Discrete Mathematics
(Set)

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## What is a set?

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>• A <strong>set</strong> is a collection of related items</td>
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<table>
<thead>
<tr>
<th>Examples</th>
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</thead>
<tbody>
<tr>
<td>• $A = {1, 2, 3, 4, 5}$</td>
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<tr>
<td>• $P = {2, 3, 5, 7, 11, 13, 17, \ldots}$</td>
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<tr>
<td>• $S$ is a set of square numbers</td>
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<tr>
<td>• $B = {m \in \mathbb{Z} \mid m = 7n + 100 \text{ for some } n \in \mathbb{Z}}$</td>
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Subsets

Definitions

- **Subset.** \( A \subseteq B \iff \forall x, \text{ if } x \in A \text{ then } x \in B \)
- **Not subset.** \( A \not\subseteq B \iff \exists x \text{ such that } x \in A \text{ and } x \notin B \)
- **Proper subset.** \( A \subset B \iff \)
  1. \( A \subseteq B \)
  2. \( \exists x \text{ such that } x \in B \text{ and } x \notin A \)

Problems

- Let \( A = \{1\} \) and \( B = \{1, \{1\}\} \).
  1. Is \( A \subseteq B \)?
  2. Is \( A \subset B \)?
- Let \( A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\} \) and \( B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\} \).
  1. Prove that \( A \subseteq B \).
  2. Disprove that \( B \subseteq A \).
Set equality

Definition

- Given sets $A$ and $B$, $A$ equals $B$, written $A = B$, if, and only if, every element of $A$ is in $B$ and every element of $B$ is in $A$.
- $A = B \iff A \subseteq B$ and $B \subseteq A$.

Problems

- $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a\}$
- $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some integer } b\}$
- Is $A = B$?
Definition

- Relationship between a small number of sets can be represented by pictures called **Venn diagrams**

Problems

- Write a Venn diagram representing sets of numbers: $\mathbb{N}, \mathbb{W}, \mathbb{Q}, \mathbb{R}$. 
**Definition**

Let $A$ and $B$ be subsets of a universal set $U$.

1. The **union** of $A$ and $B$, denoted $A \cup B$, is the set of all elements that are in at least one of $A$ or $B$.
   
   \[ A \cup B = \{ x \in U \mid x \in A \text{ or } x \in B \} \]

2. The **intersection** of $A$ and $B$, denoted $A \cap B$, is the set of all elements that are common to both $A$ and $B$.
   
   \[ A \cap B = \{ x \in U \mid x \in A \text{ and } x \in B \} \]

3. The **difference** of $B$ minus $A$ (or relative complement of $A$ in $B$), denoted $B - A$, is the set of all elements that are in $B$ and not $A$.
   
   \[ B - A = \{ x \in U \mid x \in B \text{ and } x \notin A \} \]

4. The **complement** of $A$, denoted $A'$, is the set of all elements in $U$ that are not in $A$.
   
   \[ A' = \{ x \in U \mid x \notin A \} \]
Let the universal set $U = \{a, b, c, d, e, f, g\}$.
Let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$.
Find $A \cup B$, $A \cap B$, $B - A$, and $A'$.
Operations on sets

Notations

Given real numbers $a$ and $b$ with $a \leq b$:

- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

- The symbols $\infty$ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:
  - $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
  - $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$
  - $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
  - $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

Problems

- $A = (-1, 0] = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$
- $B = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$

Find $A \cup B$, $A \cap B$, $B - A$, and $A'$.
Operations on sets

Notations

Given sets \( A_0, A_1, A_2, \ldots \) that are subsets of a universal set \( U \) and given a nonnegative integer \( n \),

- \( \bigcup_{i=0}^{n} A_i = \{ x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n \} \)
- \( \bigcup_{i=0}^{\infty} A_i = \{ x \in U \mid x \in A_i \text{ for at least one whole number } i \} \)
- \( \bigcap_{i=0}^{n} A_i = \{ x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n \} \)
- \( \bigcap_{i=0}^{\infty} A_i = \{ x \in U \mid x \in A_i \text{ for all whole numbers } i \} \)

Problems

- For each positive integer \( i \), let
  \[ A_i = \{ x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i} \} = \left( -\frac{1}{i}, \frac{1}{i} \right) \]
  Find \( A_1 \cup A_2 \cup A_3 \) and \( A_1 \cap A_2 \cap A_3 \)
  Find \( \bigcup_{i=0}^{\infty} A_i \) and \( \bigcap_{i=0}^{\infty} A_i \)
**Empty set**

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Empty set, denoted by $\phi$, is a set with no elements.</td>
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<tbody>
<tr>
<td>• ${1, 3} \cap {2, 4} = \phi$</td>
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<tr>
<td>• ${x \in \mathbb{R} \mid x^2 = -1} = \phi$</td>
</tr>
<tr>
<td>• ${x \in \mathbb{R} \mid 3 &lt; x &lt; 2} = \phi$</td>
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</table>
# Disjoint sets

## Definition

- Two sets are called **disjoint** if, and only if, they have no elements in common.
- $A$ and $B$ are disjoint $\iff A \cap B = \emptyset$

## Problems

- Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are $A$ and $B$ disjoint?
Mutually disjoint sets

**Definition**

- Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or pairwise disjoint or nonoverlapping) if, and only if, no two sets $A_i$ and $A_j$ with distinct subscripts have any elements in common.
- For all $i, j = 1, 2, 3, \ldots$
  
  $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

**Problems**

Are the following sets mutually disjoint?

- $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$.
- $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. 
# Partition of a set

## Definition

- A finite or infinite collection of nonempty sets \( \{A_1, A_2, A_3, \ldots \} \) is a **partition** of a set \( A \) if, and only if,
  1. \( A \) is the union of all the \( A_i \)
  2. The sets \( A_1, A_2, A_3, \ldots \) are mutually disjoint.

## Problems

- Let \( A = \{1, 2, 3, 4, 5, 6\} \) \( A_1 = \{1, 2\} \), \( A_2 = \{3, 4\} \), and \( A_3 = \{5, 6\} \). Is \( \{A_1, A_2, A_3\} \) a partition of \( A \)?
- Let \( \mathbb{Z} \) be the set of all integers and let
  
  \[
  T_0 = \{n \in \mathbb{Z} \mid n = 3k, \text{ for some integer } k\},
  T_1 = \{n \in \mathbb{Z} \mid n = 3k + 1, \text{ for some integer } k\},
  T_2 = \{n \in \mathbb{Z} \mid n = 3k + 2, \text{ for some integer } k\},
  \]

  Is \( \{T_0, T_1, T_2\} \) a partition of \( \mathbb{Z} \)?
### Power set

<table>
<thead>
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<tr>
<td>• Given a set $A$, the power set of $A$, denoted $P(A)$, is the set of all subsets of $A$.</td>
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<tbody>
<tr>
<td>• Find the power set of the set ${x, y}$. That is, find $P({x, y})$.</td>
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**Ordered $n$-tuple**

<table>
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<tbody>
<tr>
<td>• <strong>Ordered $n$-tuple</strong>, $(x_1, x_2, \ldots, x_n)$, consists of $x_1, x_2, \ldots, x_n$ together with the ordering.</td>
</tr>
<tr>
<td>• <strong>Ordered pair</strong> = ordered 2-tuple</td>
</tr>
<tr>
<td>• <strong>Ordered triple</strong> = ordered 3-tuple</td>
</tr>
<tr>
<td>• Two ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.</td>
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\[
(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1, \ldots, x_n = y_n.
\]

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<tbody>
<tr>
<td>• Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?</td>
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<tr>
<td>• Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?</td>
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<tr>
<td>• Is $((1, 2), 3) = (1, 2, 3)$?</td>
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**Cartesian product**

**Definition**

- Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

**Problems**

- Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$. Find:
  
  (a) $A_1 \times A_2$,
  
  (b) $(A_1 \times A_2) \times A_3$, and
  
  (c) $A_1 \times A_2 \times A_3$. 

**Table**

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Properties of sets

Definition

- Inclusion of intersection: For all sets $A$ and $B$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
- Inclusion in union: For all sets $A$ and $B$, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
- Transitive property of subsets: For all sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. 
Procedural versions of set definitions

Definition

Let $X$ and $Y$ be subsets of a universal set $U$ and suppose $x$ and $y$ are elements of $U$.

- $x \in X \cup Y \iff x \in X \text{ or } x \in Y$
- $x \in X \cap Y \iff x \in X \text{ and } x \in Y$
- $x \in X - Y \iff x \in X \text{ and } x \notin Y$
- $x \in X' \iff x \notin X$
- $(x, y) \in X \times Y \iff x \in X \text{ and } y \in Y$
<table>
<thead>
<tr>
<th>Laws</th>
<th>Formula</th>
<th>Formula</th>
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<tr>
<td>Commutative laws</td>
<td>$A \cup B = B \cup A$</td>
<td>$A \cap B = B \cap A$</td>
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<tr>
<td>Associative laws</td>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
</tr>
<tr>
<td>Distributive laws</td>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C')$</td>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C')$</td>
</tr>
<tr>
<td>Identity laws</td>
<td>$A \cup \emptyset = A$</td>
<td>$A \cap U = A$</td>
</tr>
<tr>
<td>Complement laws</td>
<td>$A \cup A' = U$</td>
<td>$A \cap A' = \emptyset$</td>
</tr>
<tr>
<td>Double comp. law</td>
<td>$(A')' = A$</td>
<td>$(A')' = A$</td>
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<tr>
<td>Idempotent laws</td>
<td>$A \cup A = A$</td>
<td>$A \cap A = A$</td>
</tr>
<tr>
<td>Uni. bound laws</td>
<td>$A \cup U = U$</td>
<td>$A \cap \emptyset = \emptyset$</td>
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<tr>
<td>De Morgan’s laws</td>
<td>$(A \cup B)' = A' \cap B'$</td>
<td>$(A \cap B)' = A' \cup B'$</td>
</tr>
<tr>
<td>Absorption laws</td>
<td>$A \cup (A \cap B) = A$</td>
<td>$A \cap (A \cup B) = A$</td>
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<tr>
<td>Complements</td>
<td>$U' = \emptyset$</td>
<td>$\emptyset' = U$</td>
</tr>
<tr>
<td>Set diff. laws</td>
<td>$A - B = A \cap B'$</td>
<td>$A - B = A \cap B'$</td>
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</tbody>
</table>
Basic method for proving that a set is a subset of another

- Let sets $X$ and $Y$ be given. To prove that $X \subseteq Y$,
  1. suppose that $x$ is a particular but arbitrarily chosen element of $X$.
  2. show that $x$ is an element of $Y$. 
Basic method for proving that two sets are equal

- Let sets $X$ and $Y$ be given. To prove that $X = Y$,
  1. Prove that $X \subseteq Y$.
  2. Prove that $Y \subseteq X$. 
Element argument

Basic method for proving a set equals the empty set

• To prove that a set $X$ is equal to the empty set $\emptyset$, prove that $X$ has no elements.
• To do this, suppose $X$ has an element and derive a contradiction.
Proof by element argument: Example 1

Proposition

- Prove that for all sets $A$, $B$, and $C$
  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
### Proposition

- Prove that for all sets $A$, $B$, and $C$
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

### Proof

We need to prove:

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
Proof by element argument: Example 1

Proof (continued)

Proof that \( A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \).

Suppose \( x \in A \cup (B \cap C) \).

\( x \in A \) or \( x \in B \cap C \) \( (\because \) defn. of union \)

• Case 1. \([x \in A.]\)
  \( x \in A \cup B \) \( (\because \) defn. of union \)
  \( x \in A \cup C \) \( (\because \) defn. of union \)
  \( x \in (A \cup B) \cap (A \cup C) \) \( (\because \) defn. of intersection \)

• Case 2. \([x \in B \cap C.]\)
  \( x \in B \) and \( x \in C \) \( (\because \) defn. of intersection \)
  \( x \in A \cup B \) \( (\because \) defn. of union \)
  \( x \in A \cup C \) \( (\because \) defn. of union \)
  \( x \in (A \cup B) \cap (A \cup C) \) \( (\because \) defn. of intersection \)
Proof by element argument: Example 1

Proof (continued)

Proof that \((A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)\).

Suppose \(x \in (A \cup B) \cap (A \cup C)\).

\(x \in A \cup B\) and \(x \in A \cup C\) \((\because \text{ defn. of intersection})\)

- **Case 1.** \([x \in A.]\)
  \(x \in A \cup (B \cap C)\) \((\because \text{ defn. of union})\)

- **Case 2.** \([x \notin A.]\)
  \(x \in A\) or \(x \in B\) \((\because \text{ defn. of union})\)
  \(x \in B\) \((\because x \notin A)\)
  \(x \in A\) or \(x \in C\) \((\because \text{ defn. of union})\)
  \(x \in C\) \((\because x \notin A)\)
  \(x \in B \cap C\) \((\because \text{ defn. of intersection})\)
  \(x \in A \cup (B \cap C)\) \((\because \text{ defn. of union})\)
Proposition

- Prove that for all sets $A$ and $B$, $(A \cup B)' = A' \cap B'$. 
Proof by element argument: Example 2

Proposition

- Prove that for all sets $A$ and $B$, $(A \cup B)' = A' \cap B'$.

Proof

We need to prove:
1. $(A \cup B)' \subseteq A' \cap B'$
2. $A' \cap B' \subseteq (A \cup B)'$
**Proof (continued)**

- **Proof that** \((A \cup B)' \subseteq A' \cap B'\).

  Suppose \(x \in (A \cup B)'\).

  \(x \notin A \cup B\) \((:: \text{defn. of complement})\)

  It is false that \((x \text{ is in } A \text{ or } x \text{ is in } B)\).

  \(x \text{ is not in } A \text{ and } x \text{ is not in } B\) \((:: \text{De Morgan’s law of logic})\)

  \(x \notin A \text{ and } x \notin B\)

  \(x \in A' \text{ and } x \in B'\) \((:: \text{defn. of complement})\)

  \(x \in (A' \cap B')\) \((:: \text{defn. of intersection})\)

  Hence, \((A \cup B)' \subseteq A' \cap B'\) \((:: \text{defn. of subset})\)
Proof by element argument: Example 2

Proof (continued)

- **Proof that** \( A' \cap B' \subseteq (A \cup B)' \).
  
  Suppose \( x \in A' \cap B' \).
  
  \( x \in A' \) and \( x \in B' \) \( (\because \text{defn. of intersection}) \)
  
  \( x \notin A \) and \( x \notin B \) \( (\because \text{defn. of complement}) \)
  
  \( x \) is not in \( A \) and \( x \) is not in \( B \)
  
  It is false that \( (x \text{ is in } A \text{ or } x \text{ is in } B) \)
  \( (\because \text{De Morgan’s law of logic}) \)

  \( x \notin A \cup B \)

  \( x \in (A \cup B)' \) \( (\because \text{defn. of complement}) \)

  Hence, \( A' \cap B' \subseteq (A \cup B)' \) \( (\because \text{defn. of subset}) \)
Proposition

- For any sets $A$ and $B$, if $A \subseteq B$, then
  
  (a) $A \cap B = A$ and (b) $A \cup B = B$. 
Proof by element argument: Example 3

Proposition

- For any sets $A$ and $B$, if $A \subseteq B$, then
  
  (a) $A \cap B = A$ and (b) $A \cup B = B$.

Proof

Part (a): We need to prove:
1. $A \cap B \subseteq A$
2. $A \subseteq A \cap B$

Part (b): We need to prove:
1. $A \cup B \subseteq B$
2. $B \subseteq A \cup B$
Proof (continued)

Part \((a)\).

1. **Proof that** \(A \cap B \subseteq A\).
   
   \(A \cap B \subseteq A\) \(\because\) inclusion of intersection

2. **Proof that** \(A \subseteq A \cap B\).
   
   Suppose \(x \in A\)
   
   \(x \in B\) \(\because\) \(A \subseteq B\)
   
   \(x \in A\) and \(x \in B\)
   
   \(x \in A \cap B\) \(\because\) defn. of intersection
Proof (continued)

**Part (b).**

1. **Proof that** $A \cup B \subseteq B$.
   
   Suppose $x \in A \cup B$
   
   $x \in A$ or $x \in B$  \((\because \text{defn. of union})\)
   
   If $x \in A$, then $x \in B$  \((\because A \subseteq B)\)
   
   $x \in B$ \((\because \text{Modus Ponens and division into cases})\)

2. **Proof that** $B \subseteq A \cup B$.
   
   $B \subseteq A \cup B$  \((\because \text{inclusion in union})\)
### Proposition

- If $E$ is a set with no elements and $A$ is any set, then $E \subseteq A$. 

**Proof by element argument:**

Suppose there exists a set $E$ with no elements and a set $A$ such that $E \not\subseteq A$. ∃$x$ such that $x \in E$ and $x \not\in A$ (∵ defn. of a subset). But there can be no such element since $E$ has no elements. Contradiction! Hence, if $E$ is a set with no elements and $A$ is any set, then $E \subseteq A$. 

**Proposition**

- If $E$ is a set with no elements and $A$ is any set, then $E \subseteq A$.

**Proof**

Proof by contradiction.

- Suppose there exists a set $E$ with no elements and a set $A$ such that $E \not\subseteq A$.
- $\exists x$ such that $x \in E$ and $x \notin A$ (∵ defn. of a subset)
- But there can be no such element since $E$ has no elements.
- Contradiction!
- Hence, if $E$ is a set with no elements and $A$ is any set, then $E \subseteq A$.
Proof by element argument: Example 5

<table>
<thead>
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<th>Proposition</th>
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<tbody>
<tr>
<td>• There is only one set with no elements.</td>
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</table>
### Proposition
- There is only one set with no elements.

### Proof
- Suppose \( E_1 \) and \( E_2 \) are both sets with no elements.
- \( E_1 \subseteq E_2 \) (\( \because \) previous proposition)
- \( E_2 \subseteq E_1 \) (\( \because \) previous proposition)
- Thus, \( E_1 = E_2 \)
Proposition

- Prove that for any set $A$, $A \cap \emptyset = \emptyset$
Proof by element argument: Example 6

<table>
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<tr>
<td>- Prove that for any set $A$, $A \cap \emptyset = \emptyset$</td>
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<table>
<thead>
<tr>
<th>Proof</th>
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<tbody>
<tr>
<td><strong>Proof by contradiction.</strong></td>
</tr>
<tr>
<td>- Suppose there is an element $x$ such that $x \in A \cap \emptyset$</td>
</tr>
<tr>
<td>- $x \in A$ and $x \in \emptyset$ (∵ defn. of intersection)</td>
</tr>
<tr>
<td>- $x \in \emptyset$</td>
</tr>
<tr>
<td>- Impossible because $\emptyset$ cannot have any elements</td>
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<tr>
<td>- Hence, the supposition is incorrect.</td>
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<tr>
<td>- So, $A \cap \emptyset = \emptyset$</td>
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</tbody>
</table>
Proof by element argument: Example 7

Proposition

- For all sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \emptyset$. 

Proof by contradiction.

Suppose there is an element $x$ such that $x \in A \cap C$.

$x \in A$ and $x \in C$ (∵ defn. of intersection)

$x \in A$ and $x \in B$ (∵ $x \in A$ and $A \subseteq B$)

$x \in C'$ (∵ $x \in B$ and $B \subseteq C'$)

$x \notin C$ (∵ defn. of complement)

Contradiction!

Hence, the supposition is incorrect.

So, $A \cap C = \emptyset$. 

Proof by element argument: Example 7

**Proposition**

For all sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \phi$.

**Proof**

Proof by contradiction.

- Suppose there is an element $x$ such that $x \in A \cap C$
- $x \in A$ and $x \in C$ (∵ defn. of intersection)
- $x \in A$
- $x \in B$ (∵ $x \in A$ and $A \subseteq B$)
- $x \in C'$ (∵ $x \in B$ and $B \subseteq C'$)
- $x \not\in C$ (∵ defn. of complement)
- $x \in C$ and $x \not\in C$
- Contradiction!
- Hence, the supposition is incorrect.
- So, $A \cap C = \phi$
Proof by counterexample: Example 1

<table>
<thead>
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<td>• For all sets $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$.</td>
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Proof by counterexample: Example 1

Proposition

- For all sets $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$. 

Draw Venn diagrams

Counterexample 1: $A = \{1\}$, $B = \emptyset$, $C = \{1\}$

Counterexample 2: $A = \emptyset$, $B = \{1\}$, $C = \emptyset$
Proof by counterexample: Example 1

Proposition

- For all sets $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$.

Disproof

- $(A - B) \cup (B - C) \neq A - C$
- Draw Venn diagrams
- Counterexample 1: $A = \{1\}$, $B = \emptyset$, $C = \{1\}$
- Counterexample 2: $A = \emptyset$, $B = \{1\}$, $C = \emptyset$
Proof by mathematical induction: Example 1

Proposition

- For all integers $n \geq 0$, if a set $X$ has $n$ elements, then $P(X)$ has $2^n$ elements.
Proof by mathematical induction: Example 1

**Proposition**

- For all integers \( n \geq 0 \), if a set \( X \) has \( n \) elements, then \( P(X) \) has \( 2^n \) elements.

**Proof**

Let \( P(n) \) denote “Any set with \( n \) elements has \( 2^n \) subsets.”

- **Basis step.** \( P(0) \) is true.
  - The only set with zero elements is the empty set. The only subset of the empty set is itself. Hence, \( 2^0 = 1 \).

- **Induction step.** Suppose that \( P(k) \) is true for any \( k \geq 0 \).
  - i.e., “Any set with \( k \) elements has \( 2^k \) subsets.”

  Now, we want to show that \( P(k + 1) \) is true.
  - i.e., “Any set with \( k + 1 \) elements has \( 2^{k+1} \) subsets.”
Proof (continued)

Suppose set $X$ has $k + 1$ elements including element $a$.

$\#\text{subsets of } X$

$= \#\text{subsets of } X \text{ without } a + \#\text{subsets of } X \text{ with } a$

$= \#\text{subsets of } (X - \{a\}) + \#\text{subsets of } X \text{ with } a$

$= \#\text{subsets of } (X - \{a\}) + \#\text{subsets of } (X - \{a\})$

($\because 1:1 \text{ correspondence}$)

$= 2 \cdot \#\text{subsets of } (X - \{a\})$

$= 2 \cdot 2^k$

$= 2^{k+1}$

Hence, $P(k + 1)$ is true.

- **1:1 correspondence.**
  Any subset $A$ of $X - \{a\}$ can be matched up with a subset $B$, equal to $A \cup \{a\}$, of $X$ that contains $a$. 
Proposition

- Construct an algebraic proof that for all sets $A$, $B$, and $C$,
  \[(A \cup B) - C = (A - C) \cup (B - C)\]
# Algebraic proof: Example 1

**Proposition**
- Construct an algebraic proof that for all sets $A$, $B$, and $C$, 
  \[(A \cup B) - C = (A - C) \cup (B - C)\]

**Proof**
- \[(A \cup B) - C\]
  \[= (A \cup B) \cap C' \quad (\because \text{set difference law})\]
  \[= C' \cap (A \cup B) \quad (\because \text{commutative law})\]
  \[= (C' \cap A) \cup (C' \cap B) \quad (\because \text{distributive law})\]
  \[= (A \cap C') \cup (B \cap C') \quad (\because \text{commutative law})\]
  \[= (A - C') \cup (B - C') \quad (\because \text{set difference law})\]
Algebraic proof: Example 2

<table>
<thead>
<tr>
<th>Proposition</th>
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<tbody>
<tr>
<td>• Construct an algebraic proof that for all sets $A$ and $B$, $A - (A \cap B) = A - B$</td>
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</table>
Proposition

- Construct an algebraic proof that for all sets $A$ and $B$, $A - (A \cap B) = A - B$

Proof

- $A - (A \cap B)$
  - $= A \cap (A \cap B)'$  (∵ set difference law)
  - $= A \cap (A' \cup B')$  (∵ De Morgan’s law)
  - $= (A \cap A') \cup (A \cap B')$  (∵ distributive law)
  - $= \emptyset \cup (A \cap B')$  (∵ complement law)
  - $= (A \cap B') \cup \emptyset$  (∵ commutative law)
  - $= A \cap B'$  (∵ identity law)
  - $= A - B$  (∵ set difference law)