Quantum Algorithms

(Introduction)

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What is inner product? What is Hilbert space?

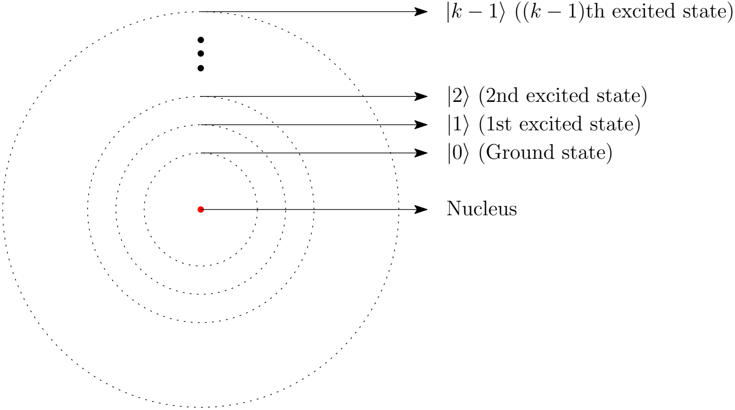
Definition

- The inner/dot/scalar product $\vec{v} \bullet \vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ is a mathematical operation between two vectors of the same dimension that returns a scalar number.

  Suppose $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Then,

  $\vec{v} \bullet \vec{w} = \overline{v_1} w_1 + \overline{v_2} w_2 + \cdots + \overline{v_n} w_n$.
- Hilbert space is a complex vector Euclidean space with well-defined inner product.

What is superposition principle?



Energy of an electron in an atom

- This is a k-level quantum mechanical system
- After measuring, electron is in exactly one of the states.
- Before measuring, electron is in all k quantum states.

What is superposition principle?

Energy of an electron in an atom

- After measuring, the electron can be in any one of $|0\rangle$, $|1\rangle$, . . ., $|k-1\rangle$ quantum energy states, where

$$|0\rangle = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, |k-1\rangle = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

  are the computational basis and they represent the orthonormal basis of a k-dimensional vector space

What is superposition principle?

Energy of an electron in an atom

- Before measuring, the electron is in a superposition of all k quantum energy states i.e.,

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + \cdots + c_{k-1}|k-1\rangle$$

$$\therefore |\psi\rangle = c_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_{k-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix},$$

where c_i's are complex numbers and $|\psi\rangle$ is a unit vector, i.e., $|c_0|^2 + |c_1|^2 + \cdots + |c_{k-1}|^2 = 1$.

What is a qubit?

Feature	Bit	Qubit
Implementation	Transistor	Quantum system
Exclusive states	0 and 1	$\|0\rangle$ and $\|1\rangle$
State after measuring	0 or 1	$\|0\rangle$ or $\|1\rangle$
State before measuring	0 or 1	Superposition of $\|0\rangle$ and $\|1\rangle$
Representation	bit $\in \{0, 1\}$	qubit $= a\|0\rangle + b\|1\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What is the Hadamard/sign basis?

Definition

- $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

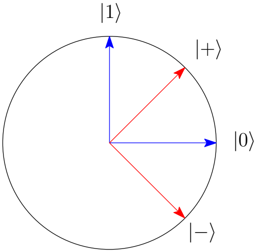
  $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

  $\implies \boxed{a|+\rangle + b|-\rangle = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle}$

- $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

  $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

  $\implies \boxed{a|0\rangle + b|1\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle}$



Quantum measurement

Probability of measurement

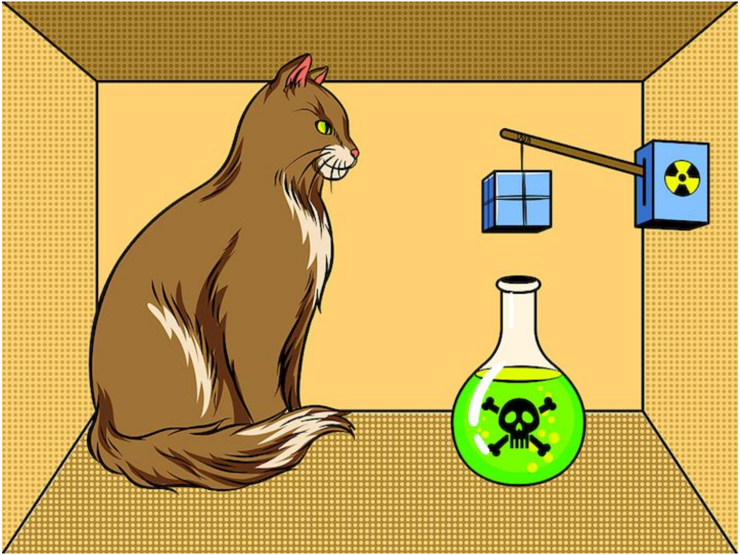
- Suppose $|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + \cdots + c_n|\phi_n\rangle$
- Probability of measuring $|\phi_i\rangle$ for the quantum system is

$$\boxed{P(\phi_i) = |c_i|^2}$$

	Collapses to state			
State	$\|0\rangle$	$\|1\rangle$	$\|+\rangle$	$\|-\rangle$
$\|0\rangle$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$\|1\rangle$	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\|+\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
$\|-\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	0	1
$a\|0\rangle + b\|1\rangle$	$\|a\|^2$	$\|b\|^2$	$\frac{\|a+b\|^2}{2}$	$\frac{\|a-b\|^2}{2}$
$a\|+\rangle + b\|-\rangle$	$\frac{\|a+b\|^2}{2}$	$\frac{\|a-b\|^2}{2}$	$\|a\|^2$	$\|b\|^2$

Probability of measurement

Schrödinger's cat



Source: `https://www.rms.com/blog/2018/09/04/schrodingers-cat-model`

Schrödinger's cat

$a\,\big|$  $\big\rangle + b\,\big|$  $\big\rangle$

Problem

- Suppose the probabilities of cat being alive and the cat being dead are the same. Then, what are the values of a and b?

What is conjugate transpose of a matrix?

Definition

- The conjugate transpose of a matrix M is a matrix $M^\dagger$ obtained by taking the complex conjugate of all elements of M and taking the transpose of the resulting matrix.

Examples

- Suppose $M = \begin{bmatrix} 1+i & 2+2i & 3+3i \\ 10-10i & 20-20i & 30-30i \end{bmatrix}$.

  Then, $M^\dagger = \begin{bmatrix} 1-i & 10+10i \\ 2-2i & 20+20i \\ 3-3i & 30+30i \end{bmatrix}$.

What is a unitary matrix?

Definition

- A unitary matrix is a matrix satisfying the property $\boxed{UU^\dagger = I}$.

Examples

- Pauli matrices
  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Hadamard matrix
  $$\boxed{H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

What is a quantum operation?

Definition

- A quantum operation transforms a quantum state to another quantum state.
- Any quantum operation can be represented by a unitary matrix. Similarly, any unitary matrix represents a possible quantum operation.

Examples

- All unitary matrices

  Pauli matrices

  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

  Hadamard matrix

  $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Quantum operation: I (Identity)

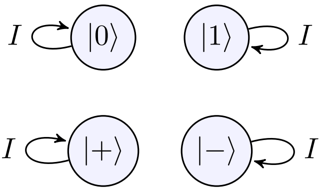
Definition

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $I(|0\rangle) = |0\rangle$ and $I(|1\rangle) = |1\rangle$

  $\implies \boxed{I(a|0\rangle + b|1\rangle) = a|0\rangle + b|1\rangle}$

  $I(|+\rangle) = |+\rangle$ and $I(|-\rangle) = |-\rangle$

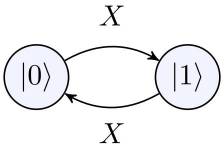
  $\implies \boxed{I(a|+\rangle + b|-\rangle) = a|+\rangle + b|-\rangle}$

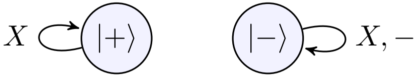


Quantum operation: X (Bit flip)

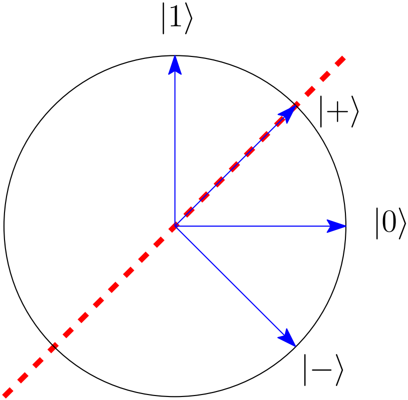
Definition

- $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $X(|0\rangle) = |1\rangle$ and $X(|1\rangle) = |0\rangle$
  $\implies \boxed{X(a|0\rangle + b|1\rangle) = b|0\rangle + a|1\rangle}$
  $X(|+\rangle) = |+\rangle$ and $X(|-\rangle) = -|-\rangle$
  $\implies \boxed{X(a|+\rangle + b|-\rangle) = a|+\rangle - b|-\rangle}$





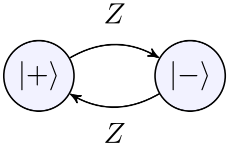
X = Rotation w.r.t $\pi/4$ axis



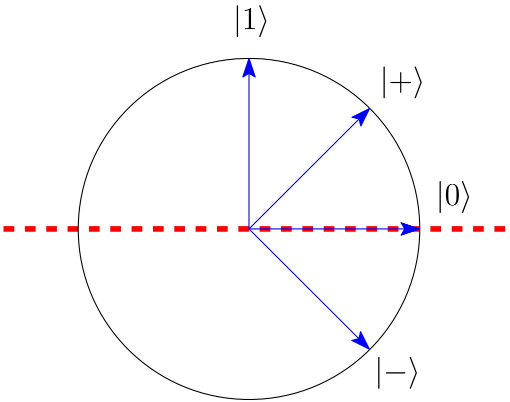
Quantum operation: Z (Phase flip)

Definition

- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $Z(|0\rangle) = |0\rangle$ and $Z(|1\rangle) = -|1\rangle$
  $\Longrightarrow$ $\boxed{Z(a|0\rangle + b|1\rangle) = a|0\rangle - b|1\rangle}$
  $Z(|+\rangle) = |-\rangle$ and $Z(|-\rangle) = |+\rangle$
  $\Longrightarrow$ $\boxed{Z(a|+\rangle + b|-\rangle) = b|+\rangle + a|-\rangle}$



Z = Rotation w.r.t 0 axis



Quantum operation: H

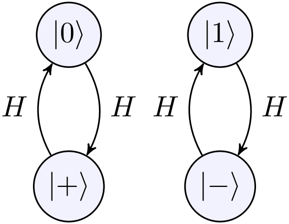
Definition

- $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $H(|0\rangle) = |+\rangle$ and $H(|1\rangle) = |-\rangle$

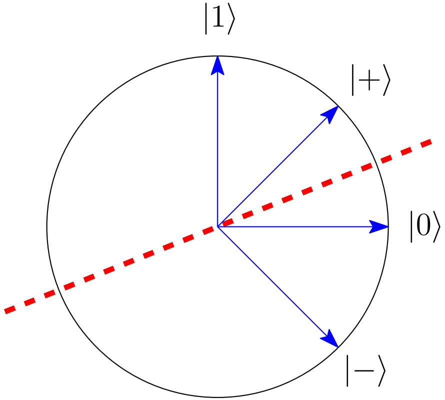
  $\Longrightarrow \boxed{H(a|0\rangle + b|1\rangle) = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle}$

  $H(|+\rangle) = |0\rangle$ and $H(|-\rangle) = |1\rangle$

  $\Longrightarrow \boxed{H(a|+\rangle + b|-\rangle) = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle}$



H = Rotation w.r.t $\pi/8$ axis



Quantum operation: Y

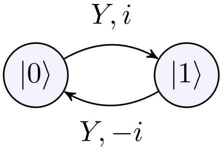
Definition

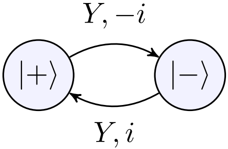
- $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- $Y(|0\rangle) = i|1\rangle$ and $Y(|1\rangle) = -i|0\rangle$

  $\implies \boxed{Y(a|0\rangle + b|1\rangle) = -ib|0\rangle + ia|1\rangle}$

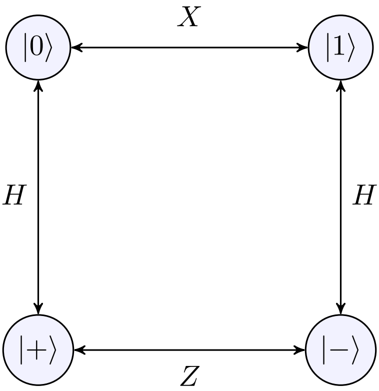
  $Y(|+\rangle) = -i|-\rangle$ and $Y(|-\rangle) = i|+\rangle$

  $\implies \boxed{Y(a|+\rangle + b|-\rangle) = ib|+\rangle - ia|-\rangle}$

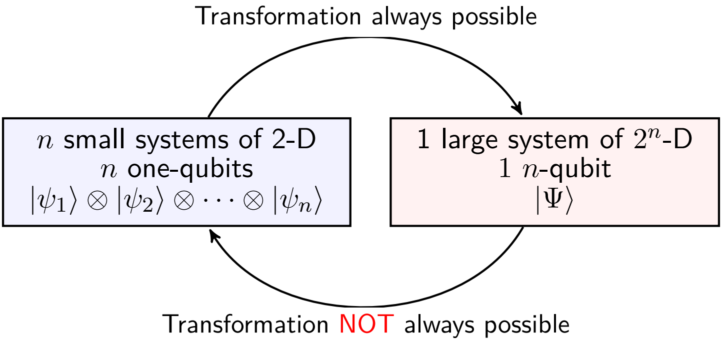




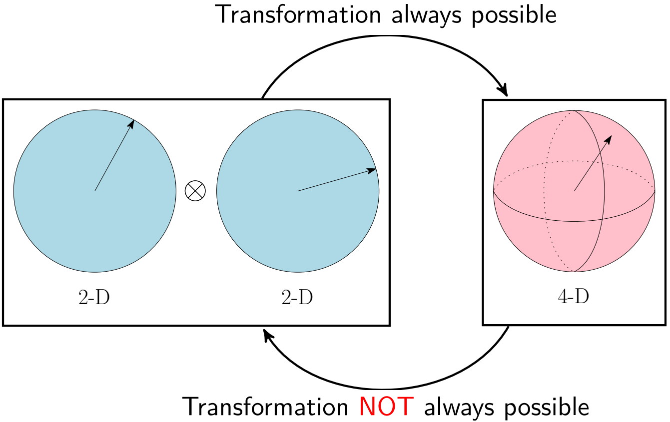
Quantum operations



Small vs. Large quantum systems



Small vs. Large quantum systems



Quantum system transformation: Small → Large

Definition

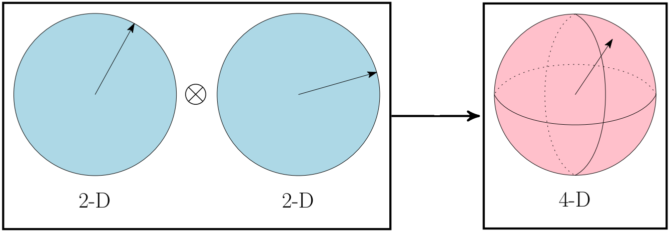
- A joint system of n small 2-D quantum systems, each having 2 quantum states can be thought as a large 2^n-D quantum mechanical system having 2^n quantum states.
- The tensor product $\otimes$ (or Kronecker product) of n one-qubits can be thought to denote a quantum mechanical system having 2^n quantum states.

Examples

- 3 qubits can be thought to denote an 8-D quantum system.

E.g.: $|1\rangle \otimes |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |101\rangle.$

Quantum system transformation: Small $\rightarrow$ Large



Examples

- Alice has quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$.
  Bob has quantum state $|\phi\rangle = c|0\rangle + d|1\rangle$.
  Then, their combined quantum state is
  $|\psi\phi\rangle = |\psi\rangle \otimes |\phi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$
  $$= \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

Quantum system transformation: Large $\rightarrow$ Small

Definition

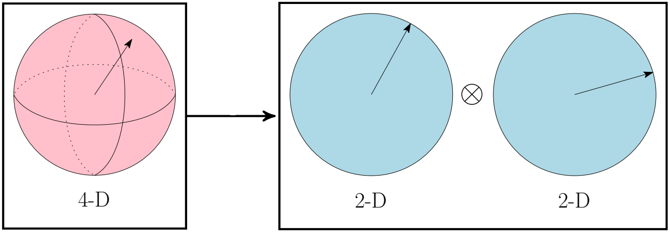
- A large 2^n-D quantum mechanical system having 2^n quantum states can be thought as a joint system of n small 2-D quantum systems, each having 2 quantum states.
- A quantum mechanical system having 2^n quantum states can be thought as a tensor product $\otimes$ (or Kronecker product) of n one-qubits.

Examples

- An 8-D quantum system can be represented using 3 qubits.

E.g.: $|101\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \otimes |0\rangle \otimes |1\rangle.$

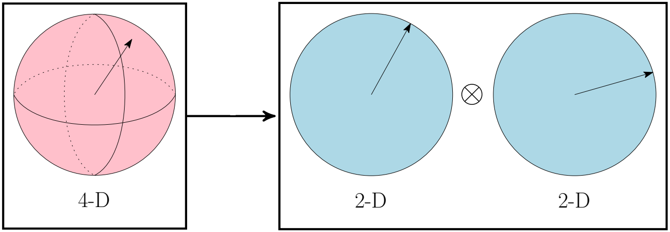
What is a separable state?



Definition

- The two-qubit state $|\Psi\rangle$ is separable if $\boxed{|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle}$ for some one-qubit states $|\psi\rangle$ and $|\phi\rangle$.

What is a separable state?



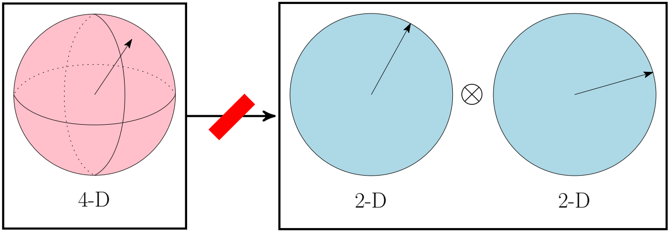
Examples

- The combined state of Alice and Bob is $|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$. Find the individual states of Alice and Bob.

  Let $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \otimes \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\ 1\\ -1\\ -1\end{bmatrix} = \begin{bmatrix}\psi_1\phi_1\\ \psi_1\phi_2\\ \psi_2\phi_1\\ \psi_2\phi_2\end{bmatrix}$.

  Solving the system of equations, we find that
  Alice's state $|\psi\rangle = |-\rangle$ and Bob's state $|\phi\rangle = |+\rangle$.

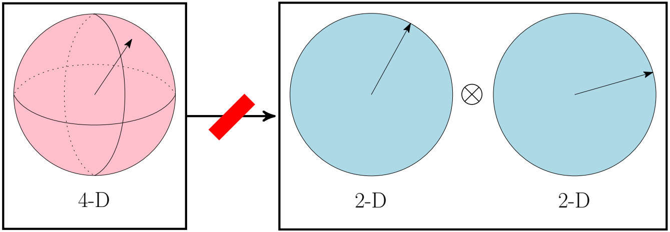
What is an entangled state?



Definition

- The two-qubit state $|\Psi\rangle$ is entangled if
  $|\Psi\rangle \neq |\psi\rangle \otimes |\phi\rangle$ for any one-qubit states $|\psi\rangle$ and $|\phi\rangle$.

What is an entangled state?



Examples

- The combined state of Alice and Bob is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Find the individual states of Alice and Bob.

  Let $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} \otimes \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1 \\ 0 \\ 0 \\ 1\end{bmatrix} = \begin{bmatrix}\psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2\end{bmatrix}$.

  The system of equations is not solvable.
  Hence, the state $|\Psi\rangle$ is entangled. This implies that it is impossible to obtain the individual states of Alice and Bob.

What are Bell states?

Definition

- The following two-qubit states are known as the Bell states. They represent an orthonormal, entangled basis for two qubits.

Bell states
$\vert\Phi^+\rangle = \frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$
$\vert\Phi^-\rangle = \frac{1}{\sqrt{2}}(\vert 00\rangle - \vert 11\rangle)$
$\vert\Psi^+\rangle = \frac{1}{\sqrt{2}}(\vert 01\rangle + \vert 10\rangle)$
$\vert\Psi^-\rangle = \frac{1}{\sqrt{2}}(\vert 01\rangle - \vert 10\rangle)$

Quantum Circuits

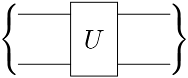
Quantum gates

Single qubit gate.

$a|0\rangle + b|1\rangle$  $a'|0\rangle + b'|1\rangle$

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = U \begin{bmatrix} a \\ b \end{bmatrix}$$

Two qubit gate.

$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$  $a'|00\rangle + b'|01\rangle + c'|10\rangle + d'|11\rangle$

$$\begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} = U \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Single qubit gates

$$a|0\rangle + b|1\rangle \quad \text{---}\boxed{I}\text{---} \quad a|0\rangle + b|1\rangle$$

$$a|0\rangle + b|1\rangle \quad \text{---}\boxed{X}\text{---} \quad b|0\rangle + a|1\rangle$$

$$a|0\rangle + b|1\rangle \quad \text{---}\boxed{Y}\text{---} \quad -ib|0\rangle + ia|1\rangle$$

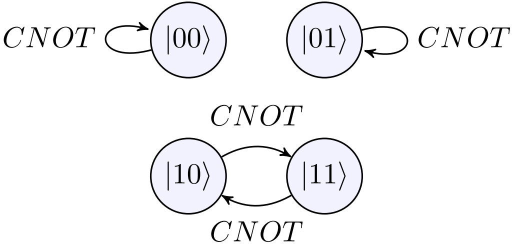
$$a|0\rangle + b|1\rangle \quad \text{---}\boxed{Z}\text{---} \quad a|0\rangle - b|1\rangle$$

$$a|0\rangle + b|1\rangle \quad \text{---}\boxed{H}\text{---} \quad a|+\rangle - b|-\rangle$$

Two qubit gate: CNOT (Controlled NOT)

Definition

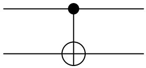
- $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- $CNOT(|00\rangle) = |00\rangle$, $CNOT(|01\rangle) = |01\rangle$
  $CNOT(|10\rangle) = |11\rangle$, $CNOT(|11\rangle) = |10\rangle$
  $\boxed{CNOT(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)}$
  $\boxed{= a|00\rangle + b|01\rangle + d|10\rangle + c|11\rangle}$



Two qubit gate: CNOT (Controlled NOT)

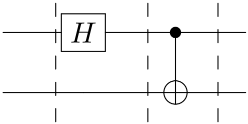
Definition

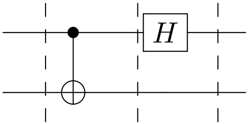
- $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- $CNOT(|00\rangle) = |00\rangle$, $CNOT(|01\rangle) = |01\rangle$
  $CNOT(|10\rangle) = |11\rangle$, $CNOT(|11\rangle) = |10\rangle$
  $\boxed{CNOT(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)}$
  $\boxed{= a|00\rangle + b|01\rangle + d|10\rangle + c|11\rangle}$



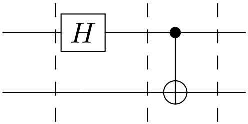
$$\begin{bmatrix} a \\ b \\ d \\ c \end{bmatrix} = CNOT \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

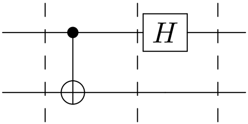
Two qubit gates to measure Bell states





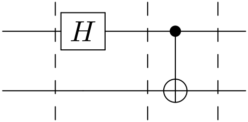
Two qubit gates to measure Bell states

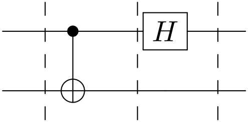




Input (Stage 0)	$H \otimes I$ (Stage 1)	$CNOT$ (Stage 2)
$\|00\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$	$\|\Phi^+\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
$\|01\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$	$\|\Phi^-\rangle = \frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
$\|10\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$	$\|\Psi^+\rangle = \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$
$\|11\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle)$	$\|\Psi^-\rangle = \frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$

Two qubit gates to measure Bell states

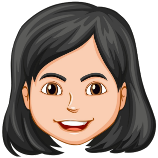




Input (Stage 0)	$H \otimes I$ (Stage 1)	$CNOT$ (Stage 2)
$\|00\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$	$\|\Phi^+\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
$\|01\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$	$\|\Phi^-\rangle = \frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
$\|10\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$	$\|\Psi^+\rangle = \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$
$\|11\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle)$	$\|\Psi^-\rangle = \frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$

Input (Stage 0)	$CNOT$ (Stage 1)	$H \otimes I$ (Stage 2)
$\|\Phi^+\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$	$\|00\rangle$
$\|\Phi^-\rangle = \frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$	$\|01\rangle$
$\|\Psi^+\rangle = \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$	$\|10\rangle$
$\|\Psi^-\rangle = \frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle)$	$\|11\rangle$

Quantum teleportation



Alice

$$|\psi\rangle = a|0\rangle + b|1\rangle \longrightarrow$$

Bob

Problem

- Alice and Bob are working on quantum experiments in two different cities. Alice wants to send her unknown quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob for an experiment.
  How can Alice achieve this?

Quantum teleportation

Solution (Attempt 1)

- Idea: Alice can figure out a and b.

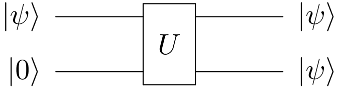
$\|\psi\rangle$ collapses to	With probability
$\|0\rangle$	$P(\|0\rangle) = \|a\|^2$
$\|1\rangle$	$P(\|1\rangle) = \|b\|^2$

- It is impossible to learn a and b given $P(|0\rangle)$ and $P(|1\rangle)$
  Hence, Alice cannot figure out a and b

Quantum teleportation

Solution (Attempt 2)

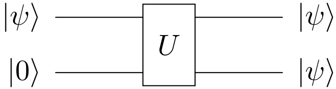
- Idea: Bob can clone Alice's qubit.



Quantum teleportation

Solution (Attempt 2)

- Idea: Bob can clone Alice's qubit.

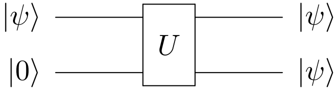


Method	$\|\psi\rangle$	Output	Reason
1	$\|0\rangle$	$\|00\rangle$	Product
	$\|1\rangle$	$\|11\rangle$	Product
	$a\|0\rangle + b\|1\rangle$	$a\|00\rangle + b\|11\rangle$	Linearity
2	$a\|0\rangle + b\|1\rangle$	$a^2\|00\rangle + ab(\|01\rangle + \|10\rangle) + b^2\|11\rangle$	Product

Quantum teleportation

Solution (Attempt 2)

- Idea: Bob can clone Alice's qubit.



Method	$\lvert\psi\rangle$	Output	Reason
1	$\lvert 0\rangle$	$\lvert 00\rangle$	Product
	$\lvert 1\rangle$	$\lvert 11\rangle$	Product
	$a\lvert 0\rangle + b\lvert 1\rangle$	$a\lvert 00\rangle + b\lvert 11\rangle$	Linearity
2	$a\lvert 0\rangle + b\lvert 1\rangle$	$a^2\lvert 00\rangle + ab(\lvert 01\rangle + \lvert 10\rangle) + b^2\lvert 11\rangle$	Product

- The output from methods 1 and 2 must be the same. So,
  $ab = 0 \implies (a = 0 \text{ or } b = 0) \implies (|\psi\rangle = |0\rangle \text{ or } |\psi\rangle = |1\rangle)$
  It is impossible to clone an unknown quantum state
  Hence, Bob cannot clone Alice's qubit

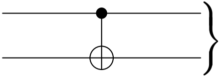
Quantum teleportation

Solution (Attempt 3)

- Step 1. Create a Bell pair.

$a|0\rangle + b|1\rangle$

$|0\rangle$

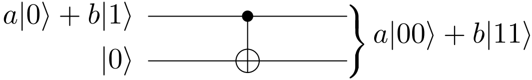


$a|00\rangle + b|11\rangle$

Quantum teleportation

Solution (Attempt 3)

- Step 1. Create a Bell pair.

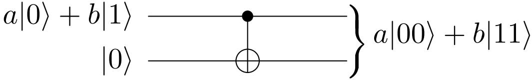


- Step 2. Alice measures.
  Idea 1: Alice measures her qubit.
  Alice's state is $|0\rangle$ or $|1\rangle$. Bob will also have the same state.

Quantum teleportation

Solution (Attempt 3)

- Step 1. Create a Bell pair.

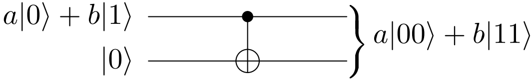


- Step 2. Alice measures.
  Idea 1: Alice measures her qubit.
  Alice's state is $|0\rangle$ or $|1\rangle$. Bob will also have the same state.
  Idea 2: Alice measures her qubit in the sign basis.
  $a|0\rangle|0\rangle + b|1\rangle|1\rangle = \frac{1}{\sqrt{2}}\left[a(|+\rangle + |-\rangle)|0\rangle + b(|+\rangle - |-\rangle)|1\rangle\right]$
  $= \frac{1}{\sqrt{2}}\left(a|0\rangle + b|1\rangle\right)|+\rangle + \frac{1}{\sqrt{2}}\left(a|0\rangle - b|1\rangle\right)|-\rangle$

Quantum teleportation

Solution (Attempt 3)

- Step 1. Create a Bell pair.



- Step 2. Alice measures.

  Idea 1: Alice measures her qubit.

  Alice's state is $|0\rangle$ or $|1\rangle$. Bob will also have the same state.

  Idea 2: Alice measures her qubit in the sign basis.

  $a|0\rangle|0\rangle + b|1\rangle|1\rangle = \frac{1}{\sqrt{2}}\left[a(|+\rangle + |-\rangle)|0\rangle + b(|+\rangle - |-\rangle)|1\rangle\right]$

  $= \frac{1}{\sqrt{2}}\left(a|0\rangle + b|1\rangle\right)|+\rangle + \frac{1}{\sqrt{2}}\left(a|0\rangle - b|1\rangle\right)|-\rangle$

- Step 3. Bob performs quantum operations.

  if (Alice's qubit collapses to $|+\rangle$) then

    Bob's qubit collapses to ψ.

  elseif (Alice's qubit collapses to $|-\rangle$) then

    Alice calls Bob and asks him to phase flip his state.

    Bob performs Z (phase flip) on his qubit to obtain ψ.

Quantum teleportation

Solution (Attempt 3)

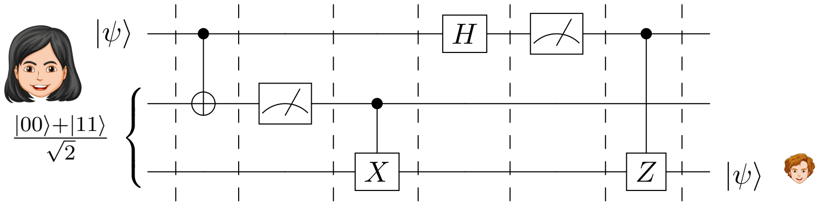
- Idea: Alice can teleport the state to Bob.
- Step 1. Create a Bell pair.
  Alice and Bob have a Bell pair setup using a CNOT gate.
  Step 2. Alice measures.
  Alice measures her qubit of the entangled Bell pair
  in the sign basis.
  Step 3. Bob performs quantum operations.
  If Alice's qubit is $|+\rangle$, Bob gets $|\psi\rangle$.
  If Alice's qubit is $|-\rangle$, Bob performs phase flip to get $|\psi\rangle$.

Quantum teleportation

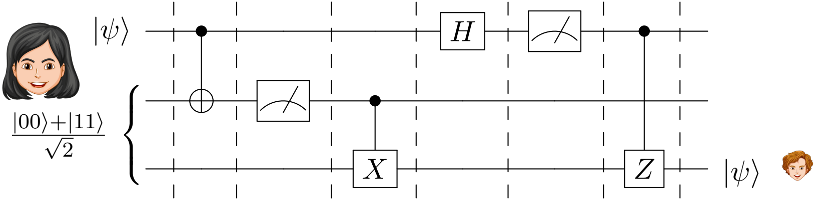
Solution (Attempt 4)

- Idea: Alice can teleport the state to Bob.
- Step 1. Create a Bell pair.
  Alice and Bob have a Bell pair setup.
  Step 2. Alice measures.
  Alice measures $|\psi\rangle$ and her qubit of the entangled Bell pair. Let's call it $b_1 b_2$. Alice's state $|\psi\rangle$ will be destroyed.
  Step 3. Bob performs quantum operations.
  Bob performs unitary rotations based on $b_1 b_2$. Bob gets $|\psi\rangle$.

Quantum teleportation

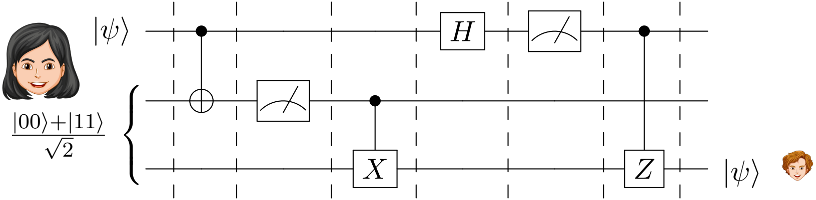


Quantum teleportation



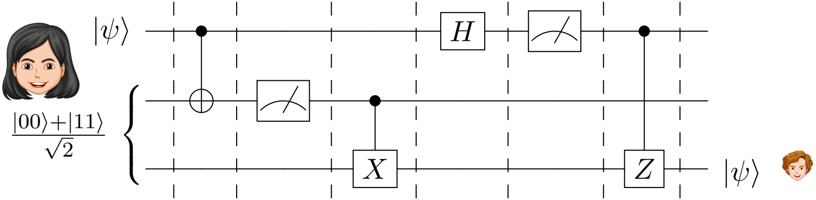
#	Operation	System state (3 qubits or 8 dimensions)
0	–	$|\psi\rangle\left(\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\right)$ $=\frac{1}{\sqrt{2}}(a|000\rangle+a|011\rangle+b|100\rangle+b|111\rangle)$

Quantum teleportation



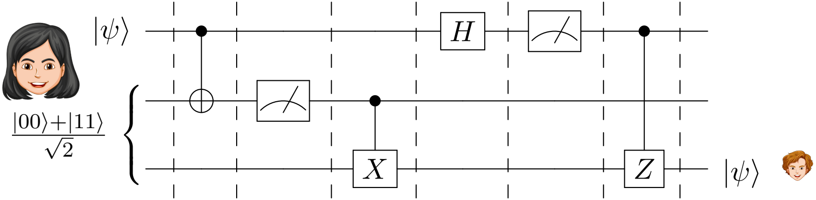
#	Operation	System state (3 qubits or 8 dimensions)
0	–	$\|\psi\rangle \left(\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)\right)$ $= \frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|100\rangle + b\|111\rangle)$
1	$CNOT(q_1, q_2)$	$\frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|110\rangle + b\|101\rangle)$

Quantum teleportation



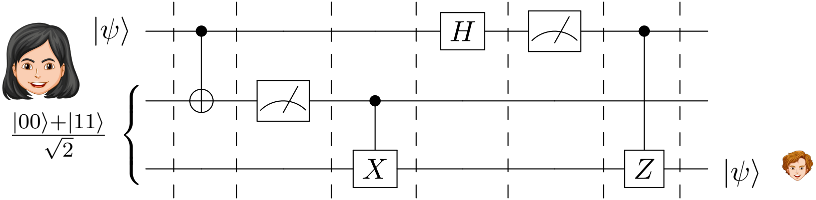
#	Operation	System state (3 qubits or 8 dimensions)
0	$-$	$\|\psi\rangle\left(\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)\right)$ $= \frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|100\rangle + b\|111\rangle)$
1	$CNOT(q_1, q_2)$	$\frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|110\rangle + b\|101\rangle)$
2	Alice measures q_2	Case $q_2 = \|0\rangle$: $a\|00\rangle + b\|11\rangle$ Case $q_2 = \|1\rangle$: $a\|01\rangle + b\|10\rangle$

Quantum teleportation



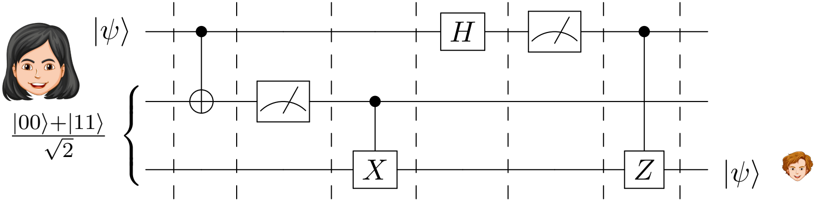
#	Operation	System state (3 qubits or 8 dimensions)
0	–	$\|\psi\rangle\left(\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)\right)$ $=\frac{1}{\sqrt{2}}(a\|000\rangle+a\|011\rangle+b\|100\rangle+b\|111\rangle)$
1	$CNOT(q_1,q_2)$	$\frac{1}{\sqrt{2}}(a\|000\rangle+a\|011\rangle+b\|110\rangle+b\|101\rangle)$
2	Alice measures q_2	Case $q_2=\|0\rangle$: $a\|00\rangle+b\|11\rangle$ Case $q_2=\|1\rangle$: $a\|01\rangle+b\|10\rangle$
3	Alice calls Bob Bob uses X if $q_2=\|1\rangle$	$a\|00\rangle+b\|11\rangle$

Quantum teleportation



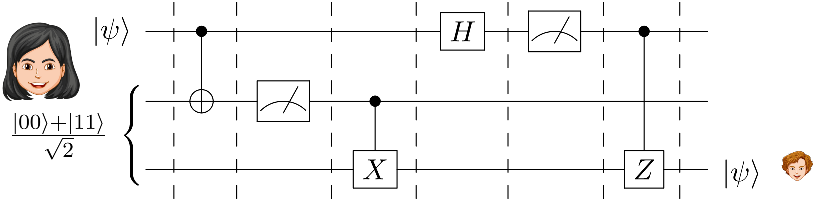
#	Operation	System state (3 qubits or 8 dimensions)
0	–	$\|\psi\rangle \left(\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)\right)$ $= \frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|100\rangle + b\|111\rangle)$
1	$CNOT(q_1, q_2)$	$\frac{1}{\sqrt{2}}(a\|000\rangle + a\|011\rangle + b\|110\rangle + b\|101\rangle)$
2	Alice measures q_2	Case $q_2 = \|0\rangle$: $a\|00\rangle + b\|11\rangle$ Case $q_2 = \|1\rangle$: $a\|01\rangle + b\|10\rangle$
3	Alice calls Bob Bob uses X if $q_2 = \|1\rangle$	$a\|00\rangle + b\|11\rangle$
4	Alice applies H	$\frac{1}{\sqrt{2}}(a\|0\rangle + b\|1\rangle)\|+\rangle + \frac{1}{\sqrt{2}}(a\|0\rangle - b\|1\rangle)\|-\rangle$

Quantum teleportation



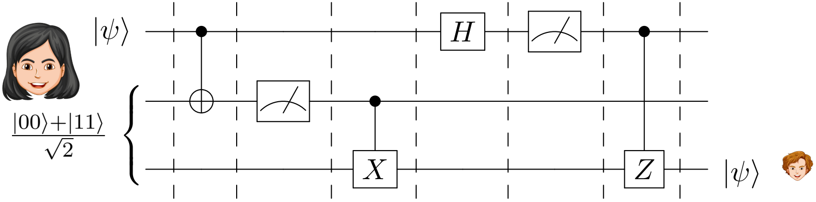
#	Operation	System state (3 qubits or 8 dimensions)
4	Alice applies H	$\frac{1}{\sqrt{2}}\left(a\|0\rangle + b\|1\rangle\right)\|+\rangle + \frac{1}{\sqrt{2}}\left(a\|0\rangle - b\|1\rangle\right)\|-\rangle$

Quantum teleportation



#	Operation	System state (3 qubits or 8 dimensions)
4	Alice applies H	$\frac{1}{\sqrt{2}}\left(a\|0\rangle + b\|1\rangle\right)\|+\rangle + \frac{1}{\sqrt{2}}\left(a\|0\rangle - b\|1\rangle\right)\|-\rangle$
5	Alice measures q_1	Case $q_1 = \|+\rangle$: $\|\psi\rangle$ Case $q_1 = \|-\rangle$: $a\|0\rangle - b\|1\rangle$

Quantum teleportation



#	Operation	System state (3 qubits or 8 dimensions)
4	Alice applies H	$\frac{1}{\sqrt{2}}(a\|0\rangle + b\|1\rangle)\|+\rangle + \frac{1}{\sqrt{2}}(a\|0\rangle - b\|1\rangle)\|-\rangle$
5	Alice measures q_1	Case $q_1 = \|+\rangle$: $\|\psi\rangle$ Case $q_1 = \|-\rangle$: $a\|0\rangle - b\|1\rangle$
6	Alice calls Bob Bob uses Z if $q_1 = \|-\rangle$	$\|\psi\rangle$

Quantum teleportation

Solution (Attempt 4)

- Idea: Alice can teleport the state to Bob.
- Step 1. Create a Bell pair.
  Alice and Bob have a Bell pair setup.
  Step 2. Alice measures.
  Alice measures $|\psi\rangle$ and her qubit of the entangled Bell pair. Let's call it $b_1 b_2$. Alice's state $|\psi\rangle$ will be destroyed.
  Step 3. Bob performs quantum operations.
  Bob performs unitary rotations based on $b_1 b_2$. Bob gets $|\psi\rangle$.