Wavelets

- Mathematical tools
  for hierarchical decomposition of functions

- Function representation

  - coarse (global) shape
  - details in different levels

- Image, curve, surface, solid

- Techniques

  - mathematical foundations
  - various wavelet functions
  - Haar wavelets
  - spline-based wavelets

- Approximation theory, signal processing

- Applications

  - image and geometric compression

- **image editing & querying**
- **multiresolution analysis and synthesis**
- **level-of-detail (LOD) control**
- **global illumination**
- **simulation and animation**

Simple Example

- Haar basis functions
- One-dimensional wavelet transform
- One-dimensional function $f(x)$

$$16 \quad 12 \quad 11 \quad 9 \quad 5 \quad 9 \quad 8 \quad 10$$

$$f(x) = \sum_{j=0}^{7} p_j B_j(x)$$

where $B_j(x)$ are piecewise constant functions

- The average of two function values

$$14 \quad 10 \quad 7 \quad 9$$

- Detailed information is lost via averaging
- Detail coefficients should be recovered!

$$2 \quad 1 \quad -2 \quad -1$$

- **Repeat the average process**

$$12 \quad 8$$

- **Detail coefficients (in the next level)**

$$2 \quad -1$$

- **Repeat the average process**

$$10$$

- **Detail coefficients (in the coarsest level)**

$$2$$

- **Wavelet transformation (decomposition)**

$$10 \quad 2 \quad 2 \quad -1 \quad 2 \quad 1 \quad -2 \quad -1$$

- **Coefficients from**
  recursively averaging and differencing operations

- **Filter bank**

- **No information is lost or gained**

- **Analysis + reconstruction**

Formulations

- **Consider functions defined over $[0, 1)$**
- **Function spaces from linear algebra**

$$V^0, V^1, V^2, V^3$$

- **Let**

$$\phi(x) = \begin{cases} 1 & 0 <= x < 1 \\ 0 & otherwise \end{cases}$$

- **Box function!**
- **Basis functions for V^j**

$$\phi_i^j(x) = \phi(2^j x - i)$$

where

$$i = 0, 1, \ldots, 2^j - 1$$

- **Scaling functions are scaled and translated "box" functions!**

- **How do box functions look like?**
- V^j**'s are nested!**
- **Define inner product on the vector space**

$$f \star g = \int_0^1 f(x)g(x)dx$$

- **Define** W^j **as the orthogonal complement of** V^j **in** V^{j+1}
- W^j **contains the details!**
- **Haar wavelets**

$$\psi_i^j(x) = \psi(2^j x - i)$$

where

$$i = 0, 1, \ldots, 2^j - 1$$

$$\psi(x) = \begin{cases} 1 & 0 <= x < 1/2 \\ -1 & 1/2 <= x < 1 \\ 0 & otherwise \end{cases}$$

- **Properties for Haar wavelets**

- W^j **is defined by basis functions** $\psi_i^j(x)$
- V^i **is defined by basis functions** $\phi_i^j(x)$
- V^{i+1} **is defined by** $\phi_i^j(x)$ **and** $\psi_i^j(x)$
- **wavelet functions are orthogonal to each other**

Wavelet Representation

- **Univariate function**

$$f(x) = \sum_{i=0}^{7} p_i^3 \phi_i^3(x)$$

- **Wavelet decomposition**

$$V^2 + W^2 = V^3$$

$$f(x) = \sum_{i=0}^{3} p_i^2 \phi_i^2(x) + \sum_{i=0}^{3} d_i^2 \psi_i^2(x)$$

$$V^1 + W^1 = V^2$$

$$f(x) = \sum_{i=0}^{1} p_i^1 \phi_i^1(x) + \sum_{i=0}^{1} d_i^1 \psi_i^1(x) + \sum_{i=0}^{3} d_i^2 \psi_i^2(x)$$

$$V^0 + W^0 = V^1$$

$$f(x) = p_0^0 \phi_0^0(x) + d_0^0 \psi_0^0(x) + \ldots + \ldots$$

- **Criteria**

- **orthogonality**
- **normalization**

Two-Dimensional Wavelets

- Standard decomposition
  - one-dimensional wavelet transformation
    for each row
  - one-dimensional wavelet transformation
    for each column

- Nonstandard decomposition
  - alternating between
    row and column operations

- Basis functions for nonstandard decomposition

- Two-dimensional Haar basis functions

- Two-dimensional scaling function

$$\phi\phi(x,y) = \phi(x)\phi(y)$$

- Three wavelet functions

$$\phi\psi(x,y) = \phi(x)\psi(y)$$

$$\psi\phi(x,y) = \psi(x)\phi(y)$$

$$\psi\psi(x,y) = \psi(x)\psi(y)$$

- **A single coarse scaling function**

$$\phi\phi^{0}_{0,0}(x,y) = \phi\phi(x,y)$$

- **Scales and translates of wavelet functions**

$$\phi\psi^{j}_{k,l}(x,y) = 2^{j}\phi\psi(2^{j}x - k, 2^{j}y - l)$$

$$\psi\phi^{j}_{k,l}(x,y) = 2^{j}\psi\phi(2^{j}x - k, 2^{j}y - l)$$

$$\psi\psi^{j}_{k,l}(x,y) = 2^{j}\psi\psi(2^{j}x - k, 2^{j}y - 1)$$

Applications

- Compression
  (simplification, approximation)
  Use smaller dataset to represent larger dataset
- Criteria
  - lossless compression
  - lossy compression
  - error metrics
- Image compression
- Multiresolution curves and surfaces
- Multiresolution subdivision-based models

Other Topics

- Mathematical theory of multiresolution analysis
- Matrix formulation for wavelets
- Bounded domain and unbounded domain
- Filter bank
- B-Spline wavelets
- Wavelets in three-dimensional space
- Volume-based wavelets
- Wavelets for arbitrary manifold