

Lagrange Basis Functions

- **Basis Functions**

$$L_i^n(u_j) = \begin{cases} 1 & i = j (i, j = 0, 1, \dots, n) \\ 0 & \text{Otherwise} \end{cases}$$

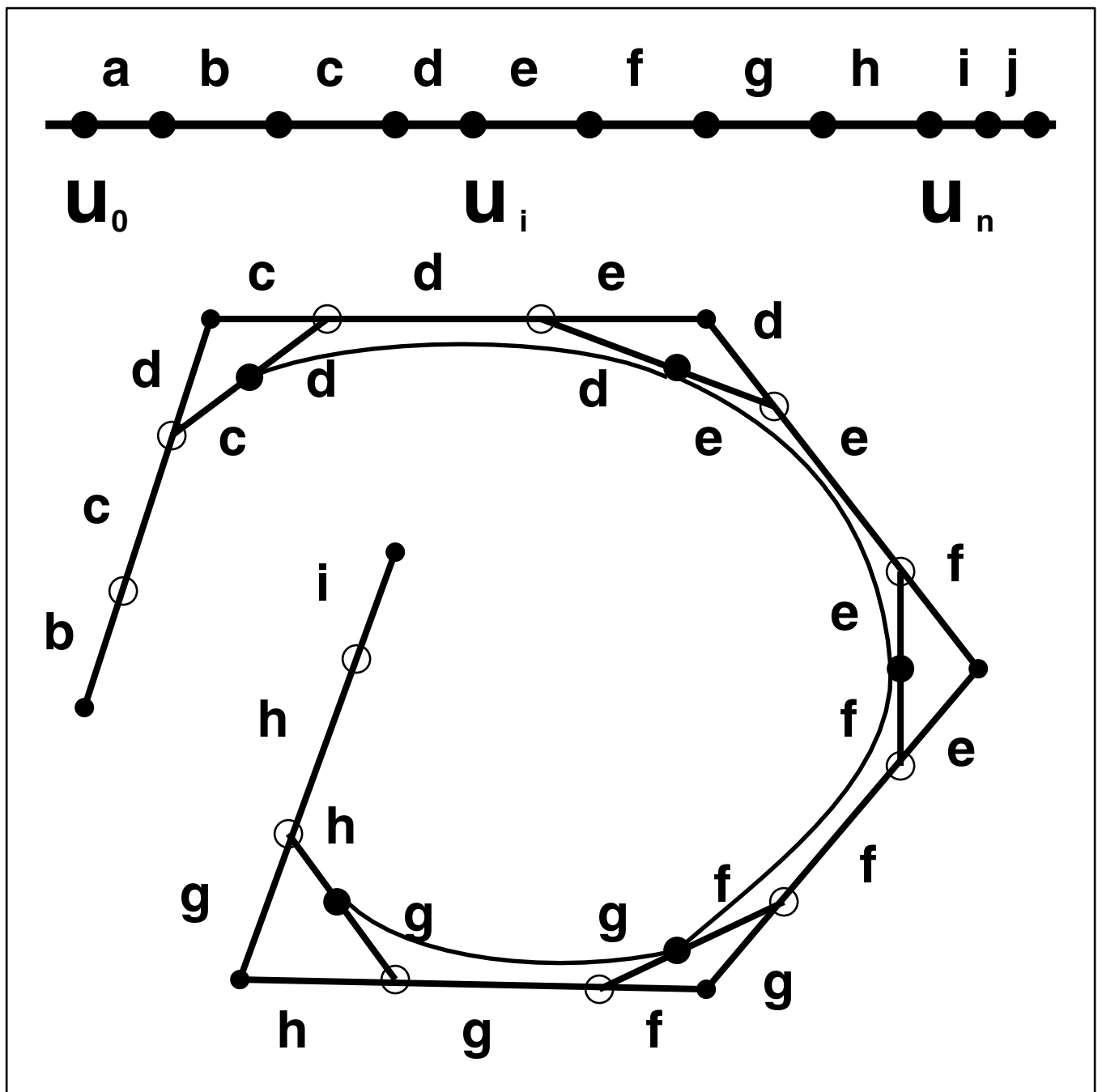
$$L_0^n(u) = \frac{(u - u_1)(u - u_2) \cdots (u - u_n)}{(u_0 - u_1)(u_0 - u_2) \cdots (u_0 - u_n)}$$

$$L_i^n(u) = \frac{(u - u_0) \cdots (u - u_{i-1})(u - u_{i+1}) \cdots (u - u_n)}{(u_i - u_0) \cdots (u_i - u_{i-1})(u_i - u_{i+1}) \cdots (u_i - u_n)}$$

$$L_n^n(u) = \frac{(u - u_0)(u - u_1)(u - u_2) \cdots (u - u_{n-1})}{(u_n - u_0)(u_n - u_1)(u_n - u_2) \cdots (u_n - u_{n-1})}$$

- **Unwanted oscillation: WHY?**

Nonuniform B-Spline



B-Spline Example

- **Knot vector:** $u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7$
- **Cubic basis functions:** $B_{0,4}, B_{1,4}, B_{2,4}, B_{3,4}$
- $B_{0,4}$ is defined over $[u_0, u_4]$
- $B_{1,4}$ is defined over $[u_1, u_5]$
- $B_{2,4}$ is defined over $[u_2, u_6]$
- $B_{3,4}$ is defined over $[u_3, u_7]$
- **The curve can be defined as**

$$c(u) = p_0 B_{0,4} + p_1 B_{1,4} + p_2 B_{2,4} + p_3 B_{3,4} =$$

$$p_0 \left\{ \frac{u - u_0}{u_3 - u_0} B_{0,3} + \frac{u_4 - u}{u_4 - u_1} B_{1,3} \right\} +$$

$$p_1 \left\{ \frac{u - u_1}{u_4 - u_1} B_{1,3} + \frac{u_5 - u}{u_5 - u_2} B_{2,3} \right\} + \dots + \dots =$$

$$p_0 \frac{u - u_0}{u_3 - u_0} B_{0,3} + (p_0 \frac{u_4 - u}{u_4 - u_1} + p_1 \frac{u - u_1}{u_4 - u_1}) B_{1,3} + \dots + \dots$$

- The B-spline curve of order k (components)...

$$p_i B_{i,k}(u) =$$

$$p_i \left(\frac{u - u_i}{u_{i+k-1} - u_i} B_{i,k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} B_{i+1,k-1}(u) \right)$$

$$p_{i+1} B_{i+1,k}(u) =$$

$$p_{i+1} \left(\frac{u - u_{i+1}}{u_{i+k} - u_{i+1}} B_{i+1,k-1} + \frac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+2}} B_{i+2,k-1} \right)$$

- Finally, the curve ...

$$\dots + \left(p_i \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} + p_{i+1} \frac{u - u_{i+1}}{u_{i+k} - u_{i+1}} \right) B_{i+1,k-1} + \dots$$

B-Spline Facts

- $n + 1$ control points: p_i
- $n + 1$ basis functions: $B_{i,k}(u)$
- Linear combination: $c(u) = \sum_{i=0}^n p_i B_{i,k}(u)$
- Important variables: i is index, k is the order, $k - 1$ is degree
- Knots: $\{u_0, \dots, u_{k-1}, \dots, u_{n+1}, \dots, u_{n+k}\}$
- The first k and last k knots do NOT contribute to the parametric domain
- Basis function $B_{i,k}$ is defined recursively!

B-Spline Discretization

- Parametric domain: $[u_{k-1}, u_{n+1}]$
- There are $n + 2 - k$ curve spans (pieces)
- Assume $m + 1$ points per span (uniform sampling)
- Total sampling points: $m(n + 2 - k) + 1 = l$
- B-spline discretization: $\mathbf{q}_0, \dots, \mathbf{q}_{l-1}$
- Corresponding parametric values: $v_0, \dots, v_{l-1}$

$$\mathbf{q}_i = \mathbf{c}(v_i) = \sum_{j=0}^n \mathbf{p}_j B_{j,k}(v_i)$$

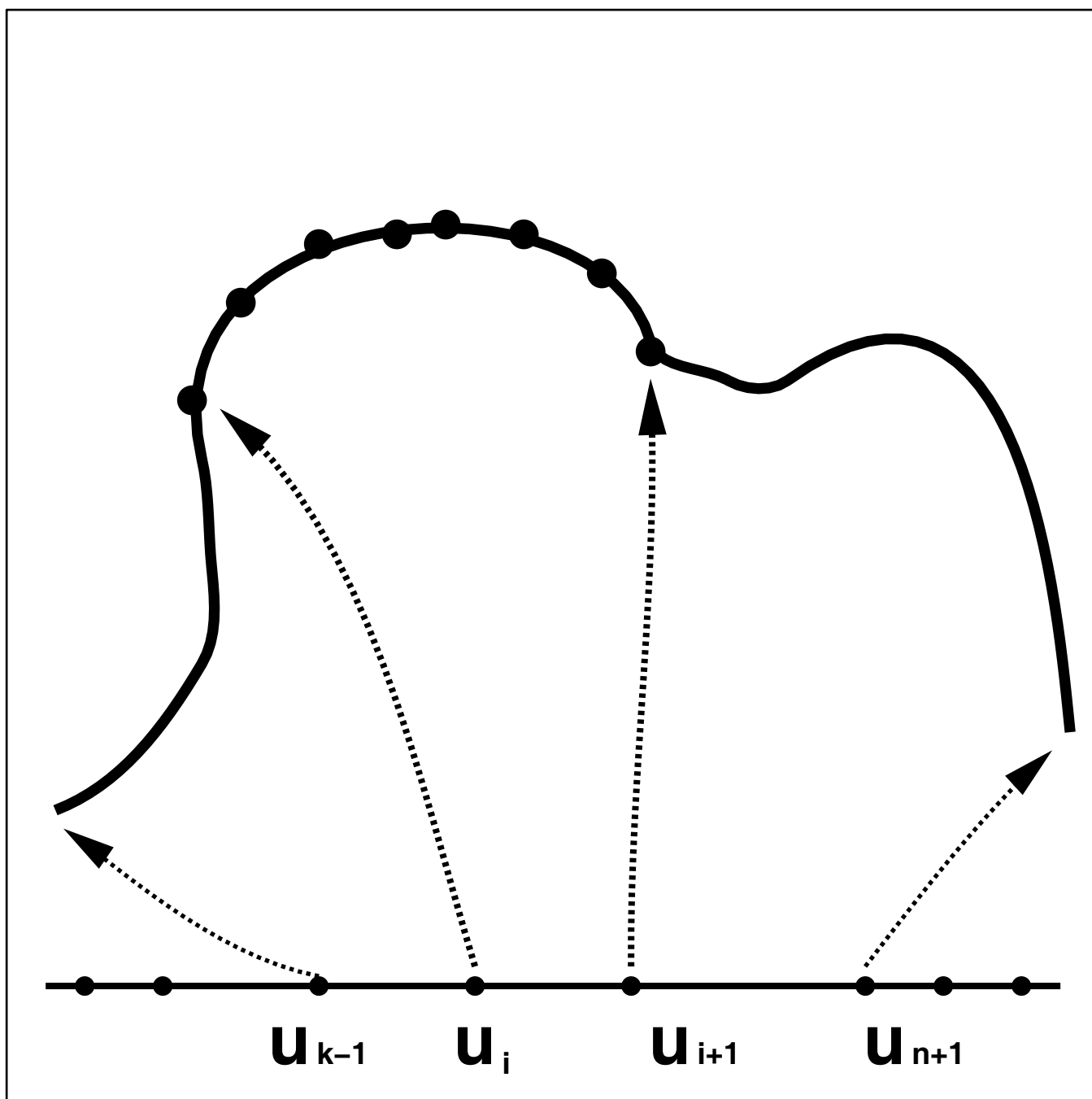
- Matrix equation

$$\begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_{l-1} \end{bmatrix} = \begin{bmatrix} B_{0,k}(v_0) & \cdots & B_{n,k}(v_0) \\ \vdots & \ddots & \vdots \\ B_{0,k}(v_{l-1}) & \cdots & B_{n,k}(v_{l-1}) \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

- $\mathbf{A}$ is $(l) \times (n + 1)$ matrix

- In general, l is much larger than $n + 1$, A is sparse
- The linear discretization for both modeling and rendering

B-Spline Discretization



Another Discretization

- From B-spline to Bezier spline
- B-spline control points: $p_0, \dots, p_n$
- Control points of piecewise Bezier curves

$$v_0, \dots, v_3, v_4, \dots, v_7, \dots, v_{4(n-3)}, \dots, v_{4(n-3)+3}$$

- Matrix expression

$$\begin{bmatrix} v_0 \\ \vdots \\ v_{4(n-3)+3} \end{bmatrix} = B \begin{bmatrix} p_0 \\ \vdots \\ p_n \end{bmatrix}$$

- The matrix structure and components of B ???

$$q = Av = ABp$$

- The matrix structure and components of A ???