Sample Problems and Solutions
Discrete Mathematics

[Propositional Logic]

1. [10 points] Give negations of the following statements using formal mathematical language. Your negations must not use conditionals or biconditionals.
   (a) [1 point] if p, then q
   (b) [1 point] p only if q
   (c) [1 point] p if and only if q
   (d) [1 point] p is necessary for q
   (e) [1 point] p is sufficient for q
   (f) [1 point] p is necessary but not sufficient for q
   (g) [1 point] p is not necessary but sufficient for q
   (h) [1 point] p is necessary and sufficient for q
   (i) [2 points] p is neither necessary nor sufficient for q

   Solution.
   (a) $p \wedge \sim q$
   (b) $p \wedge \sim q$
   (c) $p \oplus q$
   (alternate answer) $(p \wedge \sim q) \vee (\sim p \wedge q)$
   (d) $q \wedge \sim p$
   (e) $p \wedge \sim q$
   (f) $(q \wedge \sim p) \vee (\sim p \vee q)$
   (g) $(\sim q \vee p) \vee (p \wedge \sim q)$
   (h) $p \oplus q$
   (alternate answer) $(q \wedge \sim p) \vee (p \wedge \sim q)$
   (i) $(\sim q \vee p) \vee (\sim p \vee q) \equiv \mathbf{t}$

2. [5 points] Determine if the following deduction rule is valid:
   $p \to q$
   $\sim q \to \sim r$
   ———-
   $\therefore p \to r$

   Solution.

p	q	r	$p \to q$	$\sim q$	$\sim r$	$p \to q$	$\sim q \to \sim r$	$p \to r$
T	T	T	T	F	F	T	T	T
T	T	F	T	F	T	T	T	F
T	F	T	F	T	F	F	F	
T	F	F	F	T	T	F	T	
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	T	F	
F	F	F	T	T	T	T	T	T

The deduction rule is invalid.

3. [5 points] Determine if the following deduction rule is valid:
   $p \vee q$
   $\sim p \vee \sim q$
   ———-
   $\therefore p \oplus q$

   Solution.

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \oplus q$
T	T	T	F	F	F	
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	

The deduction rule is valid.

4. [5 points] Determine if the following deduction rule is valid:
   $p \to \sim q$
   $p \wedge q$
   ———-
   $\therefore p \vee q$

   Solution.

p	q	$\sim q$	$p \to \sim q$	$p \wedge q$	$p \vee q$
T	T	F	F	T	
T	F	T	T	F	
F	T	F	T	F	
F	F	T	T	F	

The deduction rule is valid (vacuous truth).

5. [5 points] Determine the validity of the following statements.
   (a) [1 point] $p\wedge \sim p \equiv \mathbf{t}$
   (b) [1 point] $p \rightarrow p \equiv \mathbf{t}$
   (c) [1 point] $p \oplus q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
   (d) [1 point] $p \rightarrow (q \rightarrow r) \equiv (p \rightarrow q) \rightarrow r$
   (e) [1 point] $(p \wedge q\wedge \sim r) \vee p \equiv p$

   Solution.
   (a) False ($p\wedge \sim p \equiv \mathbf{c}$)
   (b) True
   (c) False ($p \oplus q \equiv (\sim p \wedge q) \vee (\sim q \wedge p)$)
   (d) False (Implication is not associative)
   (e) True

6. [5 points] Determine how many ordered truth assignments satisfy the following statement:

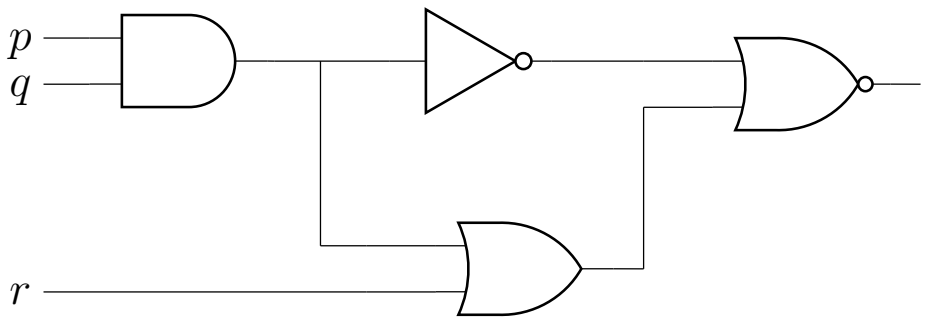
$$\sim (p \wedge q \wedge r) \vee (p\wedge \sim q \wedge r)\vee \sim p$$

   Solution.

p	q	r	$p \wedge q \wedge r$	$\sim (p \wedge q \wedge r)$	$p\wedge \sim q \wedge r$	$\sim p$	Final Expression
T	T	T	T	F	F	F	F
T	T	F	F	T	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T

   7 truth assignments.

7. [5 points] Given the following logic circuit, construct an Input-Output table.



   Solution.

p	q	r	$p \wedge q$	$\sim (p \wedge q)$	$(p \wedge q) \vee r$	$\sim (\sim (p \wedge q) \vee ((p \wedge q) \vee r))$
1	1	1	1	0	1	0
1	1	0	1	0	1	0
1	0	1	0	1	1	0
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	1	0	0	1	0	0
0	0	1	0	1	1	0
0	0	0	0	1	0	0

8. [5 points] Construct the truth table for the following logical expression:

$$((p \rightarrow q) \wedge r) \leftrightarrow \sim (p \oplus q)$$

Solution.

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \wedge r$	$p \oplus q$	$\sim (p \oplus q)$	$(p \rightarrow q) \wedge r \leftrightarrow \sim (p \oplus q)$
T	T	T	T	T	F	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	F	F
F	T	F	T	F	T	F	T
F	F	T	T	T	F	T	T
F	F	F	T	F	F	T	F

9. [5 points] Let us define a new operator $\diamond$ with the following truth table:

p	q	$p \diamond q$
T	T	T
T	F	T
F	T	F
F	F	T

Let us extend another of de Morgan's laws as shown below. Determine whether this property holds for the operator $\diamond$:

$$p \diamond (q \wedge r) \stackrel{?}{\equiv} (p \diamond q) \vee (p \diamond r)$$

Solution.

p	q	r	$q \wedge r$	$p \diamond q$	$p \diamond r$	$p \diamond (q \wedge r)$	$(p \diamond q) \vee (p \diamond r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	T	F	F	F	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	T	T	T

Yes, the property holds.