Problem 1. [20 points]
Use mathematical induction to prove the following identities.
(a) [5 points] For all integers \( n \geq 1 \),
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}
\]
(b) [5 points] For all integers \( n \geq 1 \), \( n(n^2 + 5) \) is a multiple of 6.
(c) [5 points] For all integers \( n \geq 0 \),
\[
1 +\frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = 3 \left[1 - \left(\frac{2}{3}\right)^{n+1}\right]
\]
(d) [5 points] Suppose that \( c_1, c_2, c_3, \ldots \) is a sequence defined as follows:
\[
c_1 = 3, \quad c_2 = -9
\]
\[
c_k = 7c_{k-1} - 10c_{k-2} \quad \text{for all integers } k \geq 3
\]
Prove that \( c_n = 4 \cdot 2^n - 5^n \) for all integers \( n \geq 1 \).

Problem 2. [5 points]
Mention whether the following statements are true or false without giving any reasons. Assume all sets are subsets of a universal set \( U \).
(a) [1 point] \( (A \cap B) \cap (A \cap C) = A \cap (B \cup C) \)
(b) [1 point] \( A = A \cup (A \cap B) \)
(c) [1 point] \( A \subseteq A \cup B \)
(d) [1 point] \( A \cap (A \cup B) = A \cap B \)
(e) [1 point] \( A \subseteq B \) if and only if \( A \cup B = B \)

Problem 3. [5 points]
Suppose the grading function \( f : [0, 100] \rightarrow \{A, A-, B+, B, B-, C+, C, C-, D+, D, F\} \) is defined as follows:
<table>
<thead>
<tr>
<th>Percentage</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>[93, 100]</td>
<td>A</td>
</tr>
<tr>
<td>[90, 93)</td>
<td>A−</td>
</tr>
<tr>
<td>[87, 90)</td>
<td>B+</td>
</tr>
<tr>
<td>[83, 87)</td>
<td>B</td>
</tr>
<tr>
<td>[80, 83)</td>
<td>B−</td>
</tr>
<tr>
<td>[77, 80)</td>
<td>C+</td>
</tr>
<tr>
<td>[73, 77)</td>
<td>C</td>
</tr>
<tr>
<td>[70, 73)</td>
<td>C−</td>
</tr>
<tr>
<td>[67, 70)</td>
<td>D+</td>
</tr>
<tr>
<td>[63, 67)</td>
<td>D</td>
</tr>
<tr>
<td>[0, 63)</td>
<td>F</td>
</tr>
</tbody>
</table>

Is this grading function a one-to-one correspondence? Prove or disprove.

**Problem 4. [5 points]**
The rotate-by-90-degree function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ is defined as follows.

$$f(x, y) = (y, -x)$$

(This function is used in image processing for rotating a face or an image by 90 degree in clockwise direction.) Is this function a one-to-one correspondence? Prove or disprove.

**Problem 5. [10 points]**
Mention whether the following statements are true or false without giving any reasons. Assume that the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are arbitrary functions.

(a) [1 point] $f \circ f = f$.

(b) [1 point] $f \circ g = g \circ f$.

(c) [1 point] $f$ and $g$ are both one-to-one correspondences implies that $f \circ g$ and $g \circ f$ are both one-to-one correspondences.

(d) [1 point] $f$ and $g$ are both onto does not imply that $f \circ g$ and $g \circ f$ are both onto.

(e) [1 point] $f$ and $g$ are both one-to-one implies that $f \circ g$ and $g \circ f$ are both one-to-one.

(f) [1 point] If $f \circ g$ is the identity function, then $f$ and $g$ are one-to-one correspondences.

(g) [1 point] Suppose $f^{-1}$ exists. Then $f^{-1}$ need not be an onto function.

(h) [1 point] The size of the set of all multiples of 6 is less than the size of the set of all multiples of 3.

(i) [1 point] The size of the set of rational numbers is the same as the size of the set of real numbers in the range $[0, 0.0000001]$.

(j) [1 point] The size of the set of real numbers in the range $[1, 2]$ is the same or larger than the size of the set of real numbers in the range $[1, 4]$. 