Discrete Mathematics

(Sets)

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What is a set?

Definition

- A set is a collection of related items

Examples

- $A = \{1, 2, 3, 4, 5\}$
- $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
- S is a set of square numbers
- $B = \{m \in \mathbb{Z} \mid m = 7n + 100 \text{ for some } n \in \mathbb{Z}\}$

Subsets

Definitions

- Subset. $A \subseteq B \Leftrightarrow \forall x,$ if $x \in A$ then $x \in B$
- Not subset. $A \nsubseteq B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$
- Proper subset. $A \subset B \Leftrightarrow$
  1. $A \subseteq B$
  2. $\exists x$ such that $x \in B$ and $x \notin A$

Problems

- Let $A = \{1\}$ and $B = \{1, \{1\}\}$.
  (a) Is $A \subseteq B$?
  (b) Is $A \subset B$?
- Let $A = \{m \in \mathbb{Z} \mid m = 6r + 12$ for some $r \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid n = 3s$ for some $s \in \mathbb{Z}\}$.
  (a) Prove that $A \subseteq B$.
  (b) Disprove that $B \subseteq A$.

Set equality

Definition

- Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Problems

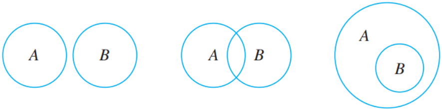
- $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a\}$
  $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some integer } b\}$
  Is $A = B$?

Venn diagrams

Definition

- Relationship between a small number of sets can be represented by pictures called Venn diagrams

(a) (b)



(a) (b) (c)

Problems

- Write a Venn diagram representing sets of numbers: $\mathbb{N}, \mathbb{W}, \mathbb{Q}, \mathbb{R}$.

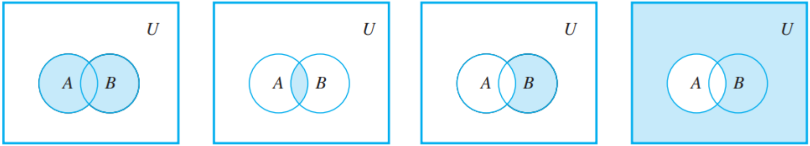
Operations on sets

Definition

Let A and B be subsets of a universal set U.

1. The union of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.
   $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. The intersection of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.
   $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
3. The difference of B minus A (or relative complement of A in B), denoted $B - A$, is the set of all elements that are in B and not A.
   $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. The complement of A, denoted A', is the set of all elements in U that are not in A.
   $A' = \{x \in U \mid x \notin A\}$

Operations on sets



Problems

- Let the universal set $U = \{a, b, c, d, e, f, g\}$.
  Let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$.
  Find $A \cup B$, $A \cap B$, $B - A$, and A'.

Operations on sets

Notations

Given real numbers a and b with $a \leq b$:

- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
  $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
  $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
  $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:
  $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
  $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$
  $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
  $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

Problems

- $A = (-1, 0] = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$
  $B = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$.
  Find $A \cup B$, $A \cap B$, $B - A$, and A'.

Operations on sets

Notations

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

- $\cup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i$ for at least one $i = 0, 1, 2, \ldots, n\}$
- $\cup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i$ for at least one whole number $i\}$
- $\cap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i$ for all $i = 0, 1, 2, \ldots, n\}$
- $\cap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i$ for all whole numbers $i\}$

Problems

- For each positive integer i, let
  $A_i = \{x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i}\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$
  Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$
  Find $\cup_{i=0}^{\infty} A_i$ and $\cap_{i=0}^{\infty} A_i$

Empty set

Definition

Empty set, denoted by ϕ, is a set with no elements.

Examples

- $\{1,3\} \cap \{2,4\} = \phi$
- $\{x \in \mathbb{R} \mid x^2 = -1\} = \phi$
- $\{x \in \mathbb{R} \mid 3 < x < 2\} = \phi$

Disjoint sets

Definition

- Two sets are called disjoint if, and only if, they have no elements in common.
- A and B are disjoint $\Leftrightarrow A \cap B = \phi$

Problems

- Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Mutually disjoint sets

Definition

- Sets $A_1, A_2, A_3, \ldots$ are mutually disjoint (or pairwise disjoint or nonoverlapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common.
- For all $i, j = 1, 2, 3, \ldots$
  $A_i \cap A_j = \phi$ whenever $i \neq j$.

Problems

Are the following sets mutually disjoint?

- $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$.
- $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$.

Partition of a set

Definition

- A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a partition of a set A if, and only if,
  1. A is the union of all the A_i
  2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Problems

- Let $A = \{1, 2, 3, 4, 5, 6\}$ $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?
- Let $\mathbb{Z}$ be the set of all integers and let
  $T_0 = \{n \in \mathbb{Z} \mid n = 3k, \text{for some integer } k\}$,
  $T_1 = \{n \in \mathbb{Z} \mid n = 3k + 1, \text{for some integer } k\}$,
  $T_2 = \{n \in \mathbb{Z} \mid n = 3k + 2, \text{for some integer } k\}$,
  Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Power set

Definition

- Given a set A, the power set of A, denoted $P(A)$, is the set of all subsets of A.

Problems

- Find the power set of the set $\{x, y\}$. That is, find $P(\{x, y\})$.

Ordered n-tuple

Definition

- Ordered n-tuple, $(x_1, x_2, \ldots, x_n)$, consists of $x_1, x_2, \ldots, x_n$ together with the ordering.
- Ordered pair $=$ ordered 2-tuple
  Ordered triple $=$ ordered 3-tuple
- Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.
  $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Leftrightarrow x_1 = y_1, \ldots, x_n = y_n$.

Problems

- Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?
- Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?
- Is $((1, 2), 3) = (1, 2, 3)$?

Cartesian product

Definition

- Given sets $A_1, A_2, \ldots, A_n$, the Cartesian product of $A_1, A_2, \ldots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.
- $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$.

Problems

- Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$. Find:
  (a) $A_1 \times A_2$,
  (b) $(A_1 \times A_2) \times A_3$, and
  (c) $A_1 \times A_2 \times A_3$.

Properties of sets

Definition

- Inclusion of intersection: For all sets A and B,
  $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
- Inclusion in union: For all sets A and B,
  $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
- Transitive property of subsets: For all sets A, B, and C,
  if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Procedural versions of set definitions

Definition

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

- $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
- $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
- $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
- $x \in X' \Leftrightarrow x \notin X$
- $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

Set identities

Laws	Formula	Formula
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \phi = A$	$A \cap U = A$
Complement laws	$A \cup A' = U$	$A \cap A' = \phi$
Double comp. law	$(A')' = A$	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Uni. bound laws	$A \cup U = U$	$A \cap \phi = \phi$
De Morgan's laws	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complements	$U' = \phi$	$\phi' = U$
Set diff. laws	$A - B = A \cap B'$	

Element argument

Basic method for proving that a set is a subset of another

- Let sets X and Y be given. To prove that $X \subseteq Y$,
  1. suppose that x is a particular but arbitrarily chosen element of X.
  2. show that x is an element of Y.

Element argument

Basic method for proving that two sets are equal

- Let sets X and Y be given. To prove that $X = Y$,
  1. Prove that $X \subseteq Y$.
  2. Prove that $Y \subseteq X$.

Element argument

Basic method for proving a set equals the empty set

- To prove that a set X is equal to the empty set ϕ, prove that X has no elements.
- To do this, suppose X has an element and derive a contradiction.

Proof by element argument: Example 1

Proposition

- Prove that for all sets A, B, and C
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof by element argument: Example 1

Proposition

- Prove that for all sets A, B, and C
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

We need to prove:

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Proof by element argument: Example 1

Proof (continued)

Proof that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Suppose $x \in A \cup (B \cap C)$.

$x \in A$ or $x \in B \cap C$ $\quad$ ($\because$ defn. of union)

- Case 1. $[x \in A.]$

  $x \in A \cup B$ $\quad$ ($\because$ defn. of union)

  $x \in A \cup C$ $\quad$ ($\because$ defn. of union)

  $x \in (A \cup B) \cap (A \cup C)$ $\quad$ ($\because$ defn. of intersection)

- Case 2. $[x \in B \cap C.]$

  $x \in B$ and $x \in C$ $\quad$ ($\because$ defn. of intersection)

  $x \in A \cup B$ $\quad$ ($\because$ defn. of union)

  $x \in A \cup C$ $\quad$ ($\because$ defn. of union)

  $x \in (A \cup B) \cap (A \cup C)$ $\quad$ ($\because$ defn. of intersection)

Proof by element argument: Example 1

Proof (continued)

Proof that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$.

$x \in A \cup B$ and $x \in A \cup C$ $\quad$ ($\because$ defn. of intersection)

- Case 1. $[x \in A.]$

  $x \in A \cup (B \cap C)$ $\quad$ ($\because$ defn. of union)

- Case 2. $[x \notin A.]$

  $x \in A$ or $x \in B$ $\quad$ ($\because$ defn. of union)

  $x \in B$ $\quad$ ($\because x \notin A$)

  $x \in A$ or $x \in C$ $\quad$ ($\because$ defn. of union)

  $x \in C$ $\quad$ ($\because x \notin A$)

  $x \in B \cap C$ $\quad$ ($\because$ defn. of intersection)

  $x \in A \cup (B \cap C)$ $\quad$ ($\because$ defn. of union)

Proof by element argument: Example 2

Proposition

- Prove that for all sets A and B, $(A \cup B)' = A' \cap B'$.

Proof by element argument: Example 2

Proposition

- Prove that for all sets A and B, $(A \cup B)' = A' \cap B'$.

Proof

We need to prove:

1. $(A \cup B)' \subseteq A' \cap B'$
2. $A' \cap B' \subseteq (A \cup B)'$

Proof by element argument: Example 2

Proof (continued)

- Proof that $(A \cup B)' \subseteq A' \cap B'$.
  Suppose $x \in (A \cup B)'$.
  $x \notin A \cup B$ $\quad$ ($\because$ defn. of complement)
  It is false that (x is in A or x is in B).
  x is not in A and x is not in B ($\because$ De Morgan's law of logic)
  $x \notin A$ and $x \notin B$
  $x \in A'$ and $x \in B'$ $\quad$ ($\because$ defn. of complement)
  $x \in (A' \cap B')$ $\quad$ ($\because$ defn. of intersection)
  Hence, $(A \cup B)' \subseteq A' \cap B'$ $\quad$ ($\because$ defn. of subset)

Proof by element argument: Example 2

Proof (continued)

- Proof that $A' \cap B' \subseteq (A \cup B)'$.
  Suppose $x \in A' \cap B'$.
  $x \in A'$ and $x \in B'$ ($\because$ defn. of intersection)
  $x \notin A$ and $x \notin B$ ($\because$ defn. of complement)
  x is not in A and x is not in B
  It is false that (x is in A or x is in B)
  ($\because$ De Morgan's law of logic)
  $x \notin A \cup B$
  $x \in (A \cup B)'$ ($\because$ defn. of complement)
  Hence, $A' \cap B' \subseteq (A \cup B)'$ ($\because$ defn. of subset)

Proof by element argument: Example 3

Proposition

- For any sets A and B, if $A \subseteq B$, then (a) $A \cap B = A$ and (b) $A \cup B = B$.

Proof by element argument: Example 3

Proposition

- For any sets A and B, if $A \subseteq B$, then (a) $A \cap B = A$ and (b) $A \cup B = B$.

Proof

Part (a): We need to prove:
1. $A \cap B \subseteq A$
2. $A \subseteq A \cap B$

Part (b): We need to prove:
1. $A \cup B \subseteq B$
2. $B \subseteq A \cup B$

Proof by element argument: Example 3

Proof (continued)

Part (a).

1. Proof that $A \cap B \subseteq A$.

   $A \cap B \subseteq A \qquad (\because$ inclusion of intersection$)$

2. Proof that $A \subseteq A \cap B$.

   Suppose $x \in A$

   $x \in B \qquad (\because A \subseteq B)$

   $x \in A$ and $x \in B$

   $x \in A \cap B \qquad (\because$ defn. of intersection$)$

Proof by element argument: Example 3

Proof (continued)

Part (b).

1. Proof that $A \cup B \subseteq B$.

   Suppose $x \in A \cup B$

   $x \in A$ or $x \in B$ ($\because$ defn. of union)

   If $x \in A$, then $x \in B$ ($\because A \subseteq B$)

   $x \in B$ ($\because$ Modus Ponens and division into cases)

2. Proof that $B \subseteq A \cup B$.

   $B \subseteq A \cup B$ ($\because$ inclusion in union)

Proof by element argument: Example 4

Proposition

- If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by element argument: Example 4

Proposition

- If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof

Proof by contradiction.

- Suppose there exists a set E with no elements and a set A such that $E \nsubseteq A$.
- $\exists x$ such that $x \in E$ and $x \notin A$ $\quad$ ($\because$ defn. of a subset)
- But there can be no such element since E has no elements.
- Contradiction!
- Hence, if E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by element argument: Example 5

Proposition

- There is only one set with no elements.

Proof by element argument: Example 5

Proposition

- There is only one set with no elements.

Proof

- Suppose E_1 and E_2 are both sets with no elements.
- $E_1 \subseteq E_2 \qquad (\because \text{ previous proposition})$
- $E_2 \subseteq E_1 \qquad (\because \text{ previous proposition})$
- Thus, $E_1 = E_2$

Proof by element argument: Example 6

Proposition

- Prove that for any set A, $A \cap \phi = \phi$

Proof by element argument: Example 6

Proposition

- Prove that for any set A, $A \cap \phi = \phi$

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap \phi$
- $x \in A$ and $x \in \phi$ $\quad$ ($\because$ defn. of intersection)
- $x \in \phi$
- Impossible because ϕ cannot have any elements
- Hence, the supposition is incorrect.
- So, $A \cap \phi = \phi$

Proof by element argument: Example 7

Proposition

- For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \phi$.

Proof by element argument: Example 7

Proposition

- For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \phi$.

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap C$
- $x \in A$ and $x \in C$ $\quad$ ($\because$ defn. of intersection)
- $x \in A$
- $x \in B$ $\quad$ ($\because x \in A$ and $A \subseteq B$)
- $x \in C'$ $\quad$ ($\because x \in B$ and $B \subseteq C'$)
- $x \notin C$ $\quad$ ($\because$ defn. of complement)
- $x \in C$ and $x \notin C$
- Contradiction!
- Hence, the supposition is incorrect.
- So, $A \cap C = \phi$

Proof by counterexample: Example 1

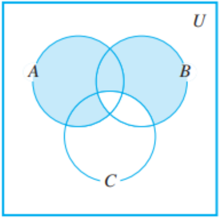
Proposition

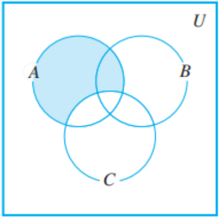
- For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.

Proof by counterexample: Example 1

Proposition

- For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.

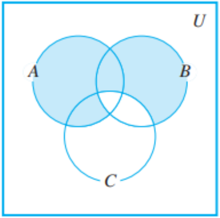


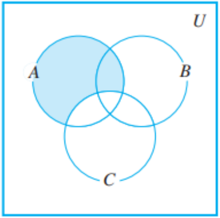


Proof by counterexample: Example 1

Proposition

- For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.





Disproof

$(A - B) \cup (B - C) \neq A - C$

- Draw Venn diagrams
- Counterexample 1: $A = \{1\}$, $B = \phi$, $C = \{1\}$
- Counterexample 2: $A = \phi$, $B = \{1\}$, $C = \phi$

Proof by mathematical induction: Example 1

Proposition

- For all integers $n \geq 0$, if a set X has n elements, then $P(X)$ has 2^n elements.

Proof by mathematical induction: Example 1

Proposition

- For all integers $n \geq 0$, if a set X has n elements, then $P(X)$ has 2^n elements.

Proof

Let $P(n)$ denote "Any set with n elements has 2^n subsets."

- Basis step. $P(0)$ is true.
  The only set with zero elements is the empty set. The only subset of the empty set is itself. Hence, $2^0 = 1$.
- Induction step. Suppose that $P(k)$ is true for any $k \geq 0$.
  i.e., "Any set with k elements has 2^k subsets."
  Now, we want to show that $P(k+1)$ is true.
  i.e., "Any set with $k+1$ elements has 2^{k+1} subsets."

Proof by mathematical induction: Example 1

Proof (continued)

Suppose set X has $k+1$ elements including element a.
#subsets of X
$=$ #subsets of X without a + #subsets of X with a
$=$ #subsets of $(X - \{a\})$ + #subsets of X with a
$=$ #subsets of $(X - \{a\})$ + #subsets of $(X - \{a\})$
($\because$ 1:1 correspondence)
$= 2\cdot$ #subsets of $(X - \{a\})$
$= 2 \cdot 2^k$
$= 2^{k+1}$
Hence, $P(k+1)$ is true.

- 1:1 correspondence.
  Any subset A of $X - \{a\}$ can be matched up with a subset B, equal to $A \cup \{a\}$, of X that contains a.

Algebraic proof: Example 1

Proposition

- Construct an algebraic proof that for all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$

Algebraic proof: Example 1

Proposition

- Construct an algebraic proof that for all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$

Proof

- $(A \cup B) - C$
  $= (A \cup B) \cap C'$ $\quad$ ($\because$ set difference law)
  $= C' \cap (A \cup B)$ $\quad$ ($\because$ commutative law)
  $= (C' \cap A) \cup (C' \cap B)$ $\quad$ ($\because$ distributive law)
  $= (A \cap C') \cup (B \cap C')$ $\quad$ ($\because$ commutative law)
  $= (A - C) \cup (B - C)$ $\quad$ ($\because$ set difference law)

Algebraic proof: Example 2

Proposition

- Construct an algebraic proof that for all sets A and B, $A - (A \cap B) = A - B$

Algebraic proof: Example 2

Proposition

- Construct an algebraic proof that for all sets A and B, $A - (A \cap B) = A - B$

Proof

- $A - (A \cap B)$
  $= A \cap (A \cap B)'$ $\quad$ ($\because$ set difference law)
  $= A \cap (A' \cup B')$ $\quad$ ($\because$ De Morgan's law)
  $= (A \cap A') \cup (A \cap B')$ $\quad$ ($\because$ distributive law)
  $= \phi \cup (A \cap B')$ $\quad$ ($\because$ complement law)
  $= (A \cap B') \cup \phi$ $\quad$ ($\because$ commutative law)
  $= A \cap B'$ $\quad$ ($\because$ identity law)
  $= A - B$ $\quad$ ($\because$ set difference law)