Discrete Mathematics

(Sequences)

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Contents

Contents

Basic Concepts on Sequences

What are sequences?

Types of sequences

- Finite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  e.g.: $1^1, 2^2, 3^2, \ldots, 100^2$
- Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots$
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Term

- Closed-form formula: $a_k = f(k)$
  e.g.: $a_k = \frac{k}{k+1}$
- Recursive formula: $a_k = g(k, a_{k-1}, \ldots, a_{k-c})$
  e.g.: $a_k = a_{k-1} + (k-1)a_{k-2}$

What are sequences?

Growth of sequences

- Increasing sequence
  e.g.: $2, 3, 5, 7, 11, 13, 17, \ldots$
- Decreasing sequence
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$
- Oscillating sequence
  e.g.: $1, -1, 1, -1, \ldots$

Problem-solving

- Compute a_k given $a_1, a_2, a_3, \ldots$
  e.g.: Compute a_k given $\frac{1}{n}, \frac{2}{n+1}, \frac{3}{n+2}, \ldots$
- Compute $a_1, a_2, a_3, \ldots$ given a_k
  e.g.: Compute $a_1, a_2, a_3, \ldots$ given $a_k = \frac{k}{k+1}$

Sums and products of sequences

Sum

- Summation form:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit

e.g.: $\sum_{k=m}^{n} \frac{(-1)^k}{k+1}$

Product

- Product form:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit

e.g.: $\prod_{k=m}^{n} \frac{k}{k+1}$

Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number

Sum

- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i < n$ where, i is between m and $n-1$ (inclusive)
- $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$
- $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

Product

- $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Change of variable

$$\begin{aligned}
\sum_{k=0}^{99} \frac{(-1)^k}{k+1} &= \sum_{j=0}^{99} \frac{(-1)^j}{j+1} \qquad \text{(Set } j = k\text{)} \\
&= \sum_{i=1}^{100} \frac{(-1)^{i-1}}{i} \qquad \text{(Set } i = j+1\text{)}
\end{aligned}$$

Factorial function

Factorial function

- The factorial of a whole number n, denoted by $n!$, is defined as follows:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 & \text{if } n > 0. \end{cases}$$

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot (n-1)! & \text{if } n > 0. \end{cases}$$

▷ Recursive definition

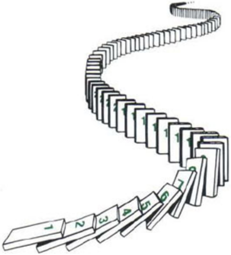
Ordinary Mathematical Induction

Proof by mathematical induction

- Mathematical induction is aesthetically beautiful and insanely powerful proof technique
- Mathematical induction is probably the greatest of all proof techniques and probably the simplest

Core idea

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.



Source: https://i.stack.imgur.com/Z3l92.jpg

Proof by mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof by mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof

- Basis step.
  Show that $P(a)$ is true.
- Induction step.
  Assume $P(k)$ is true for some integer $k \geq a$.
  (This supposition is called the inductive hypothesis.)
  Now, prove that $P(k+1)$ is true.

Proof by mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof by mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof

- $\boxed{P(a)}$ (Base case)

  $\boxed{\forall k \geq a, P(k) \to P(k+1)}$ (Induction case)

$P(a+1)$ (Conclusion)

$\forall k \geq a, P(k) \to P(k+1)$ (Induction case)

$P(a+2)$ (Conclusion)

$\forall k \geq a, P(k) \to P(k+1)$ (Induction case)

$P(a+3)$ (Conclusion)

Similarly, $P(a+4), P(a+5), \ldots$

Proof by mathematical induction: Example 0

Pattern

- $1 = \frac{1 \cdot 2}{2}$
- $1 + 2 = \frac{2 \cdot 3}{2}$
- $1 + 2 + 3 = \frac{3 \cdot 4}{2}$
- $1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2}$
- $1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \cdot 7}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{7 \cdot 8}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{8 \cdot 9}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \frac{9 \cdot 10}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{10 \cdot 11}{2}$
- $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = \frac{11 \cdot 12}{2}$

Proof by mathematical induction: Example 0

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- $1 = \frac{1\cdot 2}{2}$
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- $1+2+3 = \frac{3\cdot 4}{2}$
- $1+2+3+4 = \frac{4\cdot 5}{2}$
- $1+2+3+4+5 = \frac{5\cdot 6}{2}$
- $1+2+3+4+5+6 = \frac{6\cdot 7}{2}$
- $1+2+3+4+5+6+7 = \frac{7\cdot 8}{2}$
- $1+2+3+4+5+6+7+8 = \frac{8\cdot 9}{2}$
- $1+2+3+4+5+6+7+8+9 = \frac{9\cdot 10}{2}$
- $1+2+3+4+5+6+7+8+9+10 = \frac{10\cdot 11}{2}$
- $1+2+3+4+5+6+7+8+9+10+11 = \frac{11\cdot 12}{2}$

Proposition

- $1+2+\cdots+n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Proof by mathematical induction: Example 0

Proposition

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Proof by mathematical induction: Example 0

Proposition

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Proof

Let $P(n)$ denote $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

- Basis step. $P(1)$ is true because $1 = 1(1+1)/2$.
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$. Now, we want to show that $P(k+1)$ is true. That is,

Proof by mathematical induction: Example 0

Proposition

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  Now, we want to show that $P(k+1)$ is true. That is,
  Assume $P(k)$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for some $k \geq 1$
  Prove $P(k+1)$: $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$

Proof by mathematical induction: Example 0

Proposition

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  Assume $P(k)$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for some $k \geq 1$

  Prove $P(k+1)$: $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$

  LHS of $P(k+1)$

  $= (1 + 2 + \cdots + k) + (k+1)$

  $= \frac{k(k+1)}{2} + (k+1)$ $\quad (\because P(k)$ is true$)$

  $= \frac{(k+1)(k+2)}{2}$ $\quad (\because$ distributive law$)$

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

  LHS of $P(k+1)$

  $=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\right]\left(1-\frac{1}{k+1}\right)$

  $=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$ $\quad(\because P(k)$ is true$)$

  $=\frac{1}{k}\cdot\frac{k}{k+1}$ $\quad(\because$ common denominator$)$

  $=\frac{1}{k+1}$ $\quad(\because$ remove common factor$)$

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 2

Proposition

- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Proof by mathematical induction: Example 2

Proposition

- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Proof

Let $P(n)$ denote $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$.

- Basis step. $P(1)$ is true. ▷ How?
- Induction step.

Assume $P(k)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1}$ for some $k \geq 1$

Prove $P(k+1)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)\cdot(k+2)} = \frac{k+1}{k+2}$

LHS of $P(k+1)$

$= \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)}\right) + \frac{1}{(k+1)\cdot(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)\cdot(k+2)}$ ($\because P(k)$ is true)

$= \frac{k^2+2k+1}{(k+1)\cdot(k+2)}$ ($\because$ common denominator)

$= \frac{(k+1)^2}{(k+1)\cdot(k+2)}$ ($\because$ simplify)

$= \frac{k+1}{k+2}$ ($\because$ remove common factor)

$=$ RHS of $P(k+1)$

Problems for practice

For all integers $n \geq 1$:

- $1 + 3 + \cdots + (2n-1) = n^2$
- $2 + 4 + \cdots + 2n = n(n+1)$
- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$
- $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{n(6n^2+3n-1)}{2}$
- $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$
- $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$
- $\underbrace{111\ldots1}_{n \text{ times}} = \frac{10^n-1}{9}$

Proof by mathematical induction: Example 3

Proposition

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

Proof by mathematical induction: Example 3

Proposition

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

Proof

Let $P(n)$ denote $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$.

- Basis step. $P(0)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 0$.
  Now, we want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$
  $= (F(0)^2 + F(1)^2 + \cdots + F(k)^2) + F(k+1)^2$
  $= F(k)F(k+1) + F(k+1)^2$ ($\because$ $P(k)$ is true)
  $= F(k+1)(F(k) + F(k+1))$ ($\because$ distributive law)
  $= F(k+1)F(k+2)$ ($\because$ recursive definition)
  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 4

Proposition

- $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ for all integers $n \geq 1$.

Proof by mathematical induction: Example 4

Proposition

- $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ for all integers $n \geq 1$.

Proof

Let $P(n)$ denote $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$.

- Basis step. $P(1)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$.
  Now, we want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$
  $= (1 + r + r^2 + \cdots + r^k) + r^{k+1}$
  $= \left(\frac{r^{k+1}-1}{r-1}\right) + r^{k+1}$ $\quad(\because P(k)$ is true)
  $= \frac{(r^{k+1}-1)+r^{k+1}(r-1)}{r-1}$ $\quad(\because$ common denominator)
  $= \frac{r^{(k+1)+1}-1}{r-1}$ $\quad(\because$ simplify)
  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 5

Proposition

- $2^{2n} - 1$ is divisible by 3, for all integers $n \geq 0$.

Proof by mathematical induction: Example 5

Proposition

- $2^{2n} - 1$ is divisible by 3, for all integers $n \geq 0$.

Proof

Let $P(n)$ denote $2^{2n} - 1$ is divisible by 3.

- Basis step. $P(0)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 0$.
  Now, we want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$
  $= 2^{2(k+1)} - 1$
  $= 2^2 \cdot 2^{2k} - 1 \qquad (\because a^{b+c} = a^b \cdot a^c)$
  $= (3+1) \cdot 2^{2k} - 1 \qquad (\because \text{rewrite})$
  $= 3 \cdot 2^{2k} + (2^{2k} - 1) \qquad (\because \text{distributive law})$
  $= 3 \cdot 2^{2k} + 3r \qquad (\because P(k) \text{ is true})$
  $= 3 \cdot (2^{2k} + r) \qquad (\because \text{distributive law})$
  $= 3 \cdot \text{integer} \qquad (\because \text{addition is closed on integers})$
  $= \text{RHS of } P(k+1)$

Problems for practice

For all integers $n \geq 0$:

- $5^n - 1$ is divisible by 4
- $7^n - 1$ is divisible by 6
- $9^n + 3$ is divisible by 4
- $3^{2n} - 1$ is divisible by 8
- $7^n - 2^n$ is divisible by 5
- $7^{n+2} + 8^{2n+1}$ is divisible by 57
- $n^3 + 2n$ is divisible by 3
- $n^3 - 7n + 3$ is divisible by 3
- $17n^3 + 103n$ is divisible by 6

Proof by mathematical induction: Example 6

Proposition

- $x^n - y^n$ is divisible by $x - y$, for all integers x, y such that $x \neq y$, for all integers $n \geq 0$.

Proof by mathematical induction: Example 6

Proposition

- $x^n - y^n$ is divisible by $x - y$, for all integers x, y such that $x \neq y$, for all integers $n \geq 0$.

Proof

Let $P(n)$ denote $x^n - y^n$ is divisible by $x - y$, s.t. $x \neq y$.

- Basis step. $P(0)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 0$.
  Now, we want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$
  $= x^{k+1} - y^{k+1}$
  $= x^{k+1} - x \cdot y^k + x \cdot y^k - y^{k+1}$ ($\because$ subtract and add)
  $= x \cdot (x^k - y^k) + y^k(x - y)$ ($\because$ distributive law)
  $= x \cdot (x - y)r + y^k(x - y)$ ($\because$ $P(k)$ is true)
  $= (x - y)(xr + y^k)$ ($\because$ distributive law)
  $= (x - y) \cdot$ integer ($\because$ $+, \times,$ expo are closed on integers)
  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 7

Proposition

- $2^n < n!$, for all integers $n \geq 4$.

Proof by mathematical induction: Example 7

Proposition

- $2^n < n!$, for all integers $n \geq 4$.

Proof

Let $P(n)$ denote $2^n < n!$.

- Basis step. $P(4)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 4$.

  Now, we want to show that $P(k+1)$ is true.

  LHS of $P(k+1)$

  $= 2^{k+1}$

  $= 2^k \cdot 2 \qquad (\because a^{b+c} = a^b \cdot a^c)$

  $< k! \cdot 2 \qquad (\because P(k)$ is true$)$

  $< k! \cdot (k+1) \qquad (\because 2 < (k+1)$ for $k \geq 4)$

  $= (k+1)! \qquad (\because$ factorial recursive definition$)$

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 8

Proposition

- $n^2 < 2^n$, for all integers $n \geq 5$.

Proof by mathematical induction: Example 8

Proposition

- $n^2 < 2^n$, for all integers $n \geq 5$.

Proof

Let $P(n)$ denote $n^2 < 2^n$.

- Basis step. $P(5)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 5$.

  Now, we want to show that $P(k+1)$ is true.

  LHS of $P(k+1)$

  $= (k+1)^2 = k^2 + 2k + 1 \qquad (\because \text{expand})$

  $< k^2 + 2k + k \qquad (\because 1 < k)$

  $= k^2 + 3k \qquad (\because \text{simplify})$

  $< k^2 + k^2 \qquad (\because 3 < k)$

  $= 2k^2 \qquad (\because \text{simplify})$

  $< 2 \cdot 2^k \qquad (\because P(k) \text{ is true})$

  $= 2^{k+1} \qquad (\because a^b \cdot a^c = a^{b+c})$

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 9

Proposition

- If one square is removed from a $2^n \times 2^n$ board, the remaining squares can be completely covered by L-shaped trominoes, for all integers $n \geq 1$.

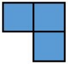
Proof by mathematical induction: Example 9

Proposition

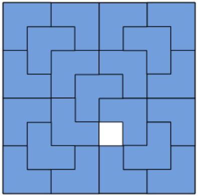
- If one square is removed from a $2^n \times 2^n$ board, the remaining squares can be completely covered by L-shaped trominoes, for all integers $n \geq 1$.

Example

- L-shaped tromino:



- L-shaped trominoes cover $2^3 \times 2^3$ board with a missing square:



Source: pd4cs.org

Proof by mathematical induction: Example 9

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Proof by mathematical induction: Example 9

Proposition

- If one square is removed from a $2^n \times 2^n$ board, the remaining squares can be completely covered by L-shaped trominoes, for all integers $n \geq 1$.

Proof

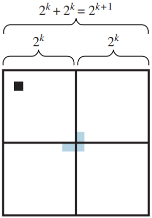
Let $P(n)$ denote "A $2^n \times 2^n$ board with a square removed can be completely covered by L-shaped trominoes."

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$. Now, we want to show that $P(k+1)$ is true.

Proof by mathematical induction: Example 9

Proof (continued)

- Induction step.
  Consider $2^{k+1} \times 2^{k+1}$ board with a square removed.
  Divide it into four equal quadrants.
  Consider the quadrant with the missing square. We can tile this quadrant with trominoes because $P(k)$ is true.
  Remaining three quadrants meet at the center. Place a tromino on the three central squares of the three quadrants.
  The three quadrants can now be tiled using trominoes because $P(k)$ is true. As all four quadrants can be covered with trominoes, $P(k+1)$ is true.



Proof by mathematical induction: Example 10

Proposition

- There are some fuel stations located on a circular road (or looping highway). The stations have different amounts of fuel. However, the total amount of fuel at all the stations is enough to make a trip around the circular road exactly once. Prove that it is possible to find an initial location from where if we start on a car with an empty tank, we can drive all the way around the circular road without running out of fuel.

Proof by mathematical induction: Example 10

Proposition

- There are some fuel stations located on a circular road (or looping highway). The stations have different amounts of fuel. However, the total amount of fuel at all the stations is enough to make a trip around the circular road exactly once. Prove that it is possible to find an initial location from where if we start on a car with an empty tank, we can drive all the way around the circular road without running out of fuel.

Proof

Let $P(n)$ denote "It is possible to find an initial location from where if we start on a car with an empty tank, we can drive all around the circular road with n fuel stations without running out of fuel."

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 1$ and any $i \in [1, k]$. We want to show that $P(k+1)$ is true.

Proof by mathematical induction: Example 10

Proof (continued)

As the total amount of fuel in all $k+1$ stations is enough for a car to make a round trip, there must be at least one fuel station A, that contains enough fuel to enable the car to reach the next fuel station X, in the direction of travel.

Suppose you transfer all fuel from X to A. The result would be a problem with k fuel stations. As $P(k)$ is true, it is possible to find an initial location for the car so that it does the round trip.

Use that location as the starting point for the car. When the car reaches A, the amount of fuel in A is enough to enable it to reach X, and once the car reaches X, the additional amount of fuel in X enables it to complete the round trip.

Hence, $P(k+1)$ is true.

Proof by mathematical induction: Example 11

Proposition

- All dogs in the world have the same color.

Proof by mathematical induction: Example 11

Proposition

- All dogs in the world have the same color.

Proof

Let $P(n)$ denote "In any collection of n dogs, all of them have the same color."

- Basis step. $P(1)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$.
  Now, we want to show that $P(k+1)$ is true.
  Consider a set of $k+1$ dogs, say, $\{d_1, d_2, \ldots, d_k, d_{k+1}\}$.
  $\{d_1, d_2, \ldots, d_k\}$ dogs have the same color. ($\because P(k)$ is true)
  $\{d_2, d_3, \ldots, d_{k+1}\}$ dogs have the same color. ($\because P(k)$ is true)
  So, all $k+1$ dogs have the same color.
  That is, $P(k+1)$ is true.

Proof by mathematical induction: Example 11

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- What's wrong?

Proof by mathematical induction: Example 12

Proposition

- All sand in the world cannot make a heap.

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Proof

Let $P(n)$ denote "n grains of sand is not a heap."

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$.
  Now, we want to show that $P(k+1)$ is true.
  If n grains of sand is not a heap of sand, then $n+1$ grains of sand is not a heap of sand either.
  Therefore, $P(k+1)$ is true.

Proof by mathematical induction: Example 12

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- All sand in the world cannot make a heap.

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Let $P(n)$ denote "n grains of sand is not a heap."

- Basis step. $P(1)$ is true. $\triangleright$ How?
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- What's wrong?

Proof by mathematical induction: Example 13

Proposition

- We are nonliving things.

Proof by mathematical induction: Example 13

Proposition

- We are nonliving things.

Proof

Let $P(n)$ denote "n atoms of matter is nonliving."

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$.
  Now, we want to show that $P(k+1)$ is true.
  Consider n atoms of matter that is not living. Add one more atom to this collection. Adding one more atom cannot suddenly bring life to the nonliving. So, $n+1$ atoms of matter is not living.
  Therefore, $P(k+1)$ is true.

Proof by mathematical induction: Example 13

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- We are nonliving things.

Proof

Let $P(n)$ denote "n atoms of matter is nonliving."

- Basis step. $P(1)$ is true. $\triangleright$ How?
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- What's wrong?

Strong Mathematical Induction

Proof by strong mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof by strong mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof

- Basis step.
  Show that $P(a), P(a+1), \ldots, P(b)$ are true.
- Induction step.
  Assume $\{P(a), P(a+1), \ldots, P(k)\}$ are true for some $k \geq b$.
  (This supposition is called the inductive hypothesis.)
  Now, prove that $P(k+1)$ is true.

Proof by strong mathematical induction

Equivalence

- Any statement that can be proved with ordinary mathematical induction can be proved with strong mathematical induction and vice versa.
- Hence, ordinary and strong mathematical induction techniques are equivalent

Applications

- Ordinary mathematical induction seems simpler.
  Then why do we care for strong mathematical induction?
- There are many propositions for which strong mathematical induction seems both simpler and more natural way of proving

Proof by mathematical induction: Example 1

Proposition

- Consider the sequence: $a_0 = 0$, $a_1 = 4$, and $a_k = 6a_{k-1} - 5a_{k-2}$ for all integers $k \geq 2$. Prove that $a_n = 5^n - 1$ for all integers $n \geq 0$.

Proof by mathematical induction: Example 1

Proposition

- Consider the sequence: $a_0 = 0$, $a_1 = 4$, and $a_k = 6a_{k-1} - 5a_{k-2}$ for all integers $k \geq 2$.
  Prove that $a_n = 5^n - 1$ for all integers $n \geq 0$.

Proof

Let $P(n)$ denote "$a_n = 5^n - 1$."

- Basis step. $P(0)$ and $P(1)$ are true. ▷ How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 1$ and any $i \in [0, k]$. We want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$

$$
\begin{aligned}
&= a_{k+1} \\
&= 6a_k - 5a_{k-1} && (\because \text{recursive definition}) \\
&= 6(5^k - 1) - 5(5^{k-1} - 1) && (\because P(k), P(k-1) \text{ are true}) \\
&= 5^{k+1} - 1 && (\because \text{simplify}) \\
&= \text{RHS of } P(k+1)
\end{aligned}
$$

Problems for practice

- Consider the sequence: $a_0 = 12$, $a_1 = 29$, and
  $a_k = 5a_{k-1} - 6a_{k-2}$ for all integers $k \geq 2$.
  Prove that $a_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.
- Consider the sequence: $a_1 = 3$, $a_2 = 5$, and
  $a_k = 3a_{k-1} - 2a_{k-2}$ for all integers $k \geq 3$.
  Prove that $a_n = 2^n + 1$ for all integers $n \geq 1$.
- Consider the sequence: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and
  $a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for all integers $k \geq 3$.
  Prove that $a_n \leq 3^n$ for all integers $n \geq 0$.
- Consider the sequence: $a_0 = 1$, $a_1 = 3$, and
  $a_k = 2a_{k-1} - a_{k-2}$ for all integers $k \geq 2$.
  Prove that $a_n = 2n + 1$ for all integers $n \geq 0$.
- Consider the sequence: $a_0 = 1$, $a_1 = 1$, and
  $a_k = 5a_{k-1} - 6a_{k-2}$ for all integers $k \geq 2$.
  Prove that $a_n = 3^n - 2^n$ for all integers $n \geq 0$.
- Consider the sequence: $a_1 = 1$, $a_2 = 8$, and
  $a_k = a_{k-1} + 2a_{k-2}$ for all integers $k \geq 3$.
  Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all integers $n \geq 1$.

Proof by mathematical induction: Example 2

Proposition

- Any integer greater than 1 is divisible by a prime number.

Proof by mathematical induction: Example 2

Proposition

- Any integer greater than 1 is divisible by a prime number.

Proof

Let $P(n)$ denote "n is divisible by a prime number."

- Basis step. $P(2)$ is true. ▷ How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 2$ and any $i \in [2, k]$. We want to show that $P(k+1)$ is true.

  Two cases:

  Case 1: [$k+1$ is prime.] $P(k+1)$ is true. ▷ How?

  Case 2: [$k+1$ is not prime.] We can write $k+1 = ab$ such that both $a, b \in [2, k]$ using the definition of a composite. This means, $k+1$ is divisible by a. We see that a is divisible by a prime due to the inductive hypothesis. As $k+1$ is divisible by a and a is divisible by a prime, $k+1$ is divisible by a prime, due to the transitivity of divisibility. Hence, $P(k+1)$ is true.

Proof by mathematical induction: Example 3

Proposition

- n cents can be obtained using a combination of 3- and 5-cent coins, for all integers $n \geq 8$.
  (Assume you have an infinite supply of 3- and 5-cent coins.)

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- n cents can be obtained using a combination of 3- and 5-cent coins, for all integers $n \geq 8$.
  (Assume you have an infinite supply of 3- and 5-cent coins.)

Proof

Let $P(n)$ denote "n cents can be obtained using a combination of 3- and 5-cent coins."

- Basis step. $P(8), P(9), P(10), P(11), P(12)$ are true. How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 12$ and any $i \in [8, k]$. We want to show that $P(k+1)$ is true.

  $$k + 1 = \underbrace{(k-4)}_{\text{Part 1}} + \underbrace{5}_{\text{Part 2}}$$

  Part 1 can be obtained by a collection of 3- and 5-cent coins because $P(k-4)$ is true and $(k-4) \geq 8$.
  Part 2 requires a 5-cent coin.
  Hence, $P(k+1)$ is true.

Proof by mathematical induction: Example 3

Proposition

- n cents can be obtained using a combination of 3 and 5 cent coins, for all integers $n \geq 8$.
  (Assume you have an infinite supply of 3- and 5-cent coins.)

Proof (improved)

Let $P(n)$ denote "n cents can be obtained using a combination of 3- and 5-cent coins."

- Basis step. $P(8)$, $P(9)$, and $P(10)$ are true. ▷ How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 10$ and any $i \in [8, k]$. We want to show that $P(k+1)$ is true.

  $$k + 1 = \underbrace{(k-2)}_{\text{Part 1}} + \underbrace{3}_{\text{Part 2}}$$

  Part 1 can be obtained by a collection of 3- and 5-cent coins because $P(k-2)$ is true and $(k-2) \geq 8$.
  Part 2 requires a 3-cent coin.
  Hence, $P(k+1)$ is true.

Proof by mathematical induction: Example 3

Proposition

- n cents can be obtained using a combination of 3 and 5 cent coins, for all integers $n \geq 8$.
  (Assume you have an infinite supply of 3 and 5 cent coins.)
  Use ordinary mathematical induction.

Proof

Let $P(n)$ denote "n cents can be obtained using a combination of 3- and 5-cent coins."

- Basis step. $P(8)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.

Proof by mathematical induction: Example 3

Proof (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.
  $$k + 1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(3 + 3 - 5)}_{\text{Part 2}}$$
  Part 1: $P(k)$ is true as $k \geq 8$.
  Part 2: Add two 3-cent coins and subtract one 5-cent coin.
  Hence, $P(k+1)$ is true.

Proof by mathematical induction: Example 3

Proof (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.
  $k+1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(3+3-5)}_{\text{Part 2}}$
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  Hence, $P(k+1)$ is true.

- Incorrect! What's wrong?

Proof by mathematical induction: Example 3

Proof (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.
  $k+1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(5+5-3-3-3)}_{\text{Part 2}}$
  Part 1: $P(k)$ is true as $k \geq 8$.
  Part 2: Add two 5-cent coins and subtract three 3-cent coins.
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Proof by mathematical induction: Example 3

Proof (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.
  $k + 1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(5 + 5 - 3 - 3 - 3)}_{\text{Part 2}}$
  Part 1: $P(k)$ is true as $k \geq 8$.
  Part 2: Add two 5-cent coins and subtract three 3-cent coins.
  Hence, $P(k+1)$ is true.

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Proof by mathematical induction: Example 3

Proof (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 8$. We want to show that $P(k+1)$ is true.
  Case 1. [There is a 5-cent coin in the set of k cents.]
  $k+1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(3+3-5)}_{\text{Part 2}}$
  Part 1: $P(k)$ is true as $k \geq 8$.
  Part 2: Add two 3-cent coins and subtract one 5-cent coin.
  Case 2. [There is no 5-cent coin in the set of k cents.]
  $k+1 = \underbrace{k}_{\text{Part 1}} + \underbrace{(5+5-3-3-3)}_{\text{Part 2}}$
  Part 1: $P(k)$ is true as $k \geq 8$.
  Part 2: Add two 5-cent coins and subtract three 3-cent coins.
  Hence, $P(k+1)$ is true.

Problems for practice

- Any collection of n people can be divided into teams of size
  - 5 and 6, for all integers $n \geq 35$
  - 4 and 7, for all integers $n \geq 18$
- Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Fibonacci sequence is: $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that:
  - $f_n \leq (7/4)^n$ for all $n \geq 0$.
  - $f_n \geq (3/2)^{n-1}$ for all $n \geq 1$.
  - $f_n \geq 2^{(n-1)/2}$ for all $n \geq 3$.
  - $f_n = (p^n - q^n)/\sqrt{5}$ for all $n \geq 1$,
    where $p = (1 + \sqrt{5})/2$ and $q = (1 - \sqrt{5})/2$.
    Hint: Note that p and q are the roots of $x^2 - x - 1 = 0$. So, $p^2 = p + 1$ and $q^2 = q + 1$.

Recursion

Recursive functions

Examples

- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
  $$f(n) = \begin{cases} 1 & \text{if } \ n = 0, \\ n \cdot f(n-1) & \text{if } \ n \geq 1. \end{cases}$$
  Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$
- Suppose $F(n) = n$th Fibonacci number. Then,
  $$F(n) = \begin{cases} 1 & \text{if } \ n = 0 \text{ or } 1, \\ F(n-1) + F(n-2) & \text{if } \ n \geq 2. \end{cases}$$
  Closed-form formula: $F(n) = ?$
- Suppose $C(n) = n$th Catalan number. Then,
  $$C(n) = \begin{cases} 1 & \text{if } \ n = 1, \\ \frac{4n-2}{n+1} \cdot C(n-1) & \text{if } \ n \geq 2. \end{cases}$$
  Closed-form formula: $C(n) = \frac{1}{n+1} \cdot \binom{2n}{n}$

Recursive functions

Examples

- Suppose $M(m, n) =$ product of $m, n \in \mathbb{N}$. Then,
  $$M(m,n) = \begin{cases} m & \text{if } \ n = 1, \\ M(m, n-1) + m & \text{if } \ n \geq 2. \end{cases}$$
  Closed-form formula: $M(m, n) = m \times n$
- Suppose $E(a, n) = a^n$, where $n \in \mathbb{W}$. Then,
  $$E(a,n) = \begin{cases} 1 & \text{if } \ n = 0, \\ E(a, n-1) \times a & \text{if } \ n \geq 1. \end{cases}$$
  Closed-form formula: $E(a, n) = a^n$
- Suppose $O(n) = n$th odd number $\in \mathbb{N}$. Then,
  $$O(n) = \begin{cases} 1 & \text{if } \ n = 1, \\ O(n-1) + 2 & \text{if } \ n \geq 2. \end{cases}$$
  Closed-form formula: $O(n) = 2n - 1$

Recursive functions

Examples

- Suppose $M(m, n)$ = product of $m, n \in \mathbb{N}$. Then,
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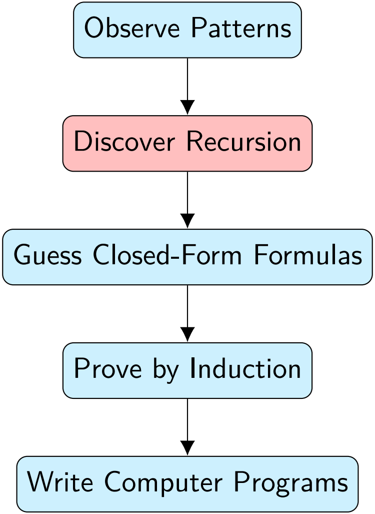
  Closed-form formula: $O(n) = 2n - 1$

- Recursive function for primes?

Relationship between induction and recursion



Relationship between induction and recursion

Recursion	Ordinary induction	Strong induction
Base case	Basis $f(a)$	Basis $f(a), f(a+1), \ldots, f(b)$
Recursive case	Induction $f(n-1)$	Induction $f(n-1), f(n-2), \ldots, f(a)$

Example: Arithmetic sequence

Definition

- Arithmetic sequence:
  $\langle a_0, a_1, a_2, \ldots, a_n \rangle = \langle a, a + d, a + 2d, \ldots, a + nd \rangle$
- Recurrence:
  $a_k = a_{k-1} + d$ for all integers $k \geq 1$
- nth term:
  $a_n = a + nd$ for all integers $n \geq 0$
- Summation:
  $a_0 + a_1 + \cdots + a_n = (n + 1)a + dn(n + 1)/2$

Example: Arithmetic sequence (Skydiving)

Problem

- A skydiver's speed upon leaving an airplane is approx. 9.8 m/sec one second after departure, $9.8 + 9.8 = 19.6$ m/sec two seconds after departure, and so forth. How fast would the skydiver be falling 60 seconds after leaving the airplane?

Example: Arithmetic sequence (Skydiving)

Problem

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Solution

- Let $s_n =$ skydiver speed (m/sec) n sec. after exiting the plane
- $s_n = s_0 + 9.8n$ for each integer $n \geq 0$
- $s_{60} = 0 + (9.8)(60) = 588$ m/sec.

Example: Geometric sequence

Definition

- Geometric sequence:
  $\langle a_0, a_1, a_2, \ldots, a_n \rangle = \langle a, ar, ar^2, \ldots, ar^n \rangle$
- Recurrence:
  $a_k = ra_{k-1}$ for all integers $k \geq 1$
- nth term:
  $a_n = ar^n$ for all integers $n \geq 0$
- Summation:
  $a_0 + a_1 + \cdots + a_n = a \left(\frac{r^{n+1} - 1}{r - 1} \right)$

Example: Geometric sequence (Compound interest)

Problem

- Suppose you deposit 100,000 dollars in your bank account for your newborn baby. Suppose you earn 3% interest compounded annually.

  How much will be the amount when your kid hits 21 years of age?

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  How much will be the amount when your kid hits 21 years of age?

Solution

- Suppose A_k = Amount in your account after k years. Then,
  $$A_k = \begin{cases} 100,000 & \text{if } \; k = 0, \\ (1 + 3\%) \times A_{k-1} & \text{if } \; k \geq 1. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
  $\boxed{A_k = ((1.03)^k \cdot 100,000) \text{ dollars}}$ ▷ How?
- Homework: Prove the formula using induction
- When your kid hits 21 years, $k = 21$, therefore
  $A_{21} = ((1.03)^{21} \cdot 100,000) \approx 186,029.46$ dollars

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- Suppose you deposit 100,000 dollars in your bank account for your newborn baby. Suppose you earn 3% interest compounded quarterly.
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Problem

- Suppose you deposit 100,000 dollars in your bank account for your newborn baby. Suppose you earn 3% interest compounded quarterly.
  How much will be the amount after 84 quarters (or periods)?

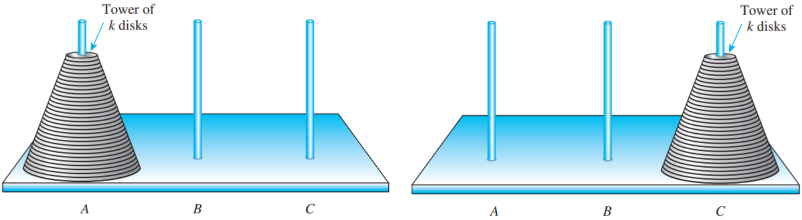
Solution

- Suppose A_k = Amount in your account after k quarters. Then,
  $$A_k = \begin{cases} 100,000 & \text{if } k = 0, \\ (1 + \frac{3}{4}\%) \times A_{k-1} & \text{if } k \geq 1. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
  $\boxed{A_k = ((1.0075)^k \cdot 100,000) \text{ dollars}}$ ▷ How?
- Homework: Prove the formula using induction
- After 84 quarters or pay periods (21 years), $k = 84$,
  $A_{84} = ((1.0075)^{84} \cdot 100,000) \approx 187,320.2$ dollars

Example: Towers of Hanoi

Problem

- There are k disks on peg 1. Your aim is to move all k disks from peg 1 to peg 3 with the minimum number of moves. You can use peg 2 as an auxiliary peg. The constraint of the puzzle is that at any time, you cannot place a larger disk on a smaller disk.
  What is the minimum number of moves required to transfer all k disks from peg 1 to peg 3?

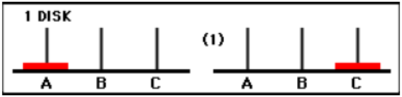


Example: Towers of Hanoi

Solution

Suppose $k = 1$. Then, the 1-step solution is:

1. Move disk 1 from peg A to peg C.



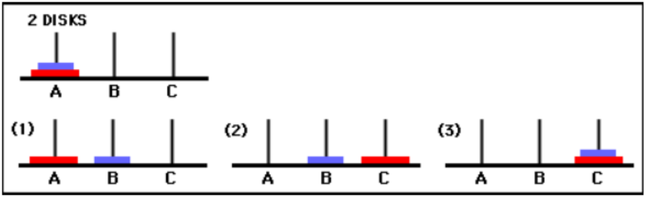
Source: http://mathforum.org/dr.math/faq/faq.tower.hanoi.html

Example: Towers of Hanoi

Solution

Suppose $k = 2$. Then, the 3-step solution is:

1. Move disk 1 from peg A to peg B.
2. Move disk 2 from peg A to peg C.
3. Move disk 1 from peg B to peg C.



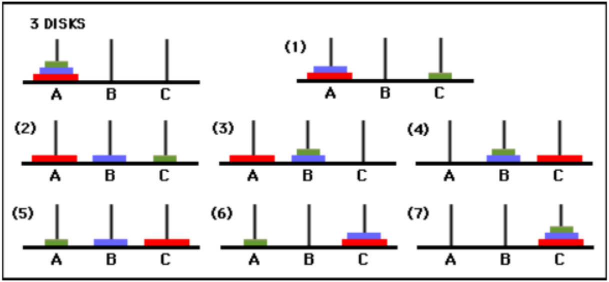
Source: http://mathforum.org/dr.math/faq/faq.tower.hanoi.html

Example: Towers of Hanoi

Solution

Suppose $k = 3$. Then, the 7-step solution is:

1. Move disk 1 from peg A to peg C.
2. Move disk 2 from peg A to peg B.
3. Move disk 1 from peg C to peg B.
4. Move disk 3 from peg A to peg C.
5. Move disk 1 from peg B to peg A.
6. Move disk 2 from peg B to peg C.
7. Move disk 1 from peg A to peg C.



Example: Towers of Hanoi

Solution

Suppose $k = 4$. Then, the 15-step solution is:

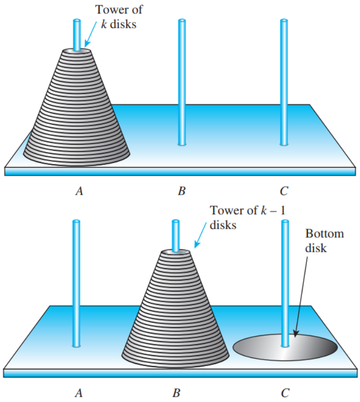
1. Move disk 1 from peg A to peg B.
2. Move disk 2 from peg A to peg C.
3. Move disk 1 from peg B to peg C.
4. Move disk 3 from peg A to peg B.
5. Move disk 1 from peg C to peg A.
6. Move disk 2 from peg C to peg B.
7. Move disk 1 from peg A to peg B.
8. Move disk 4 from peg A to peg C.
9. Move disk 1 from peg B to peg C.
10. Move disk 2 from peg B to peg A.
11. Move disk 1 from peg C to peg A.
12. Move disk 3 from peg B to peg C.
13. Move disk 1 from peg A to peg B.
14. Move disk 2 from peg A to peg C.
15. Move disk 1 from peg B to peg C.

Example: Towers of Hanoi

Solution

For any $k \geq 2$, the recursive solution is:

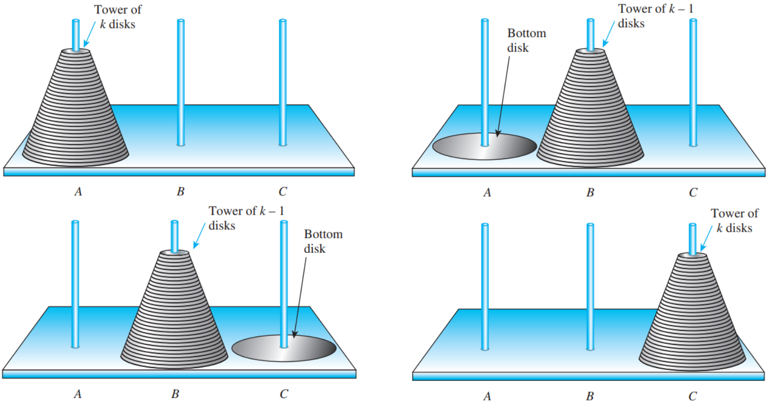
1. Transfer the top $k-1$ disks from peg A to peg B.
2. Move the bottom disk from peg A to peg C.
3. Transfer the top $k-1$ disks from peg B to peg C.



Example: Towers of Hanoi

Towers-of-Hanoi(k, A, C, B)

if $k = 1$ then
    Move disk k from A to C.
elseif $k \geq 2$ then
    Towers-of-Hanoi$(k-1, A, B, C)$
    Move disk k from A to C.
    Towers-of-Hanoi$(k-1, B, C, A)$



Example: Towers of Hanoi

Solution (continued)

- Let $M(k)$ denote the minimum number of moves required to move k disks from one peg to another peg. Then
$$M(k) = \begin{cases} 1 & \text{if } \; k = 1, \\ 2 \cdot M(k-1) + 1 & \text{if } \; k \geq 2. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
$$\boxed{M(k) = 2^k - 1}$$
▷ How?
- Homework: Prove the formula using induction

Example: Towers of Hanoi

Solution (continued)

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Generalization

- How do you solve the problem if there are p pegs instead of 3?

Example: Greatest common divisor (GCD)

Definition

- The greatest common divisor (GCD) of two integers a and b is the largest integer that divides both a and b.
- A simple way to compute GCD:
  1. Find the divisors of the two numbers
  2. Find the common divisors
  3. Find the greatest of the common divisors

Examples

- $\text{GCD}(2, 100) = 2$
- $\text{GCD}(3, 99) = 3$
- $\text{GCD}(3, 4) = 1$
- $\text{GCD}(12, 30) = 6$
- $\text{GCD}(1071, 462) = 21$

Example: Greatest common divisor (GCD)

Problem

- Compute the GCD of two integers efficiently.

Example: Greatest common divisor (GCD)

Problem

- Compute the GCD of two integers efficiently.

Solution

- Recurrence relation: Suppose $a > b$.

$$\mathsf{GCD}(a,b) = \begin{cases} a & \text{if } \ b = 0, \\ \mathsf{GCD}(b, a \bmod b) & \text{if } \ b \geq 1. \end{cases}$$

- $\mathsf{GCD}(1071, 462)$
$= \mathsf{GCD}(462, 1071 \bmod 462)$
$= \mathsf{GCD}(462, 147) \qquad (\because 1071 = 2 \cdot 462 + 147)$
$= \mathsf{GCD}(147, 462 \bmod 147)$
$= \mathsf{GCD}(147, 21) \qquad (\because 462 = 3 \cdot 147 + 21)$
$= \mathsf{GCD}(21, 147 \bmod 21)$
$= \mathsf{GCD}(21, 0) \qquad (\because 147 = 7 \cdot 21 + 0)$
$= 21$
- https://upload.wikimedia.org/wikipedia/commons/1/1c/Euclidean_algorithm_1071_462.gif

Example: Greatest common divisor (GCD)

- Recursive algorithm (Euclidean algorithm)

$\mathrm{GCD}(a, b)$

Input: Nonnegative integers a and b such that $a > b$.
Output: Greatest common divisor of a and b.

```
1. if b = 0 then
2.   return a
3. else
4.   return GCD(b, a mod b)
```

More Induction Problems

Example: $1/1^2 + 1/2^2 + \cdots + 1/n^2$

Problem

- Prove that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2$ for all natural numbers $n \geq 1$.

Example: $1/1^2 + 1/2^2 + \cdots + 1/n^2$

Problem

- Prove that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2$ for all natural numbers $n \geq 1$.

Solution

It is difficult to solve the problem directly.
It is sometimes easier to prove a stronger result.
We prove the stronger statement that $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.
Let $P(n)$ denote $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for $n \geq 2$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 2$. We need to prove that $P(k+1)$ is true.

Example: $1/1^2 + 1/2^2 + \cdots + 1/n^2$

Solution (continued)

- Induction step. Suppose that $P(k)$ is true for some $k \geq 2$.
  We need to prove that $P(k+1)$ is true.
  LHS of $P(k+1)$

$$
\begin{aligned}
&= \left(\tfrac{1}{1^2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{k^2}\right) + \tfrac{1}{(k+1)^2} \\
&< \left(2 - \tfrac{1}{k}\right) + \tfrac{1}{(k+1)^2} && (\because P(k) \text{ is true}) \\
&= 2 - \tfrac{(k+1)^2 - k}{k(k+1)^2} && (\because \text{taking common denominator}) \\
&= 2 - \tfrac{k(k+1)+1}{k(k+1)^2} && (\because \text{simplify}) \\
&< 2 - \tfrac{k(k+1)}{k(k+1)^2} && (\because \text{decrease 1 in the numerator}) \\
&= 2 - \tfrac{1}{k+1} && (\because \text{canceling common factors}) \\
&= \text{RHS of } P(k+1)
\end{aligned}
$$

Example: $x^n + 1/x^n$

Problem

- Suppose $x \in \mathbb{R}^+$ and $(x + 1/x) \in \mathbb{Z}$. Prove using strong induction that $(x^n + 1/x^n) \in \mathbb{Z}$ for all natural numbers n.

Example: $x^n + 1/x^n$

Problem

- Suppose $x \in \mathbb{R}^+$ and $(x + 1/x) \in \mathbb{Z}$. Prove using strong induction that $(x^n + 1/x^n) \in \mathbb{Z}$ for all natural numbers n.

Solution

Let $P(n)$ denote $(x^n + 1/x^n) \in \mathbb{Z}$ for $n \geq 1$.

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(i)$ is true for all $i \in [1, k]$, where $k \geq 1$. We need to prove that $P(k+1)$ is true.

  Observation: $\left(x^k + 1/x^k\right)(x + 1/x)$

  $= \left(x^{k+1} + 1/x^{k+1}\right) + \left(x^{k-1} + 1/x^{k-1}\right)$. So, we have

  LHS of $P(k+1)$

  $= \left(x^{k+1} + 1/x^{k+1}\right)$

  $= \left(x^k + 1/x^k\right)(x + 1/x) - \left(x^{k-1} + 1/x^{k-1}\right)$

  $=$ integer $\times$ integer $-$ integer $\quad$ ($\because$ inductive hypothesis)

  $=$ RHS of $P(k+1)$

Example: Chocolate bar

Problem

- Prove that breaking a chocolate bar with $n \geq 1$ pieces into individual pieces requires $n - 1$ breaks.

Example: Chocolate bar

Problem

- Prove that breaking a chocolate bar with $n \geq 1$ pieces into individual pieces requires $n-1$ breaks.

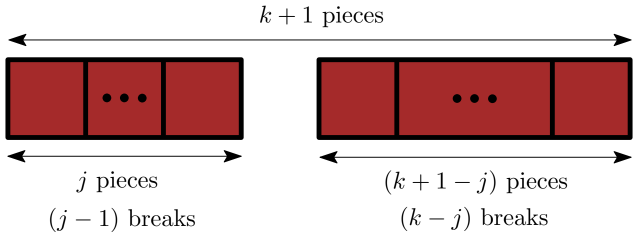
Solution

Let $P(n)$ denote "Breaking a chocolate bar with n pieces into individual pieces requires $n-1$ breaks".

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(i)$ is true for all $i \in [1, k]$, where $k \geq 1$. We need to prove that $P(k+1)$ is true.

Example: Chocolate bar

Solution (continued)



- Induction step. Suppose that $P(i)$ is true for all $i \in [1, k]$, where $k \geq 1$. We need to prove that $P(k+1)$ is true.
  Bar with $k+1$ pieces is split into two parts using 1 break.
  First part has j pieces and second part has $k+1-j$ pieces.
  #Breaks for chocolate bar with $k+1$ pieces
  $= 1+$ #Breaks for the first part $+$ #Breaks for the second part
  $= 1 + (j-1) + (k-j) \qquad (\because P(j), P(k+1-j)$ are true)
  $= k$. Hence, $P(k+1)$ is true.

Example: McCarthy's 91 function

Problem

- Let $M : \mathbb{Z} \to \mathbb{Z}$ be the following function.
  $$M(n) = \begin{cases} n - 10 & \text{if } n \geq 101, \\ M(M(n+11)) & \text{if } n \leq 100. \end{cases}$$
  Prove that $M(n) = 91$ for all integers $n \leq 100$.

Example: McCarthy's 91 function

Problem

- Let $M : \mathbb{Z} \to \mathbb{Z}$ be the following function.

$$M(n) = \begin{cases} n - 10 & \text{if } n \geq 101, \\ M(M(n+11)) & \text{if } n \leq 100. \end{cases}$$

Prove that $M(n) = 91$ for all integers $n \leq 100$.

Solution

- Basis step. $M(n) = 91$ for $n \in [90, 100]$. How?

$$\begin{aligned} M(n) &= M(M(n+11)) && (\because \text{By definition}) \\ &= M((n+11) - 10) && (\because (n+11) \geq 101) \\ &= M(n+1) \end{aligned}$$

So, $M(n) = M(101) = 91$ for $n \in [90, 100]$.

Example: McCarthy's 91 function

Solution (continued)

- Induction step. Suppose that $M(i) = 91$ for some $k \leq 90$ and any $i \in [k, 100]$. We want to show that $M(k-1) = 91$.

$$\begin{aligned} M(k-1) &= M(M(k-1+11)) \qquad (\because \text{By definition}) \\ &= M(M(k+10)) \qquad (\because \text{Simplify}) \\ &= M(91) \qquad (\because \text{Inductive hypothesis, because} \\ &\qquad\qquad\qquad k < (k+10) \leq 100) \\ &= 91 \qquad (\because \text{Base case}) \end{aligned}$$

Example: McCarthy's 91 function

Knuth's generalization

- Suppose $a \in \mathbb{Z}$ and $b, c, d \in \mathbb{N}$. Consider the function.

$$K(x) = \begin{cases} x - b & \text{if } x > a, \\ \underbrace{K(K(\cdots K(}_{c \text{ times}} x + d) \cdots)) & \text{if } x \leq a. \end{cases}$$

  Let $\Delta = (d - (c-1)b) > 0$.
- Then, the function evaluates to

$$K(x) = \begin{cases} x - b & \text{if } x > a, \\ a + \Delta - b - ((a - x) \bmod \Delta) & \text{if } x \leq a. \end{cases}$$

- Reference: https://arxiv.org/abs/cs/9301113

Example: Staircase problem

Problem

- You need to ascend a staircase consisting of n steps. The number of steps you can climb at a time is at most b. What is the number of ways of ascending the staircase?

Example: Staircase problem

Problem

- You need to ascend a staircase consisting of n steps. The number of steps you can climb at a time is at most b. What is the number of ways of ascending the staircase?

Solution

- Suppose S_k = #ways of ascending a staircase with k steps. Then,

$$S_k = \begin{cases} \boxed{?} & \text{if } k \in [1, b], \\ \boxed{?} & \text{if } k \in [b+1, n]. \end{cases}$$

Example: Staircase problem

Solution (continued)

Steps	Ways	#Ways
1	1	1
2	$1+1,\ 2$	2
3	$1+1+1,\ 1+2,\ 2+1,\ 3$	4
4	$1+1+1+1,\ 1+1+2,\ 1+2+1,\ 2+1+1,\ 2+2,$ $1+3,\ 3+1,\ 4$	8

- Base case.

  Is $S_k = 2^{k-1}$? for $k \in [1, b]$. ▷ Proof?

  $$S_k = \begin{cases} 1 & \text{if } \ k = 1, \\ S_{k-1} + \cdots + S_1 + 1 & \text{if } \ k \in [2, b]. \end{cases}$$ ▷ How?

  Solving the recurrence, we get $S_k = 2^{k-1}$ for $k \in [1, b]$.
- Recursion case.

  $S_k = S_{k-1} + S_{k-2} + \cdots + S_{k-b}$ for $k \in [b+1, n]$.

Example: Continued fractions

Problem

- Prove that every rational number can be written as a continued fraction.

Example: Continued fractions

Problem

- Prove that every rational number can be written as a continued fraction.

Solution

- A continued fraction an expression of the form: $a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots + \frac{1}{a_n}}}}$
- Formally, a continued fraction is:
  (i) integer, or
  (ii) integer + 1/(continued fraction)
- Example: Golden ratio $= \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$

Example: Continued fractions

Solution (continued)

- Given an integer n and a natural number d, we can write $n = qd + r$ such that $r \in [0, d-1]$.
- Observe that the rational number n/d can be written as: $\frac{n}{d} = q + \frac{r}{d} = q + \frac{1}{\frac{d}{r}}$
- Every rational can be written with a positive denominator.

Let $P(d)$ denote
"Any rational with denominator d has a continued fraction".

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step. Suppose that $P(i)$ is true for all $i \in [1, d]$, for some $d \geq 1$. We need to prove that $P(d+1)$ is true.

Example: Continued fractions

Solution (continued)

- Induction step. Suppose that $P(i)$ is true for all $i \in [1, d]$, for some $d \geq 1$. We need to prove that $P(d+1)$ is true.
  Consider the rational $\frac{n}{d+1}$ for some integer n.
  Using the division theorem, we have $n = q(d+1) + r$, where $r \in [0, d]$.
  We consider two cases:
  - Case $[r = 0]$. Then, $\frac{n}{d+1} = q =$ integer.
    An integer is a continued fraction.
  - Case $[r \neq 0]$. Then, $\frac{n}{d+1} = q + \frac{r}{d+1} = q + \frac{1}{\frac{d+1}{r}}$.
    $\frac{d+1}{r}$ is a continued fraction due to inductive hypothesis because $P(r)$ is true. $(\because r \in [1, d])$
    Integer + 1/(continued fraction) is a continued fraction.

  Hence, $P(d+1)$ is true.