Linear Algebra Fundamentals

- **Vector space:**
  A set of elements with structure and operations
  Certain properties hold

- **Vectors, zero vector, multiplication, addition**

- **Conventional vectors, function spaces**

- **Linear independence**

- **Vector space basis**

- **Finite dimension, infinite dimension**

- **Example: n-dimensional space**

$$R^n$$

$$e^1 = (1, 0, \ldots, 0)$$

$$e^2 = (0, 1, 0, \ldots, 0)$$

$$e^i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$e^n = (0, \ldots, 0, 1)$$

- **All continuous functions defined on** $[0, 1]$
- **Operations and properties**
  - **inner products (dot products)**

$$\mathbf{p} \star \mathbf{q} = \sum_{i=1}^{n} p_i q_i$$

$$(f \star g) = \int_0^1 f(x) g(x) dx$$

  - **orthogonality**
  - **norms (**L_2**,** L_p**, etc.)**

$$||u|| = (u \star u)^{1/2}$$

$$||u||_p = (\int |u(x)|^p dx)^{1/p}$$

$$||u||_{\text{inf}} = \max_{x \in [0,1]} |u(x)|$$

  - **normalization**

$$||u|| = 1$$

– **orthonormal basis** $u_1, u_2, \ldots,$

$$(u_i \star u_j) = \delta_{ij}$$

where δ_{ij} **is Kronecker delta**

Approximation

- Compression
- Simplification
- Mathematical foundation

$$f(x) = \sum_{i=1}^{n} p_i u_i(x)$$

$$g(x) = \sum_{i=1}^{m} q_i v_i(x)$$

such that

$$||f(x) - g(x)|| <= \epsilon$$

- Observations
  - good approximation
  - error analysis
  - approximation metric
  - arbitrary basis

– fixed basis
– various norms
– orthonormal basis
– coefficient magnitude

Multiresolution Analysis

- A nested set of vector spaces

$$V^0 \subset V^1 \subset V^2 \subset \cdots$$

- Scaling functions are basis functions
- Wavelet space W^j as
  orthogonal complement of V^j in V^{j+1}
- Basis functions for V^j

$$\Phi^j = \left[\begin{array}{ccc} \phi_0^j(x) & \cdots & \phi_{m_j-1}^j(x) \end{array} \right]$$

- wavelets for wavelet space

$$\Psi^j(x) = \left[\begin{array}{ccc} \psi_0^j(x) & \cdots & \psi_{n_j-1}^j(x) \end{array} \right]$$

- The subspace V^{j-1} is nested within V^j

$$\Phi^{j-1}(x) = \Phi^j(x) P^j$$

- Scaling functions are refinable!

- **Wavelet functions**

$$\Psi^{j-1}(x) = \Phi^{j}(x) Q^{j}$$

- **Structure of P^i and Q^j**

- **Block-matrix expression**

$$\left[\begin{array}{cc} \Phi^{j-1} & \Psi^{j-1} \end{array}\right] = \Phi^{j} \left[\begin{array}{cc} P^{j} & Q^{j} \end{array}\right]$$

- **Orthogonal requirements**

$$(\phi_k^{j-1} \star \psi_l^{j-1}) = 0$$

$$[(\Phi^{j-1} \star \Psi^{j-1})] = \mathbf{0}$$

- **Homogeneous system of equations**

$$[(\Phi^{j-1} \star \Phi^{j})] Q^{j} = A Q^{j} = \mathbf{0}$$

- **All solutions for Q^j form the null space of A**

- **Wavelet choices (bases for the null space)**
  - **orthogonal basis**
  - **semi-orthogonal basis**
  - **non-orthogonal basis**

Filter Bank

- **Functions in V^j**

$$f(x) = \Phi^j C^j$$

where C^j is a coefficient vector

- **Low-resolution approximation for $f(x)$ in V^{j-1}**

$$g(x) = \Phi^{j-1} C^{j-1}$$

- **Filtering (down-sampling)**

$$C^{j-1} = A^j C^j$$

- **Details**

$$D^{j-1} = B^j C^j$$

- **Analysis (decomposition) filters: A^j and B^j**

- **Synthesis (Reconstruction) filters**

$$C^j = P^j C^{j-1} + Q^j D^{j-1}$$

- **Multiresolution Analysis**

$$C^j, C^{j-1}, C^{j-2}, \ldots, C^0$$

$$D^{j-1}, D^{j-2}, \ldots, D^0$$

- **Filter bank**

- **Wavelet transform**

$$C^0, D^0, D^1, \ldots, D^{j-1}$$

- **Filter properties**

$$\left[\begin{array}{cc} \Phi^{j-1} & \Psi^{j-1} \end{array} \right] = \left[\begin{array}{c} A^j \\ B^j \end{array} \right]$$

$$\left[\begin{array}{c} A^j \\ B^j \end{array} \right] = \left[\begin{array}{cc} P^j & Q^j \end{array} \right]^{-1}$$

Multiresolution Framework

- **Scaling functions** $\Phi^j(x)$
  - **synthesis filters** P^j
  - **continuity requirements**
- **Inner products (determine** W^j**)**
- **Wavelets** $\Psi^j(x)$
  - **synthesis filters** Q^j
- **Analysis filters** A^j **and** B^j
- **Discussion**
  - **orthogonality**
  - **minimal local support**

Spline-Based Wavelets

- Haar basis
  - simplicity
  - orthogonality
  - small supports
  - nonoverlapping functions
    (both scaling and wavelets)
  - lack of continuity
- C^k wavelets
- B-splines
  - scaling functions
  - inner products
  - wavelets
  - filter bank