CSE 215 Practice Questions

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Truth Tables

Construct truth tables for the following statements:

1. $\sim (p \wedge r) \leftrightarrow (q \oplus r)$

p	q	r	$p \wedge r$	$\sim (p \wedge r)$	$q \oplus r$	$\sim (p \wedge r) \leftrightarrow (q \oplus r)$
T	T	T	T	F	F	T
T	T	F	F	T	T	T
T	F	T	T	F	T	F
T	F	F	F	T	F	F
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	F	F

2. $\sim (q \vee r) \rightarrow (p \oplus (r \wedge q))$

p	q	r	$q \vee r$	$\sim (q \vee r)$	$r \wedge q$	$p \oplus (r \wedge q)$	$\sim (q \vee r) \rightarrow (p \oplus (r \wedge q))$
T	T	T	T	F	T	F	T
T	T	F	T	F	F	T	T
T	F	T	T	F	F	T	T
T	F	F	F	T	F	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	F	T
F	F	T	T	F	F	F	T
F	F	F	F	T	F	F	F

3. $((p \wedge q) \to r) \to (\sim q \vee \sim r)$

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$\sim q$	$\sim r$	$\sim q \vee \sim r$	$((p \wedge q) \to r) \to (\sim q \vee \sim r)$
T	T	T	T	T	F	F	F	F
T	T	F	T	F	F	T	T	T
T	F	T	F	T	T	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	F	T	F	F	F	F
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	T	T	T

4. $(q \vee (r \oplus p)) \leftrightarrow (p \wedge (r \oplus q))$

p	q	r	$r \oplus p$	$q \vee (r \oplus p)$	$r \oplus q$	$p \wedge (r \oplus q)$	$(q \vee (r \oplus p)) \leftrightarrow (p \wedge (r \oplus q))$
T	T	T	F	T	F	F	F
T	T	F	T	T	T	T	T
T	F	T	F	F	T	T	F
T	F	F	T	T	F	F	F
F	T	T	T	T	F	F	F
F	T	F	F	T	T	F	F
F	F	T	T	T	T	F	F
F	F	F	F	F	F	F	T

5. $((p \to q) \land (q \to r)) \leftrightarrow (p \to r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \land (q \to r)$	$p \to r$	$((p \to q) \land (q \to r)) \leftrightarrow (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	F
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Deduction Rules

Determine if the following deduction rules are valid:

1. $p \to q$

   $\sim r \to p$

   $\therefore q \lor r$

p	q	r	$p \to q$	$\sim r$	$\sim r \to p$	$q \lor r$
T	T	T	T	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	
T	F	F	F	T	T	
F	T	T	T	F	T	T
F	T	F	T	T	F	
F	F	T	T	F	T	T
F	F	F	T	T	F	

Therefore, it is valid.

2. $p \leftrightarrow q$

   $\sim q \wedge \sim r$

   $\therefore \ \sim r$

p	q	r	$p \leftrightarrow q$	$\sim q$	$\sim r$	$\sim q \wedge \sim r$	$\sim r$
T	T	T	T	F	F	F	
T	T	F	T	F	T	F	
T	F	T	F	T	F	F	
T	F	F	F	T	T	T	
F	T	T	F	F	F	F	
F	T	F	F	F	T	F	
F	F	T	T	T	F	F	
F	F	F	T	T	T	T	T

Therefore, it is valid.

3. $(p \oplus q) \rightarrow r$, $(p \oplus r) \rightarrow q$, $(q \oplus r) \rightarrow p$, $\therefore p \wedge q \wedge r$

p	q	r	$p \oplus q$	$(p \oplus q) \rightarrow r$	$p \oplus r$	$(p \oplus r) \rightarrow q$	$q \oplus r$	$(q \oplus r) \rightarrow p$	$p \wedge q \wedge r$
T	T	T	F	T	F	T	F	T	T
T	T	F	F	T	T	T	T	T	F
T	F	T	T	T	F	T	T	T	F
T	F	F	T	F	T	F	F	T	
F	T	T	T	T	T	T	F	T	F
F	T	F	T	F	F	T	T	F	
F	F	T	F	T	T	F	T	F	
F	F	F	F	T	F	T	F	T	F

Therefore, it is invalid.

Logical Language

Rewrite the following sentences into two logically equivalent statements:

1. P is a necessary condition for Q. $\boxed{\sim P \to \sim Q \equiv Q \to P}$

2. P is a sufficient condition for Q. $\boxed{P \to Q \equiv \sim Q \to \sim P}$

3. P if and only if Q. $\boxed{P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)}$

4. A necessary condition for R is P and Q. $\boxed{\sim (P \wedge Q) \to \sim R \equiv R \to (P \wedge Q)}$

5. R and T are both necessary and sufficient conditions for P or Q.

   $\boxed{\sim (R \wedge T) \to \sim (P \vee Q) \equiv (P \vee Q) \to (R \wedge T)}$

Logical Rules and Fallacies

Deduce if the statements are valid. If so, state which rule. If not, state which fallacy.

1. If you study math, you are smart.

   I do not study math.

   $\therefore$ I am not smart. $\boxed{\text{Invalid: Inverse Error}}$

2. If you get above an 80 on this final, you get a B+.

   I got above an 80 on this final.

   $\therefore$ I get a B+. $\boxed{\text{Valid: Modus Ponens}}$

3. If you are a good person, you pay taxes.

   I pay taxes.

   $\therefore$ I am a good person. $\boxed{\text{Invalid: Converse Error}}$

4. If you like cats, you like furry animals.

   I do not like furry animals.

   $\therefore$ I do not like cats. $\boxed{\text{Valid: Modus Tollens}}$

Logical Deduction (Many Premises)

Use the valid arguments forms to deduce the conclusion from the premises.

1. $a \to \sim f$

   $a \vee b$

   $(b \wedge f) \to d$

   f

   $e \to \sim d$

   $\therefore \sim e$

$a \to \sim f$ (P1), f (P4)	$\sim a$ (Modus Tollens)
$a \vee b$ (P2), $\sim a$ (S1)	b (Elimination)
b (S2), f (P4)	$b \wedge f$ (Conjunction)
$(b \wedge f) \to d$ (P3), $b \wedge f$ (S3)	d (Modus Ponens)
d (S4), $e \to \sim d$ (P5)	$\sim e$ (Modus Tollens)

2. $\sim h \to f$

   $c \to \sim (f \wedge g)$

   g

   $h \to f$

   $c \vee q$

   $\therefore q$

$\sim h \to f$ (P1), $h \to f$ (P4)	f (Division into Cases)
f (S1), g (P3)	$f \wedge g$ (Conjunction)
$c \to \sim (f \wedge g)$ (P2), $f \wedge g$ (S2)	$\sim c$ (Modus Tollens)
$c \vee q$ (P5), $\sim c$ (S3)	q (Elimination)

Logic with Quantifiers

Find negations for the following statements:

1. There exists a student such that they have a higher grade than all other students.

   For every student, there is a student who has a grade that is $\geq$ than theirs.

2. For all animals, if you are a pet, then you have an owner.

   There exists an animal such that they are a pet but do not have an owner.

3. $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy \geq 0.$ $\exists x \in \mathbb{R}, \forall y \in \mathbb{Z}, xy < 0.$

4. Passing both midterms is a sufficient condition to do well in this class.

   One passed both midterms and didn't do well in this class.

5. If you get a 100% on the final or 100% on both midterms, you are going to get an A.

   One got a 100% on the final or 100% on both midterms and didn't get an A.

6. $\forall x, \forall y, \forall z, \exists \alpha, \exists \beta, \exists \zeta,\ \alpha^\beta + \zeta \geq xyz \geq \alpha^\beta - \zeta$

   $\exists x, \exists y, \exists z, \forall \alpha, \forall \beta, \forall \zeta,\ (\alpha^\beta + \zeta < xyz) \lor (\alpha^\beta - \zeta > xyz)$

Quantifiers

Deduce if the following statements are true or false:

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy \geq 0.$ True. Let $y = 0$.

2. $\forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, xy > 0.$ False. Let $x = -1$ and $y = 1$.

3. $\forall x, y \in \mathbb{Z}^+, (x^2 > y^2) \to (x > y).$ True because the domain is $\mathbb{Z}^+$.

4. $\forall x, y \in \mathbb{Z}, (\frac{x}{y} > \frac{y}{x}) \to (x \neq y).$ True. Use the previous question in the proof.

5. $\forall x, y \in \{\mathbf{c}, \mathbf{t}\}, \exists z \in \{\mathbf{c}, \mathbf{t}\}, (x \land y) \to z \equiv \mathbf{t}$ True. Let $z \equiv \mathbf{t}$.

Direct Proofs

Prove each of the following using a direct proof method:

1. The sum of any two odd integers is even.

Let two odd integers be a and b. By definition, an odd integer can be written as:

$a = 2m + 1 \quad \text{and} \quad b = 2n + 1,$ where m and n are integers.

The sum of a and b is:

$a + b = (2m + 1) + (2n + 1)$

$= 2m + 2n + 2$

$= 2(m + n + 1).$

Since $m + n + 1$ is an integer, $a + b$ is divisible by 2 and hence is even.

$\therefore$ The sum of any two odd integers is even.

2. If n and m are odd, then nm is also odd.

Let n and m be odd integers. By definition, an odd integer can be written as:

$n = 2a + 1 \quad \text{and} \quad m = 2b + 1,$ where a and b are integers.

The product of n and m is:

$nm = (2a + 1)(2b + 1)$

$= 4ab + 2a + 2b + 1$

$= 2(2ab + a + b) + 1.$

Since $2ab + a + b$ is an integer, nm is of the form $2k + 1$, where k is an integer.

$\therefore nm$ is odd.

3. The product of any two consecutive integers is even.

Let the two consecutive integers be n and $n+1$. Consider the two cases for n:

Case 1: n is even.

If n is even, we can write $n = 2k$ for some integer k. Then,

$n(n+1) = (2k)(2k+1)$

$= 2k(2k+1).$

Since $2k$ is a multiple of 2, the product $n(n+1)$ is divisible by 2, and hence even.

Case 2: n is odd.

If n is odd, we can write $n = 2k+1$ for some integer k. Then,

$n(n+1) = (2k+1)(2k+2)$

$= (2k+1)(2(k+1))$

$= 2(2k+1)(k+1).$

Here, $2(2k+1)(k+1)$ is divisible by 2, so the product $n(n+1)$ is even.

In both cases, the product of two consecutive integers is divisible by 2.

$\therefore$ The product of any two consecutive integers is even.

4. If $a|p$ and $p|q$, then $a|q$

Given: $a|p$ and $p|q$. By def. of divisibility, there exist integers k and m such that:

$p = ak \quad \text{and} \quad q = pm.$

Substitute $p = ak$ into $q = pm$:

$q = (ak)m$

$q = a(km).$

Since k and m are integers, km is also an integer. Thus, q is divisible by a.

$\therefore a|q.$

Proofs by Contrapositive

Prove each of the following using the contrapositive method:

1. If pq is even, then p or q is even.

We prove the contrapositive: If p and q are odd, then pq is odd.

Let p and q be odd integers. By definition, we can write:

$p = 2k + 1 \quad \text{and} \quad q = 2m + 1, \quad \text{where } k \text{ and } m \text{ are integers.}$

The product of p and q is:

$$pq = (2k + 1)(2m + 1)$$

$$= 4km + 2k + 2m + 1$$

$$= 2(2km + k + m) + 1.$$

Since $2km + k + m$ is an integer, pq is of the form $2n + 1$, where n is an integer.

Thus, pq is odd.

$\therefore$ The contrapositive is true, so the original statement is true.

2. If $n^2 - 6n + 5$ is even, then n is odd.

We prove the contrapositive: If n is even, then $n^2 - 6n + 5$ is odd.

Let n be an even integer. By definition, $n = 2k$ for some integer k. Then,

$$n^2 - 6n + 5 = (2k)^2 - 6(2k) + 5$$

$$= 4k^2 - 12k + 5.$$

Factor out 2 from the terms:

$$n^2 - 6n + 5 = 2(2k^2 - 6k + 2) + 1.$$

Since $2k^2 - 6k + 2$ is an integer, the expression is of the form $2m + 1$, where m is an integer.

Thus, $n^2 - 6n + 5$ is odd.

$\therefore$ The contrapositive is true, so the original statement is true.

3. If $x^2 + 5x + 6 \neq 0$, then $x \notin \{-3, -2\}$.

We prove the contrapositive: If $x \in \{-3, -2\}$, then $x^2 + 5x + 6 = 0$.

Case 1: Let $x = -3$. Substitute $x = -3$ into $x^2 + 5x + 6$:

$x^2 + 5x + 6 = (-3)^2 + 5(-3) + 6$

$= 9 - 15 + 6$

$= 0.$

Case 2: Let $x = -2$. Substitute $x = -2$ into $x^2 + 5x + 6$:

$x^2 + 5x + 6 = (-2)^2 + 5(-2) + 6$

$= 4 - 10 + 6$

$= 0.$

In both cases, $x^2 + 5x + 6 = 0$. Thus, the contrapositive is true.

$\therefore$ If $x^2 + 5x + 6 \neq 0$, then $x \notin \{-3, -2\}$.

4. If 3 doesn't divide xy, then 3 doesn't divide x and y.

We prove the contrapositive: If 3 divides x or y, then 3 divides xy.

Case 1: Suppose 3 divides x.

This means $x = 3k$ for some integer k. Then, $xy = (3k)y = 3(ky)$.

Since ky is an integer, 3 divides xy.

Case 2: Suppose 3 divides y.

This means $y = 3m$ for some integer m. Then, $xy = x(3m) = 3(xm)$.

Since xm is an integer, 3 divides xy.

In both cases, if 3 divides x or y, then 3 divides xy. Thus, the contrapositive is true.

$\therefore$ If 3 doesn't divide xy, then 3 doesn't divide x and y.

Proofs by Contradiction

1. If x^2 is irrational, then x is irrational.

We prove by contradiction: Assume x^2 is irrational, but x is rational.

Since x is rational, we can write $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, and $\gcd(p, q) = 1$.

Then, x^2 can be expressed as:

$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$.

Since p^2 and q^2 are integers, $\frac{p^2}{q^2}$ is rational.

This contradicts the assumption that x^2 is irrational.

$\therefore$ If x^2 is irrational, then x must be irrational.

2. $\sqrt{2}$ is irrational.

We prove by contradiction: Assume $\sqrt{2}$ is rational.

Then, $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$.

Squaring both sides, we have:

$2 = \frac{p^2}{q^2}$.

Rewriting, $p^2 = 2q^2$.

This implies p^2 is even, so p must also be even.

Let $p = 2k$, where $k \in \mathbb{Z}$. Then:

$p^2 = (2k)^2 = 4k^2$.

Substituting, $4k^2 = 2q^2$, or $q^2 = 2k^2$.

This implies q^2 is even, so q must also be even.

Thus, both p and q are even, contradicting the assumption that $\gcd(p, q) = 1$.

$\therefore \sqrt{2}$ is irrational.

3. If ab is irrational and a is rational, then b is irrational.

We prove by contradiction: Assume ab is irrational, a is rational, and b is rational.

Since a and b are rational, we can write $a = \frac{p}{q}$ and $b = \frac{r}{s}$, where:

$$p, q, r, s \in \mathbb{Z}, \quad q \neq 0, \quad s \neq 0.$$

Then, the product ab is:

$$ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

Since pr and qs are integers, $\frac{pr}{qs}$ is rational.

This contradicts the assumption that ab is irrational.

$\therefore$ If ab is irrational and a is rational, then b must be irrational.

4. There doesn't exist a largest number.

We prove by contradiction: Assume there exists a largest number, say M.

By definition of a largest number, M is such that for any $x \in \mathbb{R}$, $x \leq M$.

Consider the number $M + 1$. Clearly, $M + 1 > M$.

This contradicts the assumption that M is the largest number.

$\therefore$ There doesn't exist a largest number.

5. There is no smallest positive real number.

We prove by contradiction: Assume there exists a smallest positive real number, say ϵ.

By definition, ϵ is such that for all positive real numbers x, $x \geq \epsilon$.

Consider the number $\frac{\epsilon}{2}$. Clearly, $\frac{\epsilon}{2}$ is a positive real number and

$$\frac{\epsilon}{2} < \epsilon.$$

This contradicts the assumption that ϵ is the smallest positive real number.

$\therefore$ There is no smallest positive real number.

Proofs by Induction

Prove each of the following through induction:

1. The sum of the first n odd numbers is n^2.

Let $P(n) : \sum_{i=0}^{n-1}(2i+1) = n^2 \quad \text{for integers } n \geq 1.$

Base Case: $P(1)$

$P(1)$ LHS: $\sum_{i=0}^{0}(2i+1) = 1.$

$P(1)$ RHS: $(1)^2 = 1.$

$\therefore P(1)$ is true because LHS = RHS.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove $P(k+1)$.

Inductive Hypothesis: Assume $\sum_{i=0}^{k-1}(2i+1) = k^2.$

$P(k+1)$ LHS: $\sum_{i=0}^{k}(2i+1) = \sum_{i=0}^{k-1}(2i+1) + (2k+1).$

By the inductive hypothesis: $\sum_{i=0}^{k-1}(2i+1) = k^2.$

Substitute: $\sum_{i=0}^{k}(2i+1) = k^2 + (2k+1).$

Simplify: $k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2.$

Thus, $\sum_{i=0}^{k}(2i+1) = (k+1)^2.$

$\therefore P(k+1)$ is true.

Conclusion: By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

2. For all $n \geq 1$,

$$1 + x + x^2 + ... + x^n = \frac{x^{n+1} - 1}{x - 1} \quad \text{where } x \neq 1$$

Let $P(n) : 1 + x + x^2 + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ for integers $n \geq 1$ and $x \neq 1$.

Base Case: $P(1)$

$P(1)$ LHS: $1 + x = \dfrac{x^{1+1} - 1}{x - 1}$.

RHS: Expand $\dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x-1} = x + 1$.

$\therefore P(1)$ is true because LHS = RHS.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove $P(k+1)$.

Inductive Hypothesis: Assume $1 + x + x^2 + \cdots + x^k = \dfrac{x^{k+1} - 1}{x - 1}$.

$P(k+1)$ LHS: $1 + x + x^2 + \cdots + x^k + x^{k+1}$.

Using the inductive hypothesis:

$$1 + x + x^2 + \cdots + x^k + x^{k+1} = \frac{x^{k+1} - 1}{x - 1} + x^{k+1}.$$

Rewrite the second term with a common denominator:

$$\frac{x^{k+1} - 1}{x - 1} + x^{k+1} = \frac{x^{k+1} - 1 + x^{k+1}(x - 1)}{x - 1}.$$

Simplify the numerator:

$$x^{k+1} - 1 + x^{k+2} - x^{k+1} = x^{k+2} - 1.$$

Thus, $\dfrac{x^{k+1} - 1}{x - 1} + x^{k+1} = \dfrac{x^{k+2} - 1}{x - 1}$.

This matches the RHS of $P(k+1)$.

$\therefore P(k+1)$ is true.

Conclusion: By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

3. For all $n \geq 1$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

Let $P(n) : \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for integers $n \geq 1$.

Base Case: $P(1)$

$P(1)$ LHS: $\sum_{i=1}^{1} i(i+1) = 1(1+1) = 2.$

$P(1)$ RHS: $\frac{1(1+1)(1+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2.$

$\therefore P(1)$ is true because LHS = RHS.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove $P(k+1)$.

Inductive Hypothesis: Assume $\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}.$

$P(k+1)$ LHS: $\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)((k+1)+1).$

Using the inductive hypothesis:

$$\sum_{i=1}^{k+1} i(i+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2).$$

Factor $(k+1)(k+2)$:

$$\sum_{i=1}^{k+1} i(i+1) = \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}.$$

$$\sum_{i=1}^{k+1} i(i+1) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}.$$

Factor out $(k+1)(k+2)$:

$$\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+2)(k+3)}{3}. = \text{RHS of } P(k+1).$$

$\therefore P(k+1)$ is true.

Conclusion: By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

4. For all $n \geq 2$,

$$\sum_{i=2}^{n} i^2(i-1) = \frac{n(n^2-1)(3n+2)}{12}$$

Let $P(n) : \displaystyle\sum_{i=2}^{n} i^2(i-1) = \frac{n(n^2-1)(3n+2)}{12}$ for integers $n \geq 2$.

Base Case: $P(2)$

$P(2)$ LHS: $\displaystyle\sum_{i=2}^{2} i^2(i-1) = 2^2(2-1) = 4.$

$P(2)$ RHS: $\displaystyle\frac{2(2^2-1)(3\cdot 2+2)}{12} = \frac{2(4-1)(8)}{12} = \frac{2\cdot 3\cdot 8}{12} = 4.$

$\therefore P(2)$ is true because LHS = RHS.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Prove $P(k+1)$.

Inductive Hypothesis: Assume $\displaystyle\sum_{i=2}^{k} i^2(i-1) = \frac{k(k^2-1)(3k+2)}{12}.$

$P(k+1)$ LHS: $\displaystyle\sum_{i=2}^{k+1} i^2(i-1) = \sum_{i=2}^{k} i^2(i-1) + (k+1)^2(k+1-1).$

Using the inductive hypothesis:

$$\sum_{i=2}^{k+1} i^2(i-1) = \frac{k(k^2-1)(3k+2)}{12} + (k+1)^2 k.$$

Combine terms with a common denominator:

$$\sum_{i=2}^{k+1} i^2(i-1) = \frac{k(k^2-1)(3k+2) + 12k(k+1)^2}{12}.$$

Factorize the numerator:

$$k(k^2-1)(3k+2) + 12k(k+1)^2 = (k+1)((k+1)^2)(3k+2).$$

$$\therefore \sum_{i=2}^{k+1} i^2(i-1) = \frac{(k+1)((k+1)^2-1)(3(k+1)+2)}{12}.$$

This is precisely $\displaystyle\frac{(k+1)((k+1)^2-1)(3(k+1)+2)}{12}$, proving $P(k+1)$.

Conclusion: By induction, $P(n)$ holds for all $n \geq 2$.

5. For all $n \geq 1$, $5^n + 3$ is divisible by 4.

Let $P(n) : 5^n + 3$ is divisible by 4.

Base Case: $P(1)$

$P(1)$ LHS: $5^1 + 3 = 8$ is divisible by 4 $\rightarrow P(1)$ is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove $P(k+1)$.

$P(k+1)$ LHS: $5^{k+1} + 3 = 5 \cdot 5^k + 3$.

Rewrite: $5^{k+1} + 3 = 4 \cdot 5^k + 5^k + 3$.

Substitute the inductive hypothesis: $5^k + 3 = 4m$.

Thus, $5^{k+1} + 3 = 4 \cdot 5^k + 4m = 4(5^k + m)$.

Since $5^k + m$ is an integer, $5^{k+1} + 3$ is divisible by 4.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

6. For all $n \geq 1$, $4^{2n} - 1$ is divisible by 15.

Let $P(n) : 4^{2n} - 1$ is divisible by 15.

Base Case: $P(1)$

$P(1)$ LHS: $4^{2 \cdot 1} - 1 = 16 - 1 = 15$ is divisible by 15 $\rightarrow P(1)$ is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove, $P(k+1)$.

$P(k+1)$ LHS: $4^{2(k+1)} - 1 = 4^{2k+2} - 1$.

Rewrite: $4^{2k+2} - 1 = (4^2) \cdot 4^{2k} - 1 = 16 \cdot 4^{2k} - 1 = 15 \cdot 4^{2k} + 4^{2k} - 1$

Substitute the inductive hypothesis: $4^{2k} - 1 = 15m$.

Thus, $(4^{2k} - 1)(16) + 15 = (15m)(16) + 15 = 15(16m + 1)$.

Since $16m + 1$ is an integer, $4^{2(k+1)} - 1$ is divisible by 15.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

7. For all $n \geq 1$, $4^n + 6n - 1$ is divisible by 3.

Let $P(n) : 4^n + 6n - 1$ is divisible by 3.

Base Case: $P(1)$

$P(1)$ LHS: $4^1 + 6(1) - 1 = 4 + 6 - 1 = 9,$ which is divisible by 3.

Thus, $P(1)$ is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 1$. Prove $P(k+1)$.

$P(k+1)$ LHS: $4^{k+1} + 6(k+1) - 1$

$= 4 \cdot 4^k + 6k + 6 - 1$

$= 4 \cdot 4^k + 6k + 5.$

We can rewrite this as: $4 \cdot 4^k + (6k + 5)$

$= (4 \cdot 4^k + 6k - 1) + 6$

By the inductive hypothesis, we know that $4^k + 6k - 1$ is divisible by 3.

Thus, $(4 \cdot 4^k + 6k - 1)$ is divisible by 3.

Therefore, $4^{k+1} + 6(k+1) - 1 = (4 \cdot 4^k + 6k - 1) + 6$ is divisible by 3.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

Proofs by Strong Induction

Prove each of the following using strong induction:

1. For all $n \geq 1$, the n-th term of the sequence defined by

$$a_n = \begin{cases} n & \text{if } n = 1 \text{ or } n = 2, \\ a_{n-1} + 2a_{n-2} & \text{if } n \geq 3, \end{cases}$$

is given by $a_n = 2^{n-1}$.

Let $P(n) : a_n = 2^{n-1}$ for all $n \geq 1$.

Base Case: $P(1)$ and $P(2)$.

For $n = 1, \quad a_1 = 1$ (given), $\quad 2^{1-1} = 2^0 = 1$.

For $n = 2, \quad a_2 = 2$ (given), $\quad 2^{2-1} = 2^1 = 2$.

Inductive Step: Assume $P(i)$ is true for all $i \in [1, k]$ for some $k \geq 2$. Prove $P(k+1)$.

We need to show that $P(k+1)$ is true, i.e., $a_{k+1} = 2^k$.

By the recurrence relation, we have:

$a_{k+1} = a_k + 2a_{k-1}$.

Using the inductive hypothesis, $a_k = 2^{k-1}$ and $a_{k-1} = 2^{k-2}$.

Substitute these into the recurrence:

$a_{k+1} = 2^{k-1} + 2(2^{k-2})$.

$a_{k+1} = 2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^k$.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

2. For the Fibonacci sequence defined as

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3,$$

prove that $F_n < 2^n$ for all $n \geq 1$.

Let $P(n) : F_n < 2^n$ for all $n \geq 1$.

Base Case: $P(1)$ and $P(2)$.

For $n = 1$, $F_1 = 1$ and $2^1 = 2$, $F_1 < 2^1$.

For $n = 2$, $F_2 = 1$ and $2^2 = 4$, $F_2 < 2^2$.

Thus, $P(1)$ and $P(2)$ hold true.

Inductive Step: Assume $P(k)$ is true for all $i \in [1, k]$ for some $k \geq 2$. Prove $P(k+1)$.

We need to show that $P(k+1)$ is true, i.e., $F_{k+1} < 2^{k+1}$.

From the recurrence relation, we have:

$F_{k+1} = F_k + F_{k-1}$.

By the inductive hypothesis, we know that $F_k < 2^k$ and $F_{k-1} < 2^{k-1}$.

Thus,

$F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} = 2^{k-1}(2+1) = 2^{k-1} \cdot 3$.

Now observe that $2^{k-1} \cdot 3 < 2^{k+1}$ for all $k \geq 2$.

Therefore, $F_{k+1} < 2^{k+1}$.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

3. Prove that the n-th Fibonacci term can be written as

Let $P(n) : F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Base Case: $P(1)$ and $P(2)$.

For $n = 1$, $F_1 = 1$, and the right-hand side evaluates to 1.

For $n = 2$, $F_2 = 1$, and the right-hand side evaluates to 1.

Thus, $P(1)$ and $P(2)$ hold true.

Inductive Step: Assume $P(k)$ is true for all $i \in [1, k]$ for some $k \geq 2$. Prove $P(k+1)$.

We need to show that $F_{k+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right)$.

Using the inductive hypothesis:

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right),$$

$$F_{k-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right).$$

Adding these:

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\right) - \left(\left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)\right].$$

Factor out powers:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1},$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^{k+1} = \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}.$$

Thus, $F_{k+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right)$.

Conclusion: By induction, $P(n)$ is true for all $n \geq 1$.

Set Theory (Element Methods)

1. For all sets A, B, and C, prove that $(A \cup B) - (A \cap C) = (A - C) \cup (B - C)$.

To prove: $(A \cup B) - (A \cap C) = (A - C) \cup (B - C)$, show:

(1) $x \in (A \cup B) - (A \cap C) \implies x \in (A - C) \cup (B - C)$,

(2) $x \in (A - C) \cup (B - C) \implies x \in (A \cup B) - (A \cap C)$.

(1) Prove $x \in (A \cup B) - (A \cap C) \implies x \in (A - C) \cup (B - C)$.

Let $x \in (A \cup B) - (A \cap C) \implies x \in A \cup B$ and $x \notin A \cap C$.

If $x \in A \cup B$, then $x \in A$ or $x \in B$.

If $x \notin A \cap C$, then it is not true that $x \in A$ and $x \in C$.

Case 1: If $x \in A$ and $x \notin C$, then $x \in A - C$.

Case 2: If $x \in B$ and $x \notin C$, then $x \in B - C$.

Thus, $x \in (A - C) \cup (B - C)$.

(2) Prove $x \in (A - C) \cup (B - C) \implies x \in (A \cup B) - (A \cap C)$.

Let $x \in (A - C) \cup (B - C) \implies x \in A - C$ or $x \in B - C$.

Case 1: If $x \in A - C$, then $x \in A$ and $x \notin C$.

Since $x \in A$, we have $x \in A \cup B$.

Since $x \notin C$, it is not true that $x \in A \cap C$.

Thus, $x \in (A \cup B) - (A \cap C)$.

Case 2: If $x \in B - C$, then $x \in B$ and $x \notin C$.

Since $x \in B$, we have $x \in A \cup B$.

Since $x \notin C$, it is not true that $x \in A \cap C$.

Thus, $x \in (A \cup B) - (A \cap C)$.

Conclusion: $(A \cup B) - (A \cap C) = (A - C) \cup (B - C)$.

2. For all sets A, B, and C, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

To prove: $A - (B \cup C) = (A - B) \cap (A - C)$.

We will prove this using an element-based argument by showing:

(1) $x \in A - (B \cup C) \implies x \in (A - B) \cap (A - C)$,

(2) $x \in (A - B) \cap (A - C) \implies x \in A - (B \cup C)$.

(1) Prove $x \in A - (B \cup C) \implies x \in (A - B) \cap (A - C)$.

Let $x \in A - (B \cup C)$.

Then $x \in A$ and $x \notin B \cup C$.

If $x \notin B \cup C$, then $x \notin B$ and $x \notin C$.

Since $x \in A$ and $x \notin B$, it follows that $x \in A - B$.

Since $x \in A$ and $x \notin C$, it follows that $x \in A - C$.

Thus, $x \in (A - B) \cap (A - C)$.

(2) Prove $x \in (A - B) \cap (A - C) \implies x \in A - (B \cup C)$.

Let $x \in (A - B) \cap (A - C)$.

Then $x \in A - B$ and $x \in A - C$.

If $x \in A - B$, then $x \in A$ and $x \notin B$.

If $x \in A - C$, then $x \in A$ and $x \notin C$.

Since $x \notin B$ and $x \notin C$, it follows that $x \notin B \cup C$.

Thus, $x \in A$ and $x \notin B \cup C$, so $x \in A - (B \cup C)$.

Conclusion: $A - (B \cup C) = (A - B) \cap (A - C)$.

3. For all sets A, B, and C, prove that $A \times (B - C) = (A \times B) - (A \times C)$.

To prove: $A \times (B - C) = (A \times B) - (A \times C)$.

We will prove this using an element-based argument by showing:

(1) $(x, y) \in A \times (B - C) \implies (x, y) \in (A \times B) - (A \times C)$,

(2) $(x, y) \in (A \times B) - (A \times C) \implies (x, y) \in A \times (B - C)$.

(1) Prove $(x, y) \in A \times (B - C) \implies (x, y) \in (A \times B) - (A \times C)$.

Let $(x, y) \in A \times (B - C)$.

Then $x \in A$ and $y \in B - C$.

Since $y \in B - C$, we know that $y \in B$ and $y \notin C$.

Thus, $(x, y) \in A \times B$.

Since $y \notin C$, it follows that $(x, y) \notin A \times C$.

Therefore, $(x, y) \in (A \times B) - (A \times C)$.

(2) Prove $(x, y) \in (A \times B) - (A \times C) \implies (x, y) \in A \times (B - C)$.

Let $(x, y) \in (A \times B) - (A \times C)$.

Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$.

Since $(x, y) \in A \times B$, it follows that $x \in A$ and $y \in B$.

Since $(x, y) \notin A \times C$, it follows that $y \notin C$.

Thus, $y \in B$ and $y \notin C$, so $y \in B - C$.

Therefore, $(x, y) \in A \times (B - C)$.

Conclusion: $A \times (B - C) = (A \times B) - (A \times C)$.

4. For all sets A, B, and C, prove that if $A \subseteq B$ and $C \subseteq B$, then $A \times C \subseteq B \times B$.

> To prove: $A \times C \subseteq B \times B$.
>
> We are given that $A \subseteq B$ and $C \subseteq B$.
>
> We will prove this using an element-based argument.
>
> Let $(x, y) \in A \times C$.
>
> Then, by the definition of Cartesian product, $x \in A$ and $y \in C$.
>
> Since $A \subseteq B$ and $x \in A$, it follows that $x \in B$.
>
> Since $C \subseteq B$ and $y \in C$, it follows that $y \in B$.
>
> Thus, $(x, y) \in B \times B$.
>
> Therefore, $A \times C \subseteq B \times B$.

Set Theory (Algebraic Methods)

Prove each of the following using algebraic-based method:

1. For all sets A, B, and C, prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

> We want to prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
>
> First, we rewrite the left-hand side using the distributive law:
>
> $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
>
> Thus, by the distributive property, we have:
>
> $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
>
> Therefore, the identity is proven.

2. For all sets A, B, and C, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

We want to prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

First, we rewrite the left-hand side using the set difference law:

$A - (B \cup C) = A \cap (B \cup C)^c$.

Now, apply De Morgan's law:

$(B \cup C)^c = B^c \cap C^c$,

so we have:

$A - (B \cup C) = A \cap (B^c \cap C^c)$.

By the distributive property, we get:

$A - (B \cup C) = (A \cap B^c) \cap (A \cap C^c)$.

This simplifies to:

$(A - B) \cap (A - C)$.

Thus, we have shown that:

$A - (B \cup C) = (A - B) \cap (A - C)$.

3. For all sets A, B, and C, prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

We want to prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

First, we rewrite the left-hand side using the set difference law:

$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)^c$.

Now, apply De Morgan's law:

$(A \cap B)^c = A^c \cup B^c$,

so we have:

$(A \cup B) - (A \cap B) = (A \cup B) \cap (A^c \cup B^c)$.

By distributive law:

$(A \cup B) \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c)$.

Now simplify:

$A \cap A^c = \emptyset, \quad B \cap B^c = \emptyset$,

so we have:

$(A \cup B) - (A \cap B) = (A \cap B^c) \cup (B \cap A^c)$.

By the definition of set difference, this simplifies to:

$(A - B) \cup (B - A)$.

Thus, we have shown that:

$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

4. For all sets A, B, and C, prove that $(A \cap B) \cup (A - B) = A$.

We want to prove that $(A \cap B) \cup (A - B) = A$.

$A - B = A \cap B^c \implies (A \cap B) \cup (A \cap B^c)$.

Now, factor out A using the distributive law:

$(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c)$.

Since $B \cup B^c = U$ (the universal set), we have:

$A \cap (B \cup B^c) = A \cap U = A$

5. For all sets A, B, and C, prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

We want to prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

First, rewrite the left-hand side using the set difference law:

$A - (B \cap C) = A \cap (B \cap C)^c$.

Now, apply De Morgan's law to $(B \cap C)^c$:

$(B \cap C)^c = B^c \cup C^c$.

So, the expression becomes:

$A - (B \cap C) = A \cap (B^c \cup C^c)$.

Now, apply the distributive law to expand the intersection:

$A \cap (B^c \cup C^c) = (A \cap B^c) \cup (A \cap C^c)$.

By the definition of set difference, we recognize:

$A \cap B^c = A - B \quad \text{and} \quad A \cap C^c = A - C$.

Thus, we have:

$A - (B \cap C) = (A - B) \cup (A - C)$.

Therefore, we have proven the identity.

Set Theory Counterexamples

Provide a counterexample to disprove each of the following:

1. For all sets A and B, $A \cup B = A - B$ if and only if $A = B$.

Let $A = \{1, 2\}$ and $B = \{2\}$.

First, compute $A \cup B$:

$A \cup B = \{1, 2\} \cup \{2\} = \{1, 2\}$.

Next, compute $A - B$:

$A - B = \{1, 2\} - \{2\} = \{1\}$.

Clearly, $A \cup B = \{1, 2\} \neq \{1\} = A - B$.

2. For all sets A, B, and C, $(A \cup B) - C = (A - C) \cap (B - C)$.

Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2\}$.

First, compute the left-hand side:

$A \cup B = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$,

$(A \cup B) - C = \{1, 2, 3\} - \{2\} = \{1, 3\}$.

Now, compute the right-hand side:

$A - C = \{1, 2\} - \{2\} = \{1\}$,

$B - C = \{2, 3\} - \{2\} = \{3\}$,

$(A - C) \cap (B - C) = \{1\} \cap \{3\} = \emptyset$.

In this case, we observe that the left-hand side and right-hand side are not equal:

$(A \cup B) - C = \{1, 3\} \neq \emptyset = (A - C) \cap (B - C)$.

3. For all sets A, B, and C, $A - (B \cap C) = (A - B) \cap (A - C)$.

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3\}$, $C = \{3, 4\}$.

First, calculate $B \cap C$:

$B \cap C = \{2, 3\} \cap \{3, 4\} = \{3\}$.

Now, calculate the left-hand side $A - (B \cap C)$:

$A - (B \cap C) = \{1, 2, 3, 4\} - \{3\} = \{1, 2, 4\}$.

Next, calculate the right-hand side $(A - B) \cap (A - C)$:

$A - B = \{1, 2, 3, 4\} - \{2, 3\} = \{1, 4\}$,

$A - C = \{1, 2, 3, 4\} - \{3, 4\} = \{1, 2\}$,

$(A - B) \cap (A - C) = \{1, 4\} \cap \{1, 2\} = \{1\}$.

Since $\{1, 2, 4\} \neq \{1\}$, the statement is disproved.

4. For all sets A, B, and C, $A \times (B \cup C) = (A \times B) \cap (A \times C)$ holds for all A, B, and C.

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$.

First, calculate $B \cup C$:

$B \cup C = \{2\} \cup \{3\} = \{2, 3\}$.

Now, calculate the left-hand side $A \times (B \cup C)$:

$A \times (B \cup C) = \{1\} \times \{2, 3\} = \{(1, 2), (1, 3)\}$.

Next, calculate the right-hand side $(A \times B) \cap (A \times C)$:

$A \times B = \{1\} \times \{2\} = \{(1, 2)\}$,

$A \times C = \{1\} \times \{3\} = \{(1, 3)\}$,

$(A \times B) \cap (A \times C) = \{(1, 2)\} \cap \{(1, 3)\} = \emptyset$.

Finally, compare the results:

Since $\{(1, 2), (1, 3)\} \neq \emptyset$, the statement is disproved.

One-to-One Correspondence

Deduce if the following functions are one-to-one correspondences:

1. Define a function $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 2x + 1$.

1. One-to-One

A function is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Assume $f(x_1) = f(x_2)$.

Then, $2x_1 + 1 = 2x_2 + 1$.

Subtracting 1 from both sides: $2x_1 = 2x_2$.

Dividing both sides by 2: $x_1 = x_2$.

Thus, the function is one-to-one.

2. Onto

A function is onto if for every $y \in \mathbb{Z},$ there exists an $x \in \mathbb{Z}$ such that $f(x) = y$.

Let $y \in \mathbb{Z}$.

We need to find x such that $f(x) = y,$ i.e., $2x + 1 = y$.

Solving for $x:$

$$2x = y - 1,$$

$$x = \frac{y - 1}{2}.$$

For x to be an integer, $y - 1$ must be even, which means that y must be odd.

Thus, the function is not onto, because there are no x values corresponding to even y values.

3. One-to-one Correspondence

Because the function is not onto, it is not a one-to-one correspondence.

2. Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

1. One-to-One

A function is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Assume $f(x_1) = f(x_2)$.

Then, $x_1^3 = x_2^3$.

Taking the cube root of both sides: $x_1 = x_2$.

Thus, the function is one-to-one.

2. Onto

A function is onto if for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$.

Let $y \in \mathbb{R}$.

We need to find x such that $f(x) = y$, i.e., $x^3 = y$.

Solving for x :

$$x = \sqrt[3]{y}.$$

Since the cube root of any real number is defined and produces a real number,

the function is onto.

3. One-to-one Correspondence

Since the function is both one-to-one and onto, it is a one-to-one correspondence.

3. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x^2$.

1. One-to-One

A function is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Assume $f(x_1) = f(x_2)$.

Then, $x_1^2 = x_2^2$.

This implies that $x_1 = x_2$ or $x_1 = -x_2$.

However, since $x_1, x_2 \in \mathbb{N}$ (the set of natural numbers),

the possibility that $x_1 = -x_2$ is not valid.

Therefore, $x_1 = x_2$ and the function is one-to-one.

2. Onto

A function is onto if for every $y \in \mathbb{N},$ there exists an $x \in \mathbb{N}$ such that $f(x) = y$.

Let $y \in \mathbb{N}$.

We need to find x such that $f(x) = y,$ i.e., $x^2 = y$.

Solving for x, $\quad x = \sqrt{y}$.

For $x \in \mathbb{N},$ the square root of y must also be a natural number.

Thus, the function is only onto for perfect squares in $\mathbb{N}$.

Therefore, not onto, because not all elements of $\mathbb{N}$ have a corresponding x value.

3. One-to-one Correspondence

Since the function is one-to-one but not onto, it is not a one-to-one correspondence.

4. Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (4y, 2x)$.

1. One-to-One

A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is one-to-one if $f(x_1, y_1) = f(x_2, y_2) \implies (x_1, y_1) = (x_2, y_2)$.

Assume $f(x_1, y_1) = f(x_2, y_2)$.

Then, $(4y_1, 2x_1) = (4y_2, 2x_2)$.

From the first component: $4y_1 = 4y_2 \implies y_1 = y_2$.

From the second component: $2x_1 = 2x_2 \implies x_1 = x_2$.

Thus, $(x_1, y_1) = (x_2, y_2)$ and the function is one-to-one.

2. Onto

A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is onto if for every $(a, b) \in \mathbb{R} \times \mathbb{R}$,

there exists $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $f(x, y) = (a, b)$.

Let $(a, b) \in \mathbb{R} \times \mathbb{R}$.

We need to find (x, y) where $f(x, y) = (a, b)$, i.e., $(4y, 2x) = (a, b)$.

From the first component: $4y = a \implies y = \frac{a}{4}$.

From the second component: $2x = b \implies x = \frac{b}{2}$.

Thus, for every $(a, b) \in \mathbb{R} \times \mathbb{R}$, there exists $(x, y) = \left(\frac{b}{2}, \frac{a}{4}\right) \in \mathbb{R} \times \mathbb{R}$ where $f(x, y) = (a, b)$.

Therefore, the function is onto.

3. One-to-one Correspondence

Since the function is both one-to-one and onto, it is a one-to-one correspondence.

5. Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(x, y) = (2y, 3x)$.

1. One-to-One

A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is one-to-one if $f(x_1, y_1) = f(x_2, y_2) \implies (x_1, y_1) = (x_2, y_2)$.

Assume $f(x_1, y_1) = f(x_2, y_2)$.

Then, $(2y_1, 3x_1) = (2y_2, 3x_2)$.

From the first component: $2y_1 = 2y_2 \implies y_1 = y_2$.

From the second component: $3x_1 = 3x_2 \implies x_1 = x_2$.

Thus, $(x_1, y_1) = (x_2, y_2)$ and the function is one-to-one.

2. Onto

A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is onto if for every $(a, b) \in \mathbb{N} \times \mathbb{N}$,

there exists $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $f(x, y) = (a, b)$.

Let $(a, b) \in \mathbb{N} \times \mathbb{N}$.

We need to find (x, y) such that $f(x, y) = (a, b)$, i.e., $(2y, 3x) = (a, b)$.

From the first component: $2y = a \implies y = \frac{a}{2}$.

From the second component: $3x = b \implies x = \frac{b}{3}$.

For y and x to be natural numbers, a must be even and b must be divisible by 3.

If a is odd or b is not divisible by 3, there are no such x, y in $\mathbb{N}$.

Thus, the function is not onto.

3. One-to-one Correspondence

Since the function is not onto, it is not a one-to-one correspondence.

Infinite Sets

Prove or disprove the following statements regarding infinite sets:

1. $|\mathbb{N}| = |\mathbb{N} - \{2, 4, 6, 8, ...\}|$.

1. Define a Bijection

To prove $|\mathbb{N}| = |\mathbb{N} - \{2, 4, 6, 8, \dots\}|$, construct a bijection $f : \mathbb{N} \to \mathbb{N} - \{2, 4, 6, 8, \dots\}$.

Define $f(n) = 2n - 1$.

2. One-to-One

A function is one-to-one if $f(n_1) = f(n_2)$ implies that $n_1 = n_2$.

Assume $f(n_1) = f(n_2)$.

Then, $2n_1 - 1 = 2n_2 - 1$.

Adding 1 to both sides: $2n_1 = 2n_2$.

Dividing both sides by 2: $n_1 = n_2$. $\implies$ Thus, the function is one-to-one.

3. Onto

A function is onto if for every $y \in \mathbb{N} - \{2, 4, 6, 8, \dots\}, \exists n \in \mathbb{N}$ such that $f(n) = y$.

Let $y \in \mathbb{N} - \{2, 4, 6, 8, \dots\},$ so y is odd.

We need to find n such that $f(n) = y,$ i.e., $2n - 1 = y$.

Solving for n :

$$2n = y + 1,$$

$$n = \frac{y+1}{2}.$$

Since y is odd, $y + 1$ is even, and $\frac{y+1}{2} \in \mathbb{N}$. $\implies$ Thus, the function is onto.

4. Conclusion

Since f is both one-to-one and onto, it is a bijection.

Therefore, $|\mathbb{N}| = |\mathbb{N} - \{2, 4, 6, 8, \dots\}|$.

2. $|\{0, 2, 4, 6, 8, ...\}| = |\{1, 3, 5, 7, 9, ...\}|$

1. Define a Bijection

To prove $|\{0, 2, 4, 6, 8, \ldots\}| = |\{1, 3, 5, 7, 9, \ldots\}|,$ construct a bijection.

Define $f(n) = n + 1$.

2. One-to-One

A function is one-to-one if $f(n_1) = f(n_2)$ implies that $n_1 = n_2$.

Assume $f(n_1) = f(n_2)$.

Then, $n_1 + 1 = n_2 + 1$.

Subtracting 1 from both sides: $n_1 = n_2$.

Thus, the function is one-to-one.

3. Onto

A function is onto if for every $y \in \{1, 3, 5, 7, 9, \ldots\}, \exists n \in \{0, 2, 4, 6, 8, \ldots\}, f(n) = y$.

Let $y \in \{1, 3, 5, 7, 9, \ldots\}$.

We need to find n such that $f(n) = y,$ i.e., $n + 1 = y$.

Solving for $n :$

$$n = y - 1.$$

Since y is odd, $y - 1$ is even, and $n \in \{0, 2, 4, 6, 8, \ldots\}$.

Thus, the function is onto.

4. Conclusion

Since f is both one-to-one and onto, it is a bijection.

Therefore, $|\{0, 2, 4, 6, 8, \ldots\}| = |\{1, 3, 5, 7, 9, \ldots\}|$.

3. $|\mathbb{N}| < |\mathbb{R}|$

1. Cantor's Diagonal Argument:

To show $|\mathbb{N}| < |\mathbb{R}|,$ we prove that no bijection exists between $\mathbb{N}$ and $\mathbb{R}$.

Assume for contradiction that such a bijection exists, $f : \mathbb{N} \to \mathbb{R}$.

Construct the decimal expansion of each real number in $[0, 1)$ as follows:

$f(1) = 0.a_{11}a_{12}a_{13}\dots$

$f(2) = 0.a_{21}a_{22}a_{23}\dots$

$f(3) = 0.a_{31}a_{32}a_{33}\dots$

$\vdots$

Construct a new number $x = 0.b_1b_2b_3\dots$ such that $b_i \neq a_{ii}$.

By construction, x differs from $f(i)$ at the ith digit, for all i.

Thus, x is not in the range of $f,$ contradicting the assumption that f is a bijection.

2. Conclusion:

No bijection exists between $\mathbb{N}$ and $\mathbb{R}$. Therefore, $|\mathbb{N}| < |\mathbb{R}|$.

4. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$

1. Cantor's Theorem:

To show $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|,$ we prove that no bijection exists between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

Assume for contradiction that such a bijection exists, $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Define a set $S \subseteq \mathbb{N}$ as follows: $S = \{n \in \mathbb{N} : n \notin f(n)\}$.

Since $S \subseteq \mathbb{N},$ it must be in the codomain of $f,$ so there $\exists k \in \mathbb{N}$ such that $f(k) = S$.

Now, consider whether $k \in S :$

If $k \in S,$ then by definition of $S,$ we must have $k \notin f(k)$.

But $f(k) = S,$ so $k \notin S$. This is a contradiction.

If $k \notin S,$ then by definition of $S,$ we must have $k \in f(k)$.

But $f(k) = S,$ so $k \in S$. This is also a contradiction.

Thus, no such k can exist, and f cannot be a bijection.

2. Conclusion:

No bijection exists between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. Therefore, $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.

Equivalence Relations

Prove or disprove the following statements about equivalence relations:

1. Let R be a relation on the set of integers $\mathbb{Z}$ defined by $a\,R\,b$ if and only if $a-b$ is divisible by 3. Prove that R is an equivalence relation on $\mathbb{Z}$, and describe the equivalence classes of R.

1. Reflexive:

For any $a \in \mathbb{Z}$, $a - a = 0$ is divisible by 3, so $a\,R\,a$ holds.

2. Symmetric:

If $a\,R\,b,$ then $a - b = 3k$ for some $k \in \mathbb{Z}$.

Thus, $b - a = -3k$ is divisible by 3, so $b\,R\,a$ holds.

3. Transitive:

If $a\,R\,b$ and $b\,R\,c,$ then $a - b = 3k$ and $b - c = 3m$ for some $k, m \in \mathbb{Z}$.

Adding these gives $a - c = 3(k + m),$ which is divisible by 3, so $a\,R\,c$.

4. Equivalence Classes:

The equivalence classes are the sets of integers that differ by multiples of 3:

$[0] = \{\dots, -3, 0, 3, 6, \dots\}, \quad [1] = \{\dots, -2, 1, 4, 7, \dots\},$

$[2] = \{\dots, -1, 2, 5, 8, \dots\}.$

2. Let R be a relation on the set of all strings over the alphabet $\{a, b\}$ defined by $x\,R\,y$ if and only if x and y have the same length. Prove that R is an equivalence relation and describe the equivalence classes of R.

1. Reflexive:

For any string x, we have $|x| = |x|$, so $x\,R\,x$ holds.

2. Symmetric:

If $x\,R\,y$, then $|x| = |y|$.

Thus, $|y| = |x|$, so $y\,R\,x$ holds.

3. Transitive:

If $x\,R\,y$ and $y\,R\,z$, then $|x| = |y|$ and $|y| = |z|$.

Thus, $|x| = |z|$, so $x\,R\,z$ holds.

4. Equivalence Classes:

The equivalence classes are the sets of strings with the same length.

For any non-negative integer n, the equivalence class of strings of length n is:

$[n] = \{x \mid x \text{ is a string over } \{a, b\} \text{ with length } n\}$.

3. Let R be a relation on $\mathbb{R}$ defined by $a\,R\,b$ if and only if $a^2 = b^2$. Prove that R is an equivalence relation, and describe the equivalence classes of R.

1. Reflexive:

For any $a \in \mathbb{R},$ we have $a^2 = a^2,$ so $a\,R\,a$ holds.

2. Symmetric:

If $a\,R\,b,$ then $a^2 = b^2$.

Thus, $b^2 = a^2,$ so $b\,R\,a$ holds.

3. Transitive:

If $a\,R\,b$ and $b\,R\,c,$ then $a^2 = b^2$ and $b^2 = c^2$.

Thus, $a^2 = c^2,$ so $a\,R\,c$ holds.

4. Equivalence Classes:

The equivalence classes are the sets of real numbers with the same absolute value.

For any non-negative real number $r,$ the equivalence class of r is:

$[r] = \{x \in \mathbb{R} \mid |x| = r\} = \{-r, r\}$ (if $r \neq 0$), and $[0] = \{0\}$.

4. Let R be a relation on the set of all people, where $a\,R\,b$ if and only if a and b have the same birth year. Prove that R is an equivalence relation on the set of all people and describe the equivalence classes of R.

1. Reflexive:

For any person $a,$ the relation holds because a has the same birth year as a.

Thus, $a\,R\,a$ holds.

2. Symmetric:

If $a\,R\,b,$ then a and b have the same birth year.

Since having the same birth year is a mutual property, $b\,R\,a$ also holds.

3. Transitive:

If $a\,R\,b$ and $b\,R\,c,$ then a and b have the same year, and b and c have the same year.

Thus, a and c must also have the same birth year, so $a\,R\,c$ holds.

4. Equivalence Classes:

The equivalence classes of R are the sets of people born in the same year.

For any year $y,$ the equivalence class of people born in year y is:

$[y] = \{a \mid a \text{ is born in year } y\}$.

5. Let R be a relation on the set of all points in the plane $\mathbb{R}^2$ defined by $(x_1, y_1)\, R\, (x_2, y_2)$ if and only if $x_1 = x_2$ or $y_1 = y_2$. Prove that R is an equivalence relation and describe the equivalence classes of R.

1. Reflexive:

For any point $(x_1, y_1),$ we have $(x_1, y_1)\, R\, (x_1, y_1)$ since $x_1 = x_1$ and $y_1 = y_1$.

Thus, the relation is reflexive.

2. Symmetric:

If $(x_1, y_1)\, R\, (x_2, y_2),$ then either $x_1 = x_2$ or $y_1 = y_2$.

If $x_1 = x_2,$ then clearly $x_2 = x_1$.

If $y_1 = y_2,$ then clearly $y_2 = y_1$.

Thus, the relation is symmetric.

3. Transitive:

If $(x_1, y_1)\, R\, (x_2, y_2)$ and $(x_2, y_2)\, R\, (x_3, y_3),$

then either $x_1 = x_2$ or $y_1 = y_2,$ and either $x_2 = x_3$ or $y_2 = y_3$.

If $x_1 = x_2$ and $x_2 = x_3,$ then $x_1 = x_3$.

If $y_1 = y_2$ and $y_2 = y_3,$ then $y_1 = y_3$.

Thus, the relation is transitive.

4. Equivalence Classes:

The equivalence classes of R are the points where either the x-value y-value are equal.

For a given point $(x, y),$ the equivalence class of (x, y) is:

$[(x, y)] = \{(x', y') \mid x' = x \vee y' = y\}$.

Units Digit

Solve the following problems related to units digits:

1. Find the units digit of 7^{100}.

   To find the units digit of 7^{100}, observe the cycle in the units digits of powers of 7:

   $7, 9, 3, 1 \quad$ (cycle length is 4).

   Since $100 \div 4 = 25$ remainder 0, the units digit of 7^{100} is the same as that of 7^4.

   The units digit of 7^4 is 1.

   Thus, the units digit of 7^{100} is $\boxed{1}$.

2. Find the units digit of 3^{50}.

   To find the units digit of 3^{50}, observe the cycle in the units digits of powers of 3:

   $3, 9, 7, 1 \quad$ (cycle length is 4).

   Since $50 \div 4 = 12$ remainder 2, the units digit of 3^{50} matches that of 3^2.

   The units digit of 3^2 is 9.

   Thus, the units digit of 3^{50} is $\boxed{9}$.

3. Find the units digit of 12^{1234}.

   The units digits of powers of 12 repeat in a cycle: $2, 4, 8, 6$.

   To find the units digit of 12^{1234}, calculate $1234 \div 4$ which gives remainder 2.

   Thus, the units digit of 12^{1234} is the same as that of 12^2, which is $\boxed{4}$.

4. Find the units digit of 2^{987}.

   The units digits of powers of 2 repeat in a cycle: $2, 4, 8, 6$.

   To find the units digit of 2^{987}, calculate $987 \div 4$ which gives remainder 3.

   Thus, the units digit of 2^{987} is the same as that of 2^3, which is $\boxed{8}$.

5. Find the units digit of 9^{999}.

The units digits of powers of 9 repeat in a cycle of length 2: $9, 1$.

To find the units digit of 9^{999}, calculate $999 \div 2$ which gives remainder 1.

Thus, the units digit of 9^{999} is the same as that of 9^1, which is $\boxed{9}$.