ z'(u_i) \end{bmatrix}$$

  for some u_i

- What is the geometric meaning to specify derivatives

- A set of constraints

- Solve constraint equations

Cubic Polynomial example

- **Constraints: two end-points, one mid-point, and tangent at the mid-point**

$$x(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$x(0.5) = \begin{bmatrix} 0.5^3 & 0.5^2 & 0.5 & 1 \end{bmatrix} A$$

$$x'(0.5) = \begin{bmatrix} 3(0.5)^2 & 2(0.5) & 1 & 0 \end{bmatrix} A$$

$$x(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} A$$

- **In matrix form**

$$\begin{bmatrix} x(0) \\ x(0.5) \\ x'(0.5) \\ x(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.125 & 0.25 & 0.5 & 1 \\ 0.75 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} A$$

- **Solve this linear equation**

$$A = \begin{bmatrix} -4 & 0 & -4 & 4 \\ 8 & -4 & 6 & -4 \\ -5 & 4 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x(0.5) \\ x'(0.5) \\ x(1) \end{bmatrix}$$

- **Rewrite the curve expression**

$$x(u) = TM \begin{bmatrix} x(0) \\ x(0.5) \\ x'(0.5) \\ x(1) \end{bmatrix}$$

$$y(u) = TM \begin{bmatrix} y(0) \\ y(0.5) \\ y'(0.5) \\ y(1) \end{bmatrix}$$

$$z(u) = TM \begin{bmatrix} z(0) \\ z(0.5) \\ z'(0.5) \\ z(1) \end{bmatrix}$$

- **Basis functions**

$$f_1(u) = -4u^3 + 8u^2 - 5u + 1$$

$$f_2(u) = -4u^2 + 4u$$

$$f_3(u) = -4u^3 + 6u^2 - 2u$$

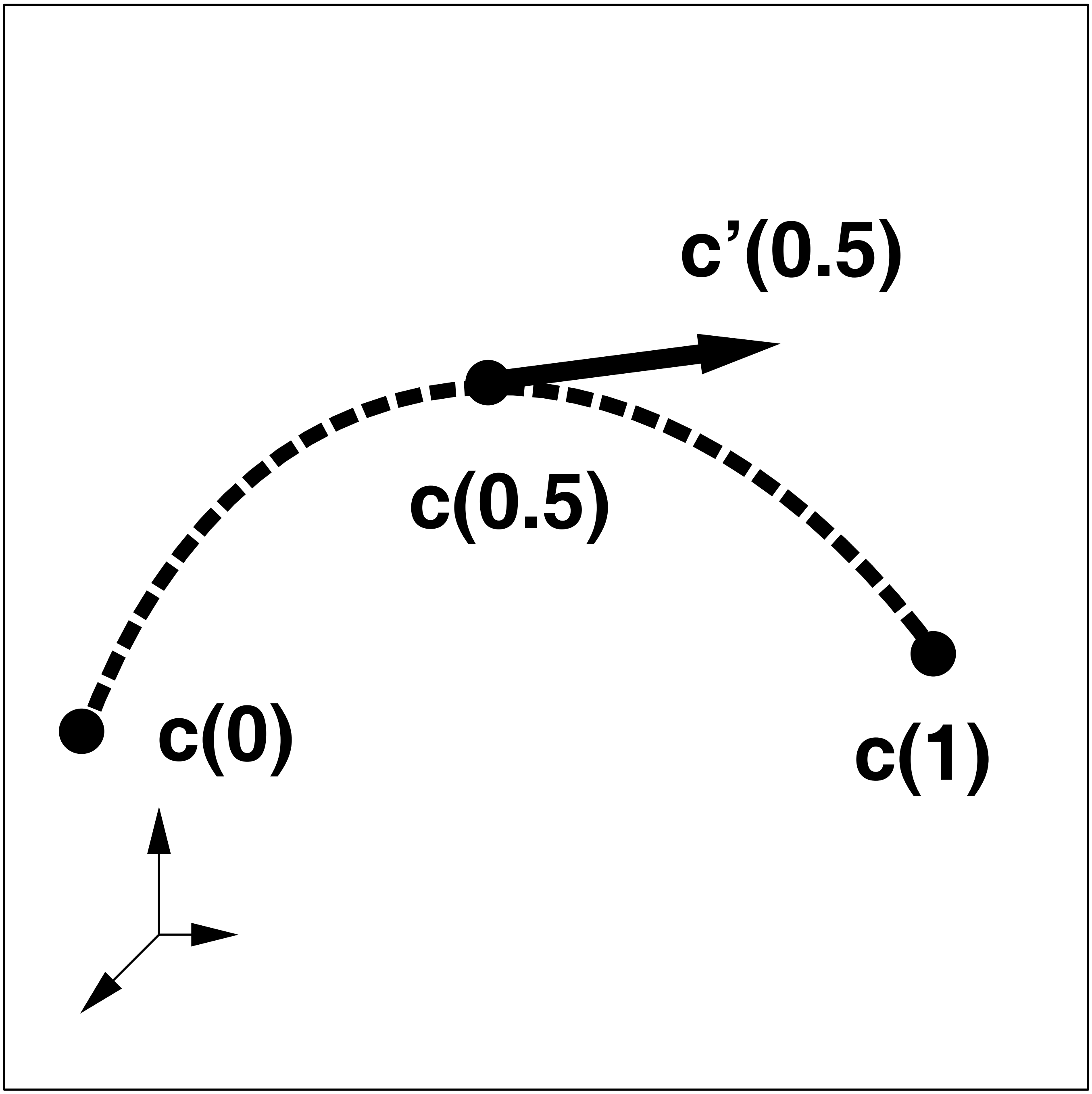
$$f_4(u) = 4u^3 - 4u^2 + 1$$

- **what is the image of these basis functions**
- **Polynomial curve can defined by**

$$\mathbf{c}(u) = \mathbf{c}(0)f_1(u) + \mathbf{c}(0.5)f_2(u) + \mathbf{c}'(0.5)f_3(u) + \mathbf{c}(1)f_4(u$$

- **Observations**
  - **More intuitive**
  - **easy to control**
  - **polynomials**

One Example



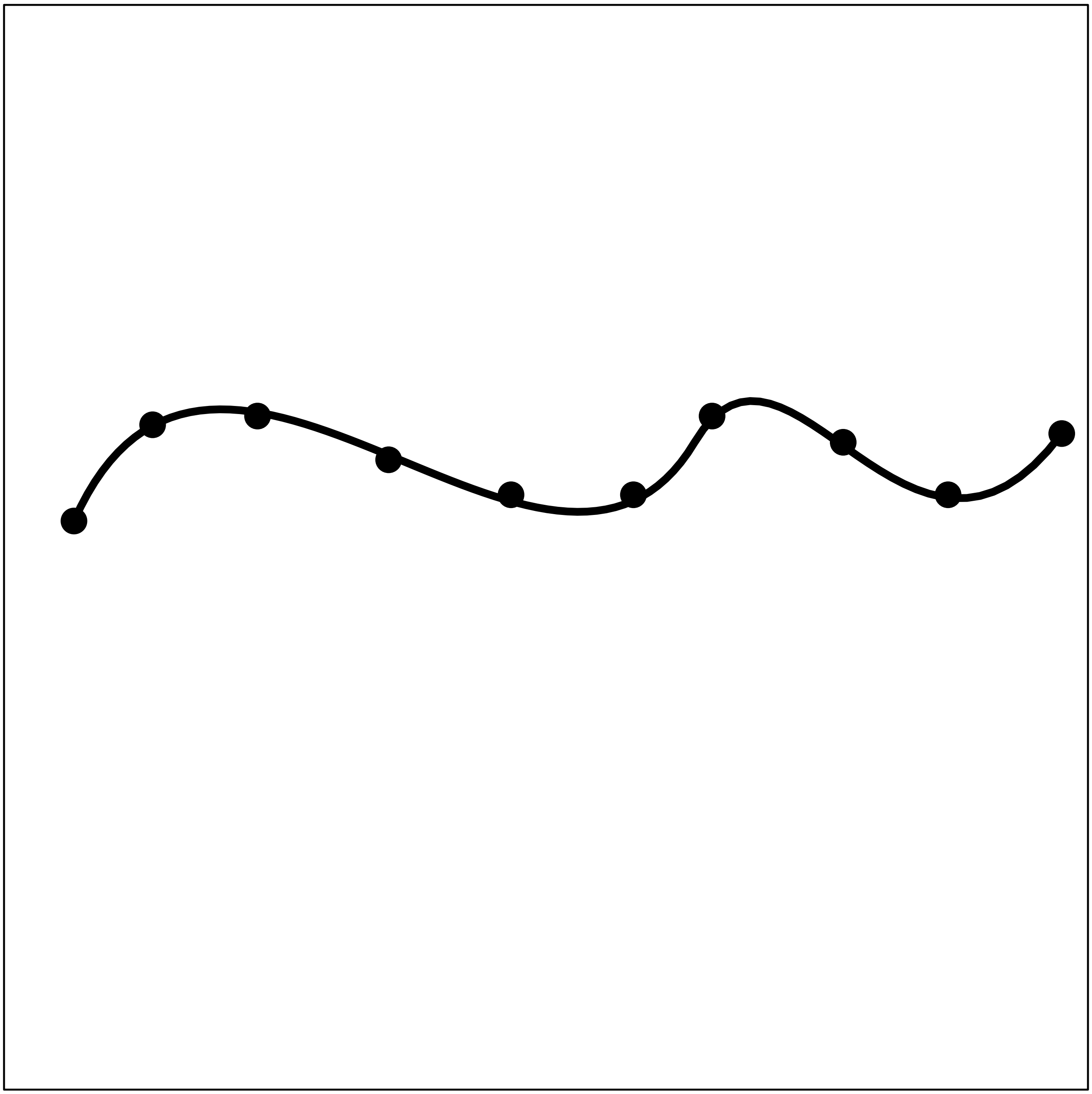
Lagrange Curves

- Curve

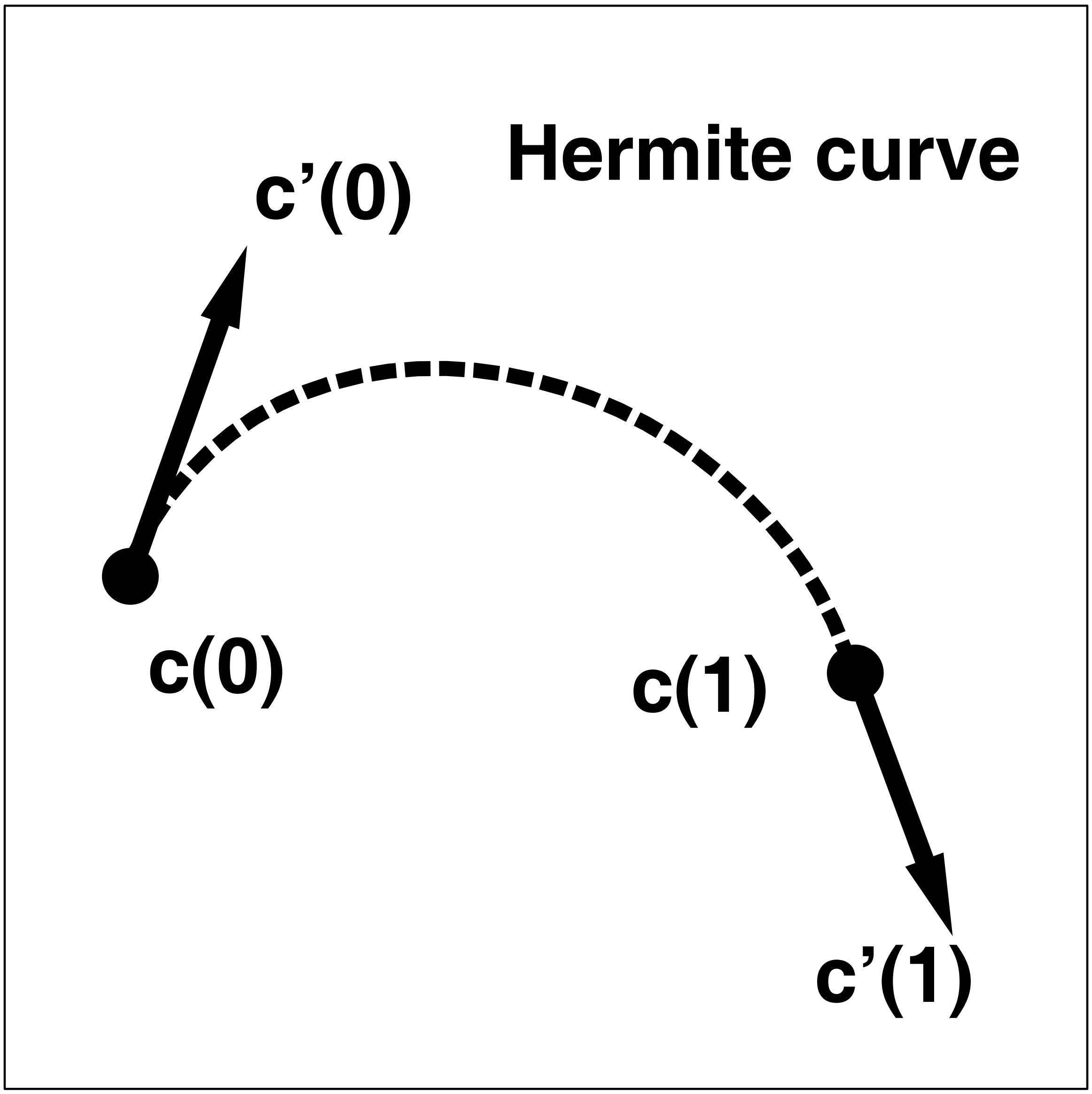
$$\mathbf{c}(u) = \begin{bmatrix} \mathbf{a}_{0,x} \\ \mathbf{a}_{0,y} \\ \mathbf{a}_{0,z} \end{bmatrix} L_0^n(u) + \ldots + \begin{bmatrix} \mathbf{a}_{n,x} \\ \mathbf{a}_{n,y} \\ \mathbf{a}_{n,z} \end{bmatrix} L_n^n(u)$$

- Lagrange polynomials of degree n: $L_i^n(u)$
- A knot sequence: $u_0, \ldots, u_n$
- $L_i^n(u_j) = \delta_{ij}$, δ is the Kronecker delta
- The curve interpolates all the $\mathbf{a}_i$'s
- Unwanted oscillation

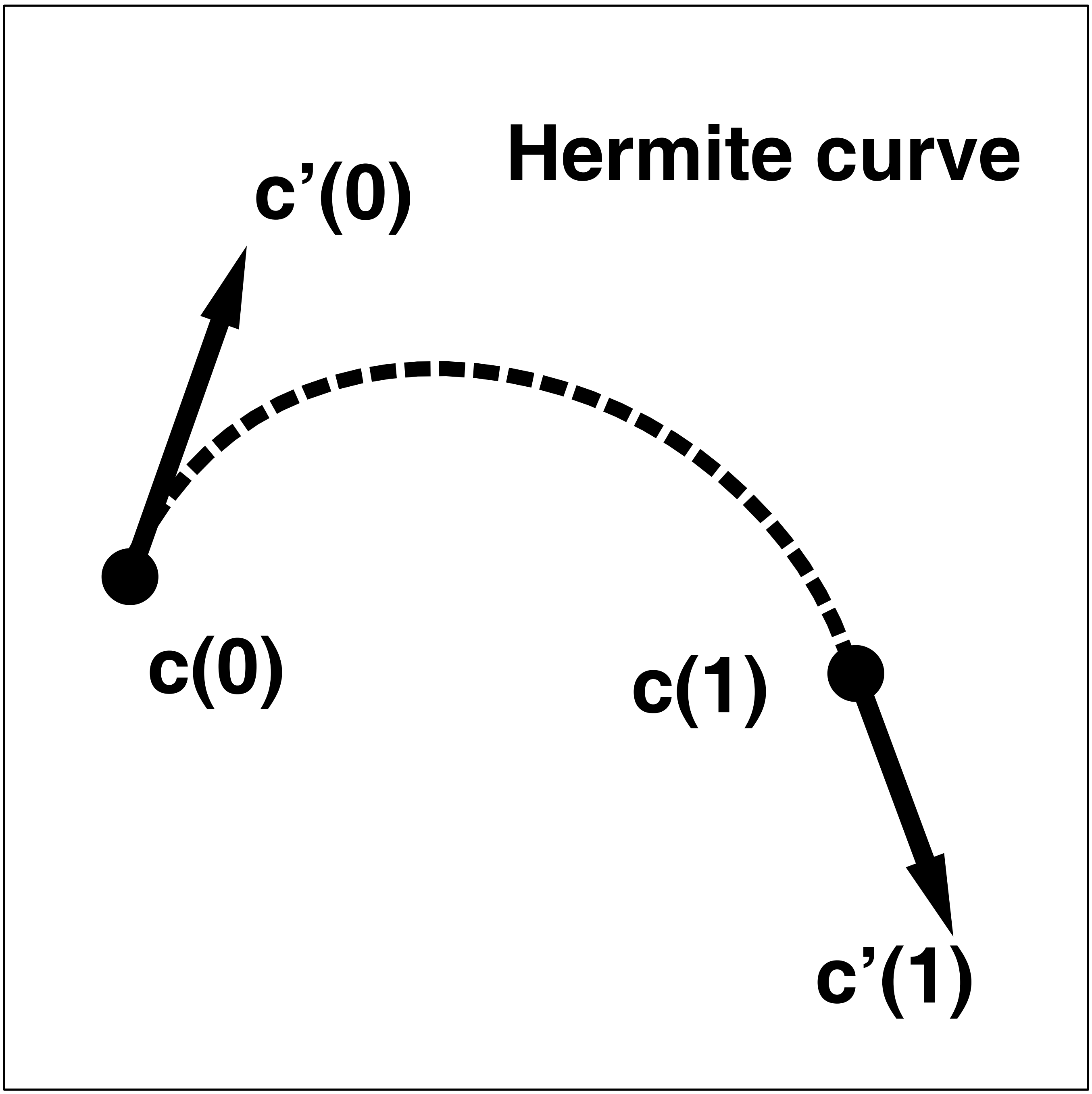
Lagrange Curve



Cubic Hermite Splines



Cubic Hermite Curve



Cubic Hermite Curve

- **Hermite curve**

$$\mathbf{c}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix}$$

- **Two end-points and two tangents at end-points**

$$\begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} A$$

- **Hermite curve**

$$x(u) = T \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix}$$

$$y(u) = TM \begin{bmatrix} y(0) \\ y(1) \\ y'(0) \\ y'(1) \end{bmatrix}$$

$$z(u) = TM \begin{bmatrix} z(0) \\ z(1) \\ z'(0) \\ z'(1) \end{bmatrix}$$

- **Basis functions**

$$f_1(u) = 2u^3 - 3u^2 + 1$$

$$f_2(u) = -2u^3 + 3u^2$$

$$f_3(u) = u^3 - 2u^2 + u$$

$$f_4(u) = u^3 - u^2$$

- **Display the image of those basis functions**

- **Hermite Curves**

$$\mathbf{c}(u) = \mathbf{c}(0)f_1(u) + \mathbf{c}(1)f_2(u) + \mathbf{c}'(0)f_3(u) + \mathbf{c}'(1)f_4(u)$$

Cubic Hermite Splines

- **Two vertices and two tangent vectors:** $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{d}_0$, **and** $\mathbf{d}_1$,

- $\mathbf{c}(0) = \mathbf{v}_0$, $\mathbf{c}(1) = \mathbf{v}_1$,

- $\mathbf{c}^{(1)}(0) = \mathbf{d}_0$, $\mathbf{c}^{(1)}(1) = \mathbf{d}_1$,

- $\mathbf{c}(u) = \mathbf{v}_0 H_0^3(u) + \mathbf{v}_1 H_1^3(u) + \mathbf{d}_0 H_2^2(u) + \mathbf{d}_1 H_3^3(u)$,

- $H_0^3(u) = 2u^3 - 3u^2 + 1$, $H_1^3(u) = -2u^3 + 3u^2$,

- $H_2^3(u) = u^3 - 2u^2 + u$, $H_3^3(u) = u^3 - u^2$

- **What do these basis functions look like?**

Hermite Curves

- **Curve**

$$\mathbf{c}(u) = \mathbf{v}_0^0 H_0^n(u) + \mathbf{v}_0^1 H_1^n(u) + \ldots + \mathbf{v}_0^{(n-1)/2} H_{(n-1)/2}^n(u)$$

$$+\mathbf{v}_1^{(n-1)/2} H_{(n+1)/2}^n(u) + \ldots + \mathbf{v}_1^1 H_{(n-1)}^n(u) + \mathbf{v}_1^0 H_n^n(u)$$

- $\mathbf{v}_0^i = \mathbf{c}^{(i)}(0)$, $\mathbf{v}_1^i = \mathbf{c}^{(i)}(1)$, $i = 0, \ldots, (n-1)/2$, **and n is odd**

- **Geometric intuition**

- **High-order derivatives are required**

Variations of Hermite Curve

- **Variations of Hermite curves**

$$\mathbf{p}_0 = \mathbf{c}(0)$$

$$\mathbf{p}_3 = \mathbf{c}(1)$$

$$\mathbf{c}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0), \mathbf{p}_1 = \mathbf{p}_0 + \frac{c'(0)}{3}$$

$$\mathbf{c}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2), \mathbf{p}_2 = \mathbf{p}_3 - \frac{c'(1)}{3}$$

- **In matrix form (x-component only)**

$$\begin{bmatrix} c(0)_x \\ c(1)_x \\ c'(0)_x \\ c'(1)_x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0,x} \\ \mathbf{p}_{1,x} \\ \mathbf{p}_{2,x} \\ \mathbf{p}_{3,x} \end{bmatrix}$$

- **Curve (x-component only)**

$$x(u) = TMM'P_x = T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} P_x$$

- **Basis functions**

$$f_0(u) = -u^3 + 3u^2 - 3u + 1 = (1-u)^3$$

$$f_1(u) = 3u^3 - 6u^2 + 3u = 3u(1-u)^2$$

$$f_2(u) = -3u^3 + 3u^2 = 3u^2(1-u)$$

$$f_3(u) = u^3$$

- **Images of basis functions**

- **Properties of basis functions**