Problem 1. [20 points]
Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=0}^{n-1} (a + ib) = n(2a + (n - 1)b)/2$$

(b) [5 points] For all integers $n \geq 1$,

$$\sum_{i=0}^{n-1} ar^i = \frac{ar^n - a}{r - 1}$$

(c) [5 points] For natural numbers $n \geq 1$,

$$8^n - 3^n$$ is a multiple of 5

(d) [5 points] Show that for natural numbers $n \geq 1$, the number of strings using the digits 0, 1, 2 with no consecutive places holding the same digit is $3 \times 2^{n-1}$. For example, there are 12 such strings for length three: 010, 012, 020, 021, 101, 102, 120, 121, 201, 202, 210, 212.

Problem 2. [5 points]
Prove that if $E$ is a set with no elements and $A$ is any set, then $E \subseteq A$.

Problem 3. [5 points]
Let $X = \{1,2,3\}$, $Y = \{1,2,3,4\}$ and $Z = \{1,2\}$. Use arrow diagrams to define functions.

1. Define a function $f : X \rightarrow Y$ that is one-to-one but not onto.
2. Define a function $g : X \rightarrow Z$ that is onto but not one-to-one.
3. Define a function $h : X \rightarrow X$ that is neither one-to-one nor onto.
4. Define a function $k : X \rightarrow X$ that is one-to-one and onto but is not the identity function on $X$.

Problem 4. [5 points]
Let $A$ and $B$ be finite sets where $|A| = |B|$. Is it possible to define a function $f : A \rightarrow B$ that is one-to-one but not onto? Is it possible to define a function $g : A \rightarrow B$ that is onto but not one-to-one?
Problem 5. [5 points]
We know that $\mathcal{P}(X)$ is a power set of set $X$. Let $A = \{1, 2, 3, \ldots, 10\}$. Consider the function $f : \mathcal{P}(A) \to \mathbb{W}$ given by $f(B) = |B|$, where $\mathbb{W}$ is a set of whole numbers (which includes 0). That is, $f$ takes a subset of $A$ as an input and it outputs the cardinality of that set.

Is $f$ one-to-one? Prove your answer.
Is $f$ onto? Prove your answer.

Problem 6. [5 points]
Prove that $\mathbb{Z}$ is countable. Come up with a function $f : \mathbb{N} \to \mathbb{Z}$ that can map a unique number of $\mathbb{N}$ to a unique number $f(n)$ of $\mathbb{Z}$. 