Differential Geometry

- Curves and surfaces
- Local properties
- Geometric foundations (critical for visual modeling and computing)
- Quantitative analysis
- Algorithm development
- Shape control and interrogation

Curves

- **Implicit forms (planar or spatial)**

$$f(x, y) = 0$$

$$\begin{cases} f_1(x, y, z) = 0 \\ f_2(x, y, z) = 0 \end{cases}$$

- **Parametric representation**

$$\mathbf{c}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix}$$

where $u \in [a, b]$
x, y, z **are differentiable functions**

- **Regular parameterization**

$$\mathbf{c}_u(u) \neq \mathbf{0}$$

- **Its geometric meaning**

- **Reparameterization**

$$v = v(u)$$

where v is a differentiable function, and regular

$$v_u(u) \neq 0$$

- **Inverse:** $u = u(v)$
- **In particular, let**

$$l(u) = \int_a^u |\mathbf{c}_u(u)| du$$

- **Reparameterization**

$$\mathbf{c}_u(u) du = \frac{d\mathbf{c}}{dv}\frac{dv}{du} du = \frac{d\mathbf{c}}{dv} dv = \mathbf{c}_v(u(v)) dv$$

- **Arc length parameterization:** $l(u)$
  - **independent of any regular reparameterization**
  - **an invariant parameter**
- **Arc length of the curve**

$$l(b) = \int_a^b |\mathbf{c}_u(u)| du$$

- **Arc element of the curve**

$$dl = |\mathbf{c}_u(u)| du$$

Arc Length

- **More intuitive way (chord length)**

$$u_i = a + i\Delta u$$

where $\Delta u > 0$

$$l = \sum_i |\mathbf{c}(u_{i+1}) - \mathbf{c}(u_i)| = \sum_i |\frac{\Delta \mathbf{c}_i}{\Delta u}|\Delta u$$

- **In the limit, chord length converges to arc length**
- **Arc length is a theory-oriented concept!**
- **Approximation using chord length is necessary**

Frenet Frame

- **Local coordinate system**
- **Local curve properties**
- **Assumption: all derivatives DO exist**
- **Taylor expansion of $\mathbf{c}(u + \Delta t)$ at $\mathbf{c}(u)$**

$$\mathbf{c}(u + \Delta u) = \mathbf{c}(u) + \mathbf{c}_u(u)\Delta u$$

$$+\mathbf{c}_{uu}(u)\frac{1}{2}(\Delta u)^2 + \mathbf{c}_{uuu}(u)\frac{1}{6}(\Delta u)^3 + \ldots + \ldots$$

- **Local coordinate system**

$$(\mathbf{c}_u, \mathbf{c}_{uu}, \mathbf{c}_{uuu})$$

- **Coordinates**

$$\begin{bmatrix} \Delta u + \ldots \\ \frac{1}{2}(\Delta u)^2 + \ldots \\ \frac{1}{6}(\Delta u)^3 + \ldots \end{bmatrix}$$

- The above system is NOT orthogonal system
- Frenet frame from orthonormalization

$$\mathbf{t} = \frac{\mathbf{c}_u}{|\mathbf{c}_u|}$$

$$\mathbf{m} = \mathbf{b} \times \mathbf{t}$$

$$\mathbf{b} = \frac{\mathbf{c}_u \times \mathbf{c}_{uu}}{|\mathbf{c}_u \times \mathbf{c}_{uu}|}$$

- Frenet frame $(\mathbf{t}, \mathbf{m}, \mathbf{b})$
  - tangent vector: $\mathbf{t}$
  - main normal vector: $\mathbf{m}$
  - binormal vector: $\mathbf{b}$
- Spatially varying orientation
- Osculating plane $\mathbf{o}$

$$(\mathbf{c}(u), \mathbf{t}, \mathbf{m})$$

- Osculating plane equation

$$(\mathbf{p} - \mathbf{c}(u)) \cdot (\mathbf{c}_u(u) \times \mathbf{c}_{uu}(u)) = 0$$

$$\mathrm{det}(\mathbf{p} - \mathbf{c}(u), \mathbf{c}_u, \mathbf{c}_{uu}) = 0$$

$$\mathbf{o}(u, v) = \mathbf{c}(u) + \Delta u \mathbf{c}_u(u) + \Delta v \mathbf{c}_{uu}(u)$$

- **Other properties**
  let

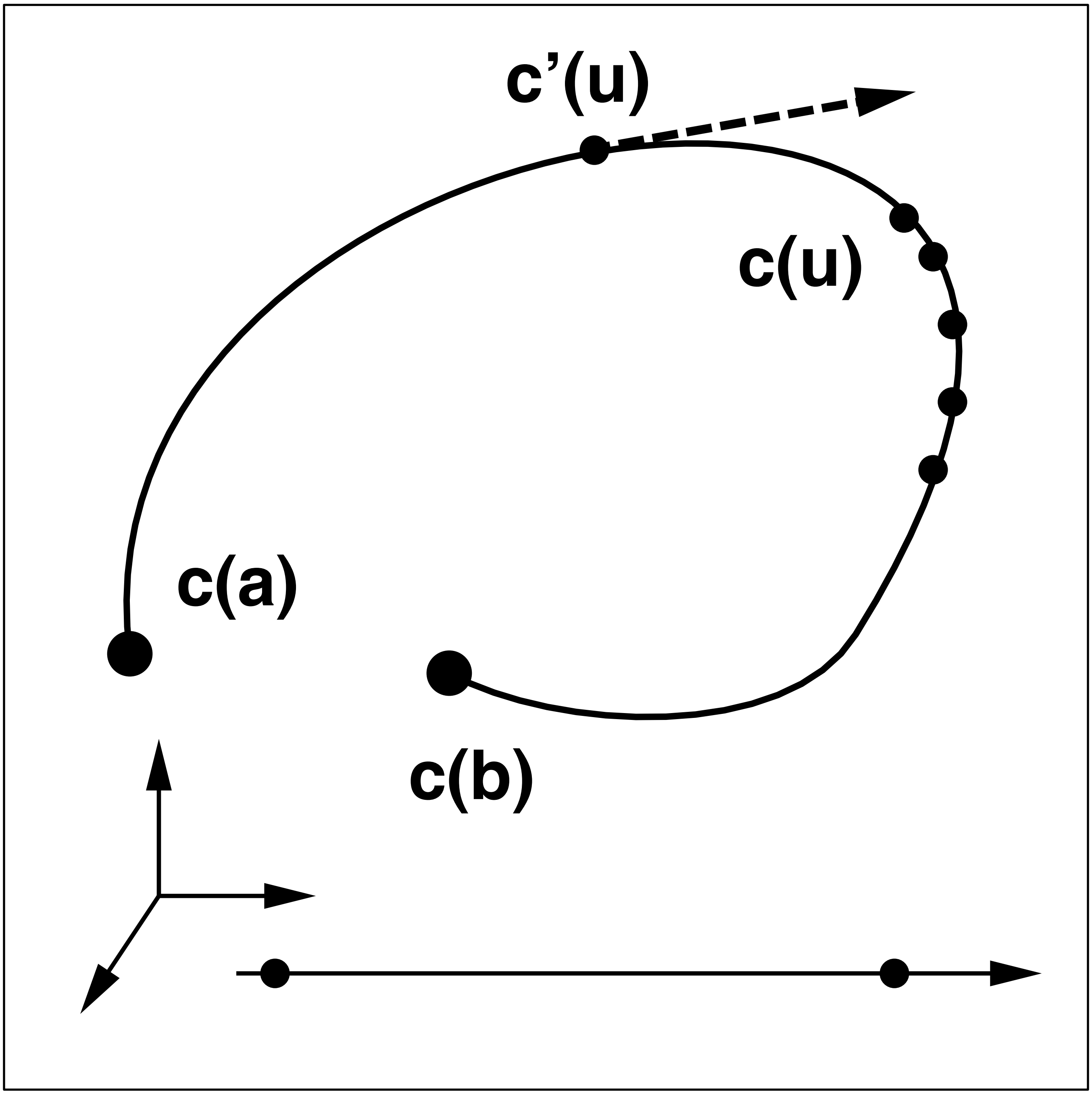
$$\mathbf{d} = (\mathbf{c}_u \mathbf{c}_u)\mathbf{c}_{uu} - (\mathbf{c}_u \mathbf{c}_{uu})\mathbf{c}_u$$

  so

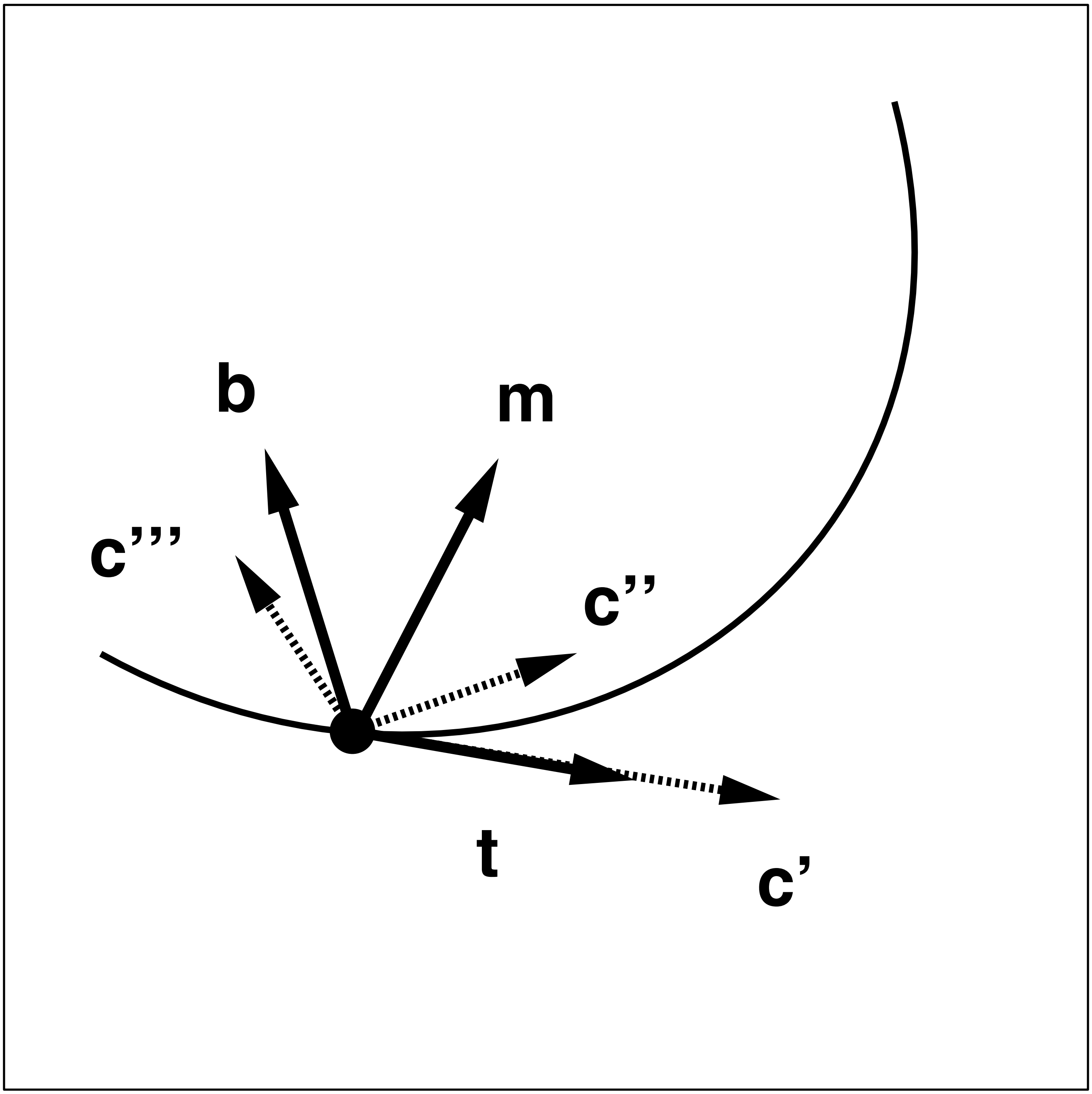
$$\mathbf{m} = \frac{\mathbf{d}}{|\mathbf{d}|}$$

- $\mathbf{b}$ **is the normal vector of the plane**

Arc Length



Frenet Frame



Moving the Frame

- The frame variation characterizes the curve's behavior
- Assume $'$: arc-length parameterization
- Basic properties

$$\mathbf{c}' \cdot \mathbf{c}' = 1$$

$$\mathbf{c}' \cdot \mathbf{c}'' = 0$$

- Unit speed traversal
- Tangent vector is perpendicular to the second-order derivative vector
- Frenet-Serret formulas

$$\begin{array}{rcl} \mathbf{t}' &=& \quad\quad +\kappa\mathbf{m} \\ \mathbf{m}' &=& -\kappa\mathbf{t} \quad\quad\quad +\tau\mathbf{b} \\ \mathbf{b}' &=& \quad\quad -\tau\mathbf{m} \end{array}$$

where

κ **is curvature**

τ **is torsion**

- **Curvature computation**

$$\kappa = \kappa(l) = |\mathbf{c}''|$$

$$\kappa = \kappa(u) = \frac{|\mathbf{c}_u \times \mathbf{c}_{uu}|}{|\mathbf{c}_u|^3}$$

- **Torsion computation**

$$\tau = \tau(l) = \frac{1}{\kappa^2} \det(\mathbf{c}', \mathbf{c}'', \mathbf{c}''')$$

$$\tau = \tau(u) = \frac{\det(\mathbf{c}_u, \mathbf{c}_{uu}, \mathbf{c}_{uuu})}{|\mathbf{c}_u \times \mathbf{c}_{uu}|^2}$$

- **Geometric meaning of Frenet-Serret Formulas**

Curvature, Torsion

- **Let $\Delta\alpha$ be the angle (in radians) between $\mathbf{t}(l)$ and $\mathbf{t}(l + \Delta l)$**

$$\kappa = \lim_{\Delta l \to 0} \frac{\Delta\alpha}{\Delta l} = \frac{d\alpha}{dl}$$

- **κ is the angular velocity of $\mathbf{t}$**

- **Let $\Delta\beta$ be the angle (in radians) between $\mathbf{b}(l)$ and $\mathbf{b}(l + \Delta l)$**

$$\tau = - \lim_{\Delta l \to 0} \frac{\Delta\beta}{\Delta l} = \frac{d\beta}{dl}$$

- **$-\tau$ measure the angular velocity of $\mathbf{b}$**

- **Properties**
  - **parameterization-independent**
  - **Euclidean invariant**
  - **independent of rigid-body motion**

- **Osculating circle**

$$\mathbf{p} = \mathbf{c} + \rho\mathbf{m}$$

where

$$\rho = \frac{1}{\kappa}$$

- **Radius of curvature**

- **This circle lies in the osculating plane** $(\mathrm{o}, \mathrm{t}, \mathrm{m})$

- **For rational Bezier curve of degree n, the curvature and torsion at the end point** p_0 **can be computed**

$$\kappa = \frac{n-1}{n}\frac{w_0 w_2}{w_1^2}\frac{b}{a^2}$$

$$\tau = \frac{n-2}{n}\frac{w_0 w_3}{w_1 w_2}\frac{c}{ab}$$

(refer to the figure for a, b, c**)**

- **Compute the curvature and torsion at any point of Bezier curve**

- **An equivalent way**

$$\kappa = 2\frac{n-1}{n}\frac{w_0 w_2}{w_1}\frac{area(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)}{dist^3(\mathbf{p}_0, \mathbf{p}_1)}$$

$$\tau = \frac{3}{2}\frac{n-2}{n}\frac{w_0 w_3}{w_1 w_2}\frac{volume(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}{area^2(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)}$$

- **Generalize to higher-order curvature of d-dimensional curves**

Other Topics

- Continuity for Composite curves
- Necessary and Sufficient conditions
- Tangent continuous
- Osculating plane continuous
- Torsion continuous
- Higher-order geometric continuity
- Generalization to Higher-dimension curve
  - $d-1$ geometric invariants
    for d-dimensional curves

From Theory to Practice

- Nice theoretical results!!!
- But, how about polygonal models
- Optimal and fast estimation for differential quantities for polygonal models
- Efficient algorithm
- Curvature and torsion computation
- Recursive subdivision models
- Error bound analysis

Parametric Surfaces

- **Surface mathematics**

$$f(x, y, z) = 0$$

$$\mathbf{s}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$$

where $(u, v) \in [a_1, a_2] \times [b_1, b_2]$

- **Regular parameterization (normal vectors are defined everywhere)**

$$\mathbf{s}_u \times \mathbf{s}_v \neq \mathbf{0}$$

- **Reparameterization will not change the surface shape**

- **A curve on the surface**

$$\mathbf{s}(u(t), v(t))$$

where $(u(t), v(t)) \in [a_1, a_2] \times [b_1, b_2]$

- **Basic features (first-order)**
  - **curve arc**
  - **surface area**

Curve Arc

- **Curve derivative**

$$\frac{d\mathrm{s}}{dt} = \mathrm{s}_t(u(t), v(t)) = \mathrm{s}_u u_t + \mathrm{s}_v v_t$$

- **Arc element**

$$dl^2 = (dl)^2 = |\mathrm{s}_t|^2 dt^2 = (\mathrm{s}_u^2 u_t^2 + 2\mathrm{s}_u \mathrm{s}_v u_t v_t + \mathrm{s}_v^2 v_t^2) dt^2$$

$$dl^2 = E du^2 + 2F du dv + G dv^2$$

where

$$E = E(u, v) = \mathrm{s}_u \mathrm{s}_u$$

$$F = F(u, v) = \mathrm{s}_u \mathrm{s}_v$$

$$G = G(u, v) = \mathrm{s}_v \mathrm{s}_v$$

- **The squared arc element: first fundamental form**

- **Invariant for curve parameterization**

- **Arc length of the (surface) curve**

$$\int_{t_0}^{t} |s_t(u(t), v(t))| dt = \int_{t_0}^{t} \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2} dt$$

Surface Area

- **Area element**

$$dA = |\mathbf{s}_u du \times \mathbf{s}_v dv| = |\mathbf{s}_u \times \mathbf{s}_v| du dv$$

- **Vector-computation review**

$$|\mathbf{a} \times \mathbf{b}|^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

$$|\mathbf{s}_u \times \mathbf{s}_v| = \sqrt{EG - F^2}$$

- **Surface area**

$$A = \int\int \sqrt{EG - F^2} du dv$$

- **u-isoparametric and v-isoparametric lines are orthogonal, when**

$$F = 0$$

Coordinate System

- **Local frame for every point on the surface**
- $\mathbf{s}_u$ **and** $\mathbf{s}_v$ **define the tangent plane**
- **Tangent plane equation (implicit)**

$$(\mathbf{p} - \mathbf{s}(u, v)) \cdot (\mathbf{s}_u \times \mathbf{s}_v) = 0$$

$$\det(\mathbf{p} - \mathbf{s}, \mathbf{s}_u, \mathbf{s}_v) = 0$$

- **Parametric equation**

$$\mathbf{p}(u, v) = \mathbf{s} + \Delta u \mathbf{s}_u + \Delta v \mathbf{s}_v$$

- **Surface normal**

$$\mathbf{n} = \frac{\mathbf{s}_u \times \mathbf{s}_v}{D}$$

- **Local coordinate system (every point)**

$$(\mathbf{s}_u, \mathbf{s}_v, \mathbf{n})$$

Second-Order Properties

- **Curvature of a surface curve**
- **Surface curvature**

Curve Curvature

- **The surface curve (arc-length parameterization)**

$$\mathrm{s}(u(l), v(l))$$

- **Assume $'$: arc-length derivative**

$$\mathrm{t} = \mathrm{s}' = \frac{d\mathrm{s}}{dl}$$

$$u' = \frac{du}{dl}$$

$$v' = \frac{dv}{dl}$$

- **Curve curvature**

$$\mathrm{t}' = \kappa \mathrm{m}$$

- **Radius of curvature**

$$\kappa = \frac{1}{\rho}$$

- **Formulate the (curve) curvature in surface terms**

$$\mathbf{t}' = \mathbf{s}'' = \mathbf{s}_{uu}(u')^2 + 2\mathbf{s}_{uv}u'v' + \mathbf{s}_{vv}(v')^2 + \mathbf{s}_u u'' + \mathbf{s}_v v''$$

- **Let ϕ be the angle between $\mathbf{m}$ and $\mathbf{n}$**

$$\mathbf{t}'\mathbf{n} = \kappa \cos\phi$$

$$\kappa\cos\phi = \mathbf{n}\mathbf{s}_{uu}(u')^2 + 2\mathbf{n}\mathbf{s}_{uv}u'v' + \mathbf{n}\mathbf{s}_{vv}(v')^2$$

note

$$\mathbf{n}\mathbf{s}_u = \mathbf{n}\mathbf{s}_v = 0$$

furthermore,

$$\mathbf{n}_u\mathbf{s}_u + \mathbf{n}\mathbf{s}_{uu} = 0$$

- **New abbreviations**

$$L = L(u,v) = -\mathbf{s}_u\mathbf{n}_u = \mathbf{n}\mathbf{s}_{uu}$$

$$M = M(u,v) = -\frac{1}{2}(\mathbf{s}_u\mathbf{n}_v + \mathbf{s}_v\mathbf{n}_u) = \mathbf{n}\mathbf{s}_{uv}$$

$$N = N(u,v) = -\mathbf{s}_v\mathbf{n}_v = \mathbf{n}\mathbf{s}_{vv}$$

- **Second fundamental form**

$$\kappa\cos\phi\, dl^2 = L du^2 + 2M du dv + N dv^2$$

- **Curvature computation**
  - **curve tangent (from du/dv)**
  - **curve normal (from ϕ)**
  - **first fundamental form**
  - **second fundamental form**
- **Curve curvature depends on ϕ and t only**

Normal Curvature

- **If** $\phi = 0$

$$\mathbf{m} = \mathbf{n}$$

- **(Curve) osculating plane perpendicular to (surface) tangent plane**

- **Normal curvature of the surface in the direction of** t

$$\kappa_0$$

note t **is defined by** du/dv

- **Formulation**

$$\kappa_0 = \kappa_0(\mathrm{s}, \mathrm{t}) = \frac{1}{\rho} = \frac{2nd fundamental form}{1st fundamental form}$$

- **Radius of curvature**

$$\rho = \rho_0 \cos \phi$$

- **Geometric meaning (interpretation)**

- **All osculating circles at s for surface curves with the same tangent t form a sphere**

- **This sphere and the surface have a common tangent plane**

- **The radius of the sphere is ρ_0**

- **It is sufficient to only investigate curves with $\mathrm{m} = \mathrm{n}$**

- **Normal sections (intersection of the surface with a plane containing n)**

- **Normal sections are planar**

Lines of Curvature

- **Normal curvature κ**
  (Surface normal is on the osculating plane)

$$\mathbf{m} = \mathbf{n}$$

$$\kappa = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

where

$$\lambda = \frac{dv}{du} = \tan(\alpha)$$

- **One special case**

$$L : M : N = E : F : G$$

  - κ **is independent of** λ
  - **umbilical points**

- **General case: κ is a function of λ**

- **Two extreme values can be found**

$$\kappa_1, \kappa_2$$

- **The quantities λ_1 and λ_2 define two directions in the (u,v) plane**

- **The corresponding directions in the tangent plane are called principal directions**

- **Lines with these directions are called lines of curvature**

- **Useful for surface parameterization**

$$F = 0$$

$$M = 0$$

- **Example: lines of curvature on a torus!**

Principal Curvature

- **Principal curvatures (two extreme values)**

$$\kappa_1, \kappa_2$$

- **Gaussian curvature**

$$K = \kappa_1 \kappa_2$$

$$\kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

- **Mean curvature**

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$\kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2}$$

- **Shape (point, region) classification**
  - **elliptic points (regions), when** $K > 0$ **(e.g. ellipsoid)**
  - **hyperbolic points (regions), when** $K < 0$

(e.g. hyperboloid)
- parabolic points (regions), when either $\kappa_1 = 0$ or κ_2 (e.g. cylinder)
- flat points (regions), when $K = 0$ and $H = 0$ (e.g. plane)

- Another important result (due to Gauss)
  - K is determined by E,F, and G and their derivatives!
  - K does not change if surface deformation does NO change any length measurement!

- Developable surface: $K = 0$

- Sphere points are umbilical, both Gaussian and mean curvatures are constant

Euler's Theorem

- **Normal curvatures are NOT independent of each other**
- **For simplicity, assume isoparametric curves are lines of curvature**

$$F = 0$$

$$M = 0$$

$$\kappa_1 = \frac{L}{E}$$

$$\kappa_2 = \frac{N}{G}$$

$$\kappa(\lambda) = \frac{L + N\lambda^2}{E + G\lambda^2} = \kappa_1 \frac{E}{E + G\lambda^2} + \kappa_2 \frac{G\lambda^2}{E + G\lambda^2}$$

- **Geometric meaning: Φ is the angle between $\mathbf{s}_u$ and $\mathbf{s}_t = \mathbf{s}_u u_t + \mathbf{s}_v v_t$**

$$\cos\Phi = \frac{\mathbf{s}_t \mathbf{s}_u}{|\mathbf{s}_t||\mathbf{s}_u|}$$

$$\sin \phi = \frac{\mathbf{s}_t \mathbf{s}_v}{|\mathbf{s}_t||\mathbf{s}_v|}$$

where $\lambda = (dv/dt)/(du/dt)$

$$\kappa = \kappa_1 \cos^2 \Phi + \kappa_2 \sin^2 \Phi$$

Other Concepts

- Dupin's indicatrix
- Asymptotic lines
- Conjugate direction
- Differential analysis for
  - ruled surfaces
  - developable surfaces
- Continuity for composite surfaces
- Curvature continuous
- Tangent plane continuous

Applications

- Basic components
  - tangent, curvature, torsion
  - arc-length, normal curvature
  - continuity
  - much more!!!
- Scientific and engineering tools
  - analysis
  - synthesis
- Shape classification
- Shape quality
- Shape interrogation (inspection)
- Shape manipulation and control
- Example: curvature plot!

- **Polyhedron-based objects**
- **Subdivision-based objects**
- **Constructive (procedural) objects**
- **Volumetric objects**
- **Algorithm**
  - **precision**
  - **efficiency**
  - **robustness**
  - **error tolerance**
  - **generality**