Discrete Mathematics (Propositional Logic)

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Puzzle

A beautiful princess and an intelligent philosopher were in love. After learning the relationship between the princess and the philosopher, the king vowed to give the philosopher a death sentence. The next day, the philosopher was brought to the court.

The king asked the philosopher to give one sentence.

Puzzle (continued)

The death options for the philosopher were as follows:

- If the sentence is true, then he would be hanged.
- If the sentence is false, then he would be beheaded.
- If the sentence cannot be assigned a truth value, then he would be fed to lions.
- If the sentence is inherently self-contradictory, then he would be shot with an arrow.
- If the sentence is an unproven mathematical statement, then he would be left to die from starvation.
- If the sentence is not stated within ten minutes, then he would be poisoned.
- If the sentence cannot be understood easily, then he would thrown from a tall mountain.

Puzzle (continued)



The philosopher was ready to face death for his love. He used his extraordinary philosophic acumen and came up with a sentence. The sentence is so profound that the king let the philosopher free, united the princess and the philosopher, and finally made the philosopher his successor to the throne.

What was the philosopher's sentence?

Propositional logic

Contents

- Compound Statements
- Logical Arguments
- Digital Logic Circuits

Definition

• Branch of mathematics that deals with the verification of truth/falsity of mathematical statements/assertions

Scope

- Study of valid arguments, inference (induction + deduction), proofs, proof techniques, proving theorems, and, paradoxes and fallacies
- Logic in philosophy, mathematics, computer science, and everyday life are somewhat different
- Our focus: Logic in mathematics (started since mid-19th century)

Uses and applications

- Everyday life. To reduce problems caused by ignorance.
- Philosophy. Logic models reasoning.
 "Why study logic?" = "Why know truth?"
- Remove ambiguity.

English is ambiguous:

- "The cat chased the mouse until it fell."
- "My mother never made cake, which we hated."
- "Let's stop controlling people."
- "Time flies like an arrow, fruit flies like a peanut." Mathematics and programming languages are unambiguous. (Can you think of some examples for ambiguous math statements or computer programs?)

Uses and applications

Mathematics.

To prove theorems.

• Computer science.

To prove correctness of software/hardware.

Used in computer circuit design.

Used in modeling programming languages.

Used in the design of expert systems, robots, and artificial in-telligence.

Rigor	Truth type	Field	Truth teller
0	Word of God	Religion	God/Priests
1	Authoritative truth	Business	Boss
2	Legal truth	Judiciary	Law/Judge/Law makers
3	Philosophical truth	Philosophy	Plausible argument
4	Scientific truth	Physical sciences	Experiments/observation
5	Statistical truth	Statistics	Data sampling
6	Mathematical truth	Mathematics	Logical deduction

Order	Name	Topics
0	Propositional logic	Variables $+$ logical connectives
1	Predicate logic	Previous + quantifiers
2	Predicate logic	Previous + more quantifiers
3	Predicate logic	Previous + more quantifiers
Higher	Predicate logic	Previous + more quantifiers

What is a statement?

Definition

• A statement or proposition is a sentence for which a truth value (either true or false) can be assigned

Classes of statements

- True statements. "The atomic number of Oxygen is 8."
- False statements. "1 + 1 = 3."
- Not propositions.

• "
$$x + y = 5$$
." \triangleright Why

• "
$$A \times B = 0$$
 implies $A = 0$ or $B = 0$." \triangleright Why

- "Natasha is beautiful."
- Truth value currently unknown.
 - Goldbach's conjecture
- Truth values change with time/scenarios.
 - "Today is Sunday." written on a paper slip
- Logical paradoxes.
 - "This statement is false."

⊳ Why?

 \triangleright Why?

 \triangleright Why?

 \triangleright Why?



Definition

- A propositional variable is a lowercase letter such as *p*, *q*, or *r* used to represent a proposition.
- A propositional variable can be true (T) or false (F)
- Algebra uses numeric variables; Logic uses propositional variables

Compound Statements

What is a compound statement?

Definition

• A compound statement is a complex sentence that is obtained by joining propositional variables using logical connectives

Logical operator	Notation	Read as
Negation	$\sim p$	not p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	$p \operatorname{or} q$
Conditional	$p \rightarrow q$	p implies $qif p, then qp$ only if qq if pq provided that p
Biconditional	$p \leftrightarrow q$	p if and only if q
Logical equivalence	$p \equiv q$	p logically equivalent to q

• How do you find the truth values of a compound statement?

What is a compound statement?

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Logical equivalence	$p \equiv q$	p logically equivalent to q

• How do you find the truth values of a compound statement? Truth tables!

Negation $(\sim p)$

Definition

• Negation of a statement p, denoted by $\sim p,$ is a statement obtained by changing the truth value of p.



Examples

• p: "x is a real number such that x < 4." $\sim p:$ "x is a real number such that $x \ge 4$." • p: "The equation $a^4 + b^4 + c^4 = d^4$ has no solution when a, b, c, d are positive integers." $\sim p:$ "The equation $a^4 + b^4 + c^4 = d^4$ has a solution when a, b, c, d are positive integers."

Conjunction $(p \land q)$

Definition

• Conjunction of statements p and q, denoted by $p \land q$, is a statement such that it is true if both p and q are true and it is false, otherwise.



- p : "I will get an accept from MIT."
 - q : "I will get an accept from Stanford."
 - $p \wedge q$: "I will get accepts from both MIT and Stanford."
- Write $0 \le x \le 1$ using conjunction $(x \ge 0) \land (x \le 1)$

Disjunction $(p \lor q)$

Definition

Disjunction of statements p and q, denoted by p∨q, is a statement such that it is false if both p and q are false and it is true, otherwise.

p	q	$p \lor q$
Т	Т	Т
Т	F	Т
F	Т	Т
F	F	F

- p : "I will get an accept from MIT."
 - q: "I will get an accept from Stanford."
 - $p \lor q$: "I will get an accept from either MIT or Stanford."
- "Either you get heads or I get heads." (inclusive or)

Exclusive or $(p \oplus q)$

Definition

• Exclusive or of statements p and q, denoted by $p \oplus q$, is defined as p or q but not both. It is computed as $(p \lor q) \land \sim (p \land q)$

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \lor q) \land \sim (p \land q)$
Т	Т	Т	Т	F	F
Т	F	Т	F	Т	Т
F	Т	Т	F	Т	Т
F	F	F	F	Т	F

 In English, the term "or" can mean either inclusive or (i.e., logical or) or exclusive or based on contexts. Which or is this?

- p : "I will choose MIT."
 - q : "I will choose Stanford."
 - $p \oplus q$: "I will choose MIT or Stanford."
- "I get either heads or tails." (exclusive or)

What is a statement form?

Definition

 Statement form or propositional form is a compound statement with propositional variables (such as p, q, r) and logical connectives (such as ∼, ∧, ∨).

Examples

•
$$(p \lor q) \land \sim (\sim p \land r)$$

•
$$(\sim p \land q \land r) \lor (q \lor \sim r)$$

Evaluation

- How should we evaluate the logical forms? Write truth tables starting from the innermost parenthesized expressions and then the next innermost parenthesized expressions and so on
- A truth table with n propositional variables will have 2^n rows

What is the precedence of the logical operators?

Priority	Operator	Comments
1	~	Evaluate \sim first
2	\wedge	Evaluate \land and \lor next; Use
	V	parenthesis to avoid ambiguity
3 →		$Evaluate \to and \leftrightarrow next; Use$
	\leftrightarrow	parenthesis to avoid ambiguity
4	≡	$Evaluate \equiv last$

What is logical equivalence $(p \equiv q)$?

Definition

• Two statement forms p and q are logically equivalent, denoted by $p \equiv q$, if and only if they have the same truth values for all possible combination of truth values for the propositional variables

Checking logical equivalence

- 1. Construct and compare truth tables (most powerful)
- 2. Use logical equivalence laws

Problem

• Show that
$$p \wedge (q \vee r) \not\equiv (p \wedge q) \vee r$$

$\left(\begin{array}{c} p \end{array} \right)$	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \land q) \lor r$
Т	Т	Т	Т	Т	Т	Т
Т	Т	F	Т	Т	Т	Т
Т	F	Т	Т	Т	F	Т
Т	F	F	F	F	F	F
F	Т	Т	Т	F	F	Т
F	Т	F	Т	F	F	F
F	F	Т	Т	F	F	Т
F	F	F	F	F	F	F

• 5th column \neq 7th column

Problem

• Show that
$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \land q) \lor (p \land r)$
Т	Т	Т	Т	Т	Т	Т	Т
Т	Т	F	Т	Т	Т	F	Т
Т	F	Т	Т	Т	F	Т	Т
Т	F	F	F	F	F	F	F
F	Т	Т	Т	F	F	F	F
F	Т	F	Т	F	F	F	F
F	F	Т	Т	F	F	F	F
F	F	F	F	F	F	F	F

• 5th column = 8th column

Definitions

- A tautology is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables.
- A contradication is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables.

$\bullet \ p \lor \sim p$	▷ Tautology
• $p \wedge \sim p$	▷ Contradiction

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \lor q \equiv q \lor p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \lor q) \lor r \equiv p \lor (q \lor r)$
Distributive laws	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \lor \mathbf{c} \equiv p$
Negation laws	$p \vee \sim p \equiv \mathbf{t}$	$p\wedge \sim p \equiv \mathbf{c}$
Double neg. law	$\sim (\sim p) \equiv p$	
Idempotent laws	$p \wedge p \equiv p$	$p \lor p \equiv p$
Uni. bound laws	$p \lor \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan's laws	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \lor q) \equiv \sim p \land \sim q$
Absorption laws	$p \lor (p \land q) \equiv p$	$p \land (p \lor q) \equiv p$
Negations	$\sim {f t} \equiv {f c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Definition

 Conditional or implication is a compound statement of the form "if p, then q". It is denoted by p → q and read as "p implies q". It is false when p is true and q is false, and it is true, otherwise.

p	q	$p \rightarrow q$
Т	Т	Т
Т	F	F
F	Т	Т
F	F	Т

- "If $b^2 4ac > 0$, then $ax^2 + bx + c = 0$ has two distinct real solutions."
- Write "All polynomials are differentiable." as an implication. "If P is a polynomial, then P is differentiable."

- In $p \to q$, p is called hypothesis/premise/antecendent, and q is called conclusion/consequence
- $p \to q$ is called conditional because the truth of q is conditioned upon the truth of p
- $p \rightarrow q$ is also called:

f "if p , then q "	" p implies q "
" p only if q "	"if <i>p</i> , <i>q</i> "
" q follows from p "	" q , provided that p "
"not p unless q "	" q if/when/whenever q "
" p is sufficient for q "	"a sufficient condition for q is p "
" q is necessary for p "	"a necessary condition for p is q "



Tomo is not sunny Did not go to the beach Promise is not broken (T)	L	renne is net sainij		
	ĺ	Tomo is not sunny	Did not go to the beach	Promise is not broken (T)

• $p \to q$ being true because p is false is called vacuously true or true by default

Definitions

• Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

• Converse of
$$p \to q$$
 is $q \to p$

• Inverse of
$$p \rightarrow q$$
 is $\sim p \rightarrow \sim q$

Identities

- Conditional \equiv Contrapositive
- $\bullet \ \ {\sf Conditional} \not\equiv {\sf Converse}$
- $\bullet \ \ Conditional \not\equiv Inverse$
- Converse \equiv Inverse

Formulas

•
$$(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$$

• $(p \rightarrow q) \not\equiv (q \rightarrow p)$
• $(p \rightarrow q) \not\equiv (\sim p \rightarrow \sim q)$
• $(q \rightarrow p) \equiv (\sim p \rightarrow \sim q)$

▷ Useful for proofs

 \triangleright Useful for proofs



Definitions • p is a sufficient condition for q means if p then q • p is a necessary condition for q means if $\sim p$ then $\sim q$ or if q then p • p only if q means if $\sim q$ then $\sim p$ or if p then q Example • For real x, x = 1 is a sufficient condition for $x^2 = 1$ i.e., If x = 1 then $x^2 = 1$ \triangleright True • For real $x, x^2 = 1$ is a necessary condition for x = 1i.e., If $x^2 \neq 1$ then $x \neq 1$ \triangleright True • For real x, x = 1 only if $x^2 = 1$ i.e., If $x^2 \neq 1$, then $x \neq 1$ \triangleright True

Definitions

• The biconditional of p and q is of the form "p if and only if q" and is denoted by $p \leftrightarrow q$. It is true when p and q have the same truth value, and it is false, otherwise.

$$\bullet \ p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \to q) \land (q \to p)$
Т	Т	Т	Т	Т
Т	F	F	Т	F
F	Т	Т	F	F
F	F	Т	Т	Т

Examples

• Assume x and y are real numbers.

" $x^2 + y^2 = 0$ if and only if x = 0 and y = 0."

• A polygon is a triangle if and only if it has exactly three sides."

Logical Arguments

Definitions

- Logical argument. Sequence of statements aimed at demonstrating the truth of an assertion
- Conclusion. Last statement in an argument
- Premises. Last-but-one statements in an argument
- Formal logic. The framework for determining the validity or invalidity of arguments
- Proof/Derivation/Logical deduction. A valid sequence of statements used for establishing new mathematical truths (or conclusions or propositions or theorems) from acceptable/established mathematical truths (premises or axioms or assumptions)


If $\mathsf{Premise}_1$ and $\mathsf{Premise}_2$ and \cdots and $\mathsf{Premise}_m$, then $\mathsf{Conclusion}$.

Definition

- An argument is valid if the conclusion follows necessarily from the premises
- An argument is valid means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true





Examples	
• If $x > 2$, then $x^2 > 4$.	
x > 2.	
Therefore, $x^2 > 4$.	▷ Valid argument
• If $x > 2$, then $x^2 > 4$.	
$x^2 > 4.$	
Therefore, $x > 2$.	Invalid argument
• If $x > 2$, then $x^2 > 4$.	
$x^2 \le 4.$	
Therefore, $x \leq 2$.	Valid argument
• If $x > 2$, then $x^2 > 4$.	
$x \leq 2.$	
Therefore, $x^2 \leq 4$.	Invalid argument

Examples	
• If p , then q .	
p.	
Therefore, q .	Valid argument
• If p , then q .	
q.	
Therefore, p .	Invalid argument
• If p , then q .	
$\sim q$.	
Therefore, $\sim p$.	Valid argument
• If p , then q .	
$\sim p.$	
Therefore, $\sim q$.	Invalid argument

Method 1: Construct a truth table

- 1. Identify the premises and conclusion
- 2. Construct a truth table for premises and conclusion
- 3. A row of the truth table in which all the premises are true is called a critical row.

If there is a critical row in which the conclusion is false, then the argument is invalid. If the conclusion in every critical row is true, then the argument is valid.

Method 2: Find a counterexample

- 1. If there is a counterexample that has all premises true and false conclusion, then the argument is invalid.
- 2. Failing to find a counterexample does not prove that the argument is valid.

How to determine if an argument is valid/invalid?

Problem

• Determine the validity of the argument:

$$\begin{array}{l} p \rightarrow q \lor \sim r \\ q \rightarrow p \wedge r \end{array}$$

$$\therefore p \to r$$

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \rightarrow r$
Т	Т	Т	F	Т	Т	Т	Т	Т
Т	Т	F	Т	Т	F	Т	F	
Т	F	Т	F	F	Т	F	Т	
Т	F	F	Т	Т	F	Т	Т	F
F	Т	Т	F	Т	F	Т	F	
F	Т	F	Т	Т	F	Т	F	
F	F	Т	F	F	F	Т	Т	Т
F	F	F	Т	Т	F	Т	Т	Т

• 4th row (a critical row) has false conclusion. Invalid argument.

What is a rule of inference?

Definition

• A rule of inference is a valid argument form that can be used to establish logical deductions

Name	Rule	Name	Rule	
Modus Ponens	$p \to q$	Elimination	$p \vee q$	$p \vee q$
	p		$\sim q$	$\sim p$
	$\therefore q$		$\therefore p$	$\therefore q$
Modus Tollens	$p \to q$	Transitivity	$p \to q$	
	$\sim q$		$q \to r$	
	$\therefore \sim p$		$\therefore p \to r$	
Proof by division	$p \vee q$	Generalization	p	q
into cases	$p \to r$		$\therefore p \lor q$	$\therefore p \lor q$
	$q \to r$	Specialization	$p \wedge q$	$p \wedge q$
	$\therefore r$		$\therefore p$	$\therefore q$
Conjunction	p	Contradiction	$\sim p \to c$	
	q		$\therefore p$	
l	$\therefore p \land q$			

Definition • A syllogism is a rule of inference with two premises and a conclusion. Argument Premise₁ (Major premise) Premise₂ (Minor premise) .:. Conclusion Examples • Modus ponens Modus tollens

Rule of inference: Modus ponens

Definition

- It has the form: If p, then q p
 - $\therefore q$
- The term modus ponens in Latin means "method of affirming"

p	q	$p \rightarrow q$	p	q
Т	Т	Т	Т	Т
Т	F	F	Т	
F	Т	Т	F	
F	F	Т	F	

Example

• If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

The sum of the digits of 371,487 is divisible by 3.

 \therefore 371,487 is divisible by 3.

Rule of inference: Modus tollens

Definition

- It has the form: If p, then q
 - $\sim q$
 - $\therefore \sim p$
- The term modus tollens in Latin means "method of denying"

p	q	$p \rightarrow q$	$\sim q$	$\sim p$
Т	Т	Т	F	
Т	F	F	Т	
F	Т	Т	F	
F	F	Т	Т	Т

Example

- If Zeus is human, then Zeus is mortal. Zeus is not mortal.
 - .:. Zeus is not human.

Rule of inference: Generalization



Rule of inference: Specialization



Example

• Ana knows numerical analysis and Ana knows graph algorithms.

F

∴ (in particular) Ana knows graph algorithms

FF

Rule of inference: Conjunction

Definition

- It has the form:
 - $egin{array}{c} p \ q \ \therefore p \wedge q \end{array}$



Example

- Lily loves mathematics.
 - Lily loves algorithms.
 - ... Lily loves both mathematics and algorithms.

Rule of inference: Elimination

Definition

• It has the form:

```
p \vee q
```

```
\sim q
```

```
\therefore p
```

• Intuition: When you have only two possibilities and you can rule one out, the other must be the case

p	q	$p \vee q$	$\sim q$	p
Т	Т	Т	F	Г
Т	F	Т	Т	Т
F	Т	Т	F	F
F	F	F	Т	F

Example

• Suppose
$$x - 3 = 0$$
 or $x + 2 = 0$.

Also, suppose x is nonnegative.

$$\therefore x = 3.$$

Definition

- It has the form:
 - $p \to q$
 - $q \to r$
 - $\therefore p \to r$
- Can be generalized to a chain with any number of conditionals

Example

- If 18,486 is divisible by 18, then 18,486 is divisible by 9. If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.
 - \therefore If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Definition
• It has the form:
$p \lor q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore r$
Example
• x is positive or x is negative.
If x is positive, then $x^2 > 0$.
If x is negative, then $x^2 > 0$.
$\therefore x^2 > 0.$

Definition

• It has the form:

```
\sim p \to \mathbf{c}
```

```
\therefore p
```

- Intuition: If an assumption leads to a contradiction, then that assumption must be false
- Extensively used in a proof technique called proof by contradiction



What is a fallacy?

Definition

• A fallacy is an error in reasoning that results in an invalid argument

Types

- 1. Using ambiguous premises, and treating them as if they were unambiguous
- 2. Circular reasoning (assuming what is to be proved without having derived it from the premises)
- 3. Jumping to a conclusion (without adequate grounds)
- 4. Converse error
- 5. Inverse error

Fallacy: Converse error

Definition

• It has the form:

```
\begin{array}{c} p \rightarrow q \\ q \end{array}
```

```
\therefore p
```

- Intuition: A conditional is not equivalent to its converse
- Also known as the fallacy of affirming the consequent
- Superficially resembles modus ponens but is invalid



Example

• If x > 2, then $x^2 > 4$. $x^2 > 4$. $\therefore x > 2$.

Fallacy: Inverse error

Definition

• It has the form:

$$p \to q$$

$$\sim p$$

$$\therefore \sim q$$

- Intuition: A conditional is not equivalent to its inverse
- Also known as the fallacy of denying the antecedent
- Superficially resembles modus tollens but is invalid

p	q	$p \rightarrow q$	$\sim p$	$\sim q$
Т	Т	Т	F	F
Т	F	F	F	Т
F	Т	Т	Т	F
F	F	Т	Т	Т

Example

• If
$$x > 2$$
, then $x^2 > 4$.
 $x < 2$

$$\therefore x^2 \le 4$$

Validity \neq Truthfulness; Invalidity \neq Falsity

Types

- A valid argument can have a false conclusion
- An invalid argument can have a true conclusion

Examples

• Valid argument with false conclusion (Modus ponens) If Isaac Newton was a scientist, then Albert Einstein was not a scientist.

Isaac Newton was a scientist.

- ... Albert Einstein was not a scientist.
- Invalid argument with true conclusion (Converse error) If New York is a big city, then New York has tall buildings. New York has tall buildings.
 - \therefore New York is a big city.

Definition

- A sound argument is an argument that is valid and has true premises
- Validity shows that an argument is logical Soundness shows that an argument is truthful

Examples

• Valid argument with true premises (Modus ponens)

If Isaac Newton was a scientist, then Albert Einstein was a scientist.

Isaac Newton was a scientist.

: Albert Einstein was a scientist.

Problem

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

- (a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
- (b) If my glasses are on the kitchen table, then I saw them at breakfast.
- (c) I did not see my glasses at breakfast.
- (d) I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
- (e) If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Solution
Let:
• RK = I was reading the newspaper in the kitchen.
• $GK = My$ glasses are on the kitchen table.
• SB = I saw my glasses at breakfast.
• RL = I was reading the newspaper in the living room.
• GC = My glasses are on the coffee table.
Given statements:
(a) $RK \to GK$
(b) $GK \to SB$
$(c) \sim SB$
(d) RL ∨ RK
(e) $RL \rightarrow GC$

Solution

Sequence of steps to draw the conclusion: \triangleright by (a) 1. $\mathsf{RK} \to \mathsf{GK}$ $\mathsf{GK} \to \mathsf{SB}$ \triangleright by (b) $\cdot \text{ RK} \rightarrow \text{SB}$ \triangleright by transitivity 2. RK \rightarrow SB \triangleright by the conclusion of (1) $\sim SB$ \triangleright by (c) $\therefore \sim \mathsf{RK}$ \triangleright by modus tollens 3 RI \vee RK \triangleright by (d) $\sim RK$ \triangleright by the conclusion of (2) ∴ RL \triangleright by elimination 4. RI \rightarrow GC \triangleright by (e) RI \triangleright by the conclusion of (3) .: GC \triangleright by modus ponens Thus, the glasses are on the coffee table.

Problem

• There is an island containing two types of people: truth tellers who always tell the truth and liars who always lie. You visit the island and are approached by two natives who speak to you as follows:

A says: B is a truth teller.

B says: A and I are of opposite type.

• What are A and B?

Example: Truth tellers and liars

Solution

- Suppose A is a truth teller.
- \therefore What A says is true.
- ∴ B is also a truth teller. ∴ What B says is true. ▷ That's what A said. ▷ by definition of truth teller
- \therefore A and B are of opposite types. \triangleright That's what B said.
- \therefore Contradiction: A and B are both truth tellers and A and B are of opposite type.
- \therefore Initial assumption is false. \triangleright by the contradiction rule
- \therefore A is not a truth teller. \triangleright negation of assumption
- \therefore A is a liar. \triangleright by elimination: All inhabitants are truth tellers or liars. so since A is not a truth teller, A is a liar.
- \therefore What A says is false.
- \therefore B is not a truth teller.
- $\therefore B$ is also a liar.
- Final answer: A and B are both liars

 \triangleright by elimination

 \triangleright by definition of truth teller

Digital Logic Circuits

Idea: Circuits and logic are related



Open or off or false

Closed or on or true

Idea: Circuits and logic are related





Switches		Light bulb
P	Q	State
closed	closed	on
closed	open	on
open	closed	on
open	open	off

History
 1930s: Physical switches were used in circuit design (Physical states: closed and open) Late 1930s: Great idea that mathematical logic (or Boolean algebra) can be used to analyze switches 1940s and 1950s: Electronic switches were used in circuit design (Electronic states: high and low)

Application

- This technology led to the development of electronic computers, electronic telephone switching systems, traffic light controls, electronic calculators, and the control mechanisms
- Electronic switches to implement logic is the fundamental concept that underlies all electronic digital computers

Complicated logic gates as black boxes



Input			Output
P	Q	R	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

A black box focuses on the functionality and ignores the hardware implementation details

Simple logic gates

Method

• Complicated logic gates can be built using a collection of simple logic gates such as NOT-gate, AND-gate, and OR-gate

R

R

1

1

0



What is a combinational circuit

Definition

• A combinational circuit is a circuit designed by combining logic gates such that the output at any given time depends only on the input at that time and not on previous inputs.

Rules

- 1. Never combine two input wires.
- 2. A single input wire can be split partway and used as input for two separate gates.
- 3. An output wire can be used as input.
- 4. No output of a gate can eventually feed back into that gate.

Other types of circuits

1. A sequential circuit violates the 4th rule and hence its output depends on previous inputs, too.
Problem-solving in digital logic circuits



Problem-solving in digital logic circuits

Problem
• Circuit \rightarrow Table
Solution
 Logic circuit → Boolean expression Simplify Boolean expression Boolean expression → Input-output table
Problem
• Table \rightarrow Circuit
Solution
 Input-output table → Boolean expression Simplify Boolean expression Boolean expression → Logic circuit



$\textbf{Circuit} \rightarrow \textbf{Table}$



Problem							
• Determine the logic circuit for the given input-output table.							
	Input			Output			
	P	Q	R	S			
	1	1	1	1			
	1	1	0	0			
	1	0	1	1			
	1	0	0	1			
	0	1	1	0			
	0	1	0	0			
	0	0	1	0			
	0	0	0	0			

$\textbf{Table} \rightarrow \textbf{Circuit}$

Solution

1. Table \rightarrow expression

$$(P \land Q \land R) \lor (P \land \sim Q \land R) \lor (P \land \sim Q \land \sim R)$$

Disjunctive normal form or sum-of-products form

Input		Output	Expression	
P	Q	R	S	S
1	1	1	1	$P \wedge Q \wedge R$
1	1	0	0	$P \wedge Q \wedge \sim R$
1	0	1	1	$P\wedge \sim Q\wedge R$
1	0	0	1	$P\wedge \sim Q\wedge \sim R$
0	1	1	0	$\sim P \wedge Q \wedge R$
0	1	0	0	$\sim P \wedge Q \wedge \sim R$
0	0	1	0	$\sim P \wedge \sim Q \wedge R$
0	0	0	0	$\sim P \wedge \sim Q \wedge \sim R$

$\textbf{Table} \rightarrow \textbf{Circuit}$





What is a recognizer circuit?

Definition

• A recognizer is a circuit that outputs a 1 for exactly one particular combination of input signals and outputs 0's for all other combinations.



Equivalence of logic circuits

Definition

- Two digital logic circuits are called equivalent if and only if their input-output tables are identical
- Approach: Simplify the Boolean expression

Problem

• Show that the following two logic circuits are equivalent.



Equivalence of logic circuits



NAND and NOR gates

Definition

• NAND: $P \mid Q \equiv \sim (P \land Q)$ NOR: $P \downarrow Q \equiv \sim (P \lor Q)$

▷ Sheffer stroke
▷ Pierce arrow

• Any Boolean expression is equivalent to an expression written entirely with NAND gates or written entirely with NOR gates



Universal logic gates: NAND and NOR

Expression	Using NAND gates only
$\sim P$	$\equiv P \mid P$
$P \wedge Q$	$\equiv (P \mid Q) \mid (P \mid Q)$
$P \lor Q$	$\equiv (P \mid P) \mid (Q \mid Q)$
$P \rightarrow Q$	$\equiv P \mid (P \mid Q) \equiv P \mid (Q \mid Q)$
$P \leftrightarrow Q$	$\equiv (P \mid Q) \mid ((P \mid P) \mid (Q \mid Q))$

Expression	Using NOR gates only
$\sim P$	$\equiv P \downarrow P$
$P \wedge Q$	$\equiv (P \downarrow P) \downarrow (Q \downarrow Q)$
$P \lor Q$	$\equiv (P \downarrow Q) \downarrow (P \downarrow Q)$
$P \rightarrow Q$	$\equiv ((P \downarrow P) \downarrow Q) \downarrow ((P \downarrow P) \downarrow Q)$

Take-Home Lessons

Problem

• Are these two computer programs equivalent?

if ((a == b+1 && c < d) || (a == c || b == c)) { ... }

Problem

• Are these two computer programs equivalent?

Solution

- Answer. Yes.
- Reason.

Let p: (a == b+1 && c < d) and q: (a == c || b == c) $p \lor (\sim p \land q) \equiv p \lor q$

Problem

• What is the output of this computer program?

Problem

• What is the output of this computer program?

- Answer. Hey
- Reason. Let p: (a < b) and q: (c == d)
 - $((p \lor q) \land (\sim p \lor q) \land (p \lor \sim q) \land (\sim p \lor \sim q)) \equiv p \land \sim p \equiv \mathbf{c}$

Problem

• Is the second computer program simplified/optimized version of the first computer program?

if (a < b && a < c && b < c)
{ ... }</pre>

if (a < b && b < c) { ... }

Problem

• Is the second computer program simplified/optimized version of the first computer program?

- Answer. No.
- Reason. Let p: (a < b), q: (a < c), and <math>r: (b < c) $p \land q \land r \neq p \land r.$ (Counterexample: p = T, q = F, r = T.)

Problem

• Is the second computer program simplified/optimized version of the first computer program?

- Answer. No.
- Reason. Let p: (a < b), q: (a < c), and r: (b < c) $p \land q \land r \neq p \land r$. (Counterexample: p = T, q = F, r = T.)
- Incorrect answer and incorrect reasoning. What's wrong?

Logic models circuit design



Logic models circuit design





Problem

• Is John's conclusion logical?

- Answer. No.
- Reason. Logical fallacy: Converse error.



Problem

• Is John's conclusion logical?

- Answer. Yes.
- Reason. Rule of inference: Modus ponens.



Problem

• Is John's conclusion logical?

- Answer. No.
- Reason. Logical fallacy: Inverse error.



Problem

• Is John's conclusion logical?

- Answer. Yes.
- Reason. Rule of inference: Modus tollens.

Expectations

- Understand logical expressions
- Understand negations, conditionals, and equivalence laws
- Construct truth tables
- Check equivalence of logical expressions
- \bullet Understand valid/logical and invalid/illogical arguments
- Solve problems using valid arguments
- Understand sound/truthful and unsound/false arguments
- $\bullet\,$ Convert circuit diagrams to I/O tables and vice versa
- Check equivalence of circuit diagrams