CSE 215 Practice Questions

By Nicholas Smirnov

Truth Tables

Construct truth tables for the following statements:

1. $\sim (p \wedge r) \leftrightarrow (q \oplus r)$

2. $\sim (q \vee r) \rightarrow (p \oplus (r \wedge q))$

3. $((p \wedge q) \rightarrow r) \rightarrow (\sim q \vee \sim r)$

4. $(q \vee (r \oplus p)) \leftrightarrow (p \wedge (r \oplus q))$

5. $((p \rightarrow q) \wedge (q \rightarrow r)) \leftrightarrow (p \rightarrow r)$

Deduction Rules

Determine if the following deduction rules are valid:

1. $p \rightarrow q$

   $\sim r \rightarrow p$

   $\therefore q \vee r$

2. $p \leftrightarrow q$

   $\sim q \wedge \sim r$

   $\therefore \ \sim r$

3. $(p \oplus q) \rightarrow r$

   $(p \oplus r) \rightarrow q$

   $(q \oplus r) \rightarrow p$

   $\therefore p \wedge q \wedge r$

Logical Language

Rewrite the following sentences into two logically equivalent statements:

1. P is a necessary condition for Q.
2. P is a sufficient condition for Q.
3. P if and only if Q.
4. A necessary condition for R is P and Q.
5. R and T are both a necessary and sufficient condition for P or Q.

Logical Rules and Fallacies

Deduce if the statements are valid. If so, state which rule. If not, state which fallacy.

1. If you study math, you are smart.

   I do not study math.

   $\therefore$ I am not smart.

2. If you get above an 80 on this final, you get a B+.

   I got above an 80 on this final.

   $\therefore$ I get a B+.

3. If you are a good person, you pay taxes.

   I pay taxes.

   $\therefore$ I am a good person.

4. If you like cats, you like furry animals.

   I do not like furry animals.

   $\therefore$ I do not like cats.

Logical Deduction (Many Premises)

Use the valid arguments forms to deduce the conclusion from the premises.

1. $a \to \sim f$

   $a \vee b$

   $(b \wedge f) \to d$

   f

   $e \to \sim d$

   $\therefore \sim e$

2. $\sim h \to f$

   $c \to \sim (f \wedge g)$

   g

   $h \to f$

   $c \vee q$

   $\therefore q$

Logic with Quantifiers

Find negations for the following statements:

1. There exists a student such that they have a higher grade than all other students.
2. For all animals, if you are a pet, then you have an owner.
3. $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy \geq 0.$
4. Passing both midterms is a sufficient condition to do well in this class.
5. If you get a 100% on the final or 100% on both midterms, you are going to get an A.
6. $\forall x, \forall y, \forall z, \exists \alpha, \exists \beta, \exists \zeta,\ \alpha^\beta + \zeta \geq xyz \geq \alpha^\beta - \zeta$

Quantifiers

Deduce if the following statements are true or false:

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy \geq 0.$

2. $\forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, xy > 0.$

3. $\forall x, y \in \mathbb{Z}^+, (x^2 > y^2) \rightarrow (x > y).$

4. $\forall x, y \in \mathbb{Z}, (\frac{x}{y} > \frac{y}{x}) \rightarrow (x \neq y).$

5. $\forall x, y \in \{\mathbf{c}, \mathbf{t}\}, \exists z \in \{\mathbf{c}, \mathbf{t}\}, (x \wedge y) \rightarrow z \equiv \mathbf{t}$

Direct Proofs

Prove each of the following using a direct proof method:

1. The sum of any two odd integers is even.

2. If n and m are odd, then nm is also odd.

3. The product of any two consecutive integers is even.

4. If $a|p$ and $p|q$, then $a|q$

Proofs by Contrapositive

Prove each of the following using the contrapositive method:

1. If pq is even, then p or q is even.

2. If $n^2 - 6n + 5$ is even, then n is odd.

3. If $x^2 + 5x + 6 \neq 0$, then $x \notin \{-3, -2\}$.

4. If 3 doesn't divide xy, then 3 doesn't divide x and y.

Proofs by Contradiction

Prove each of the following using the contradiction method:

1. If x^2 is irrational, then x is irrational.

2. $\sqrt{2}$ is irrational.

3. If ab is irrational and a is rational, then b is irrational.

4. There doesn't exist a largest number.

5. There is no smallest positive real number.

Proofs by Induction

Prove each of the following through induction:

1. The sum of the first n odd numbers is n^2.

2. For all $n \geq 1$,
$$1 + x + x^2 + ... + x^n = \frac{x^{n+1} - 1}{x - 1} \quad \text{where } x \neq 1$$

3. For all $n \geq 1$,
$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

4. For all $n \geq 2$,
$$\sum_{i=2}^{n} i^2(i-1) = \frac{n(n^2-1)(3n+2)}{12}$$

5. For all $n \geq 1$, $5^n + 3$ is divisible by 4.

6. For all $n \geq 1$, $4^{2n} - 1$ is divisible by 15.

7. For all $n \geq 1$, $4^n + 6n - 1$ is divisible by 3.

Proofs by Strong Induction

Prove each of the following using strong induction:

1. For all $n \geq 1$, the n-th term of the sequence defined by

$$a_n = \begin{cases} n & \text{if } n = 1 \text{ or } n = 2, \\ a_{n-1} + 2a_{n-2} & \text{if } n \geq 3, \end{cases}$$

   is given by $a_n = 2^{n-1}$.

2. For the Fibonacci sequence defined as

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3,$$

   prove that $F_n < 2^n$ for all $n \geq 1$.

3. Prove that the n-th Fibonacci term can be written as $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Set Theory (Element Methods)

Prove each of the following using an element-based argument:

1. For all sets A, B, and C, prove that $(A \cup B) - (A \cap C) = (A - C) \cup (B - C)$.

2. For all sets A, B, and C, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

3. For all sets A, B, and C, prove that $A \times (B - C) = (A \times B) - (A \times C)$.

4. For all sets A, B, and C, prove that if $A \subseteq B$ and $C \subseteq B$, then $A \times C \subseteq B \times B$.

Set Theory (Algebraic Methods)

Prove each of the following using algebraic-based method:

1. For all sets A, B, and C, prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
2. For all sets A, B, and C, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.
3. For all sets A, B, and C, prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
4. For all sets A, B, and C, prove that $(A \cap B) \cup (A - B) = A$.
5. For all sets A, B, and C, prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

Set Theory Counterexamples

Provide a counterexample to disprove each of the following:

1. For all sets A and B, $A \cup B = A - B$ if and only if $A = B$.
2. For all sets A, B, and C, $(A \cup B) - C = (A - C) \cap (B - C)$.
3. For all sets A, B, and C, $A - (B \cap C) = (A - B) \cap (A - C)$.
4. For all sets A, B, and C, $A \times (B \cup C) = (A \times B) \cap (A \times C)$ holds for all A, B, and C.

One-to-One Correspondence

Deduce if the following functions are one-to-one correspondences:

1. Define a function $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 2x + 1$.
2. Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.
3. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x^2$.
4. Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (4y, 2x)$.
5. Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(x, y) = (2y, 3x)$.

Infinite Sets

Prove or disprove the following statements regarding infinite sets:

1. $|\mathbb{N}| = |\mathbb{N} - \{2, 4, 6, 8, ...\}|$
2. $|\{0, 2, 4, 6, 8, ...\}| = |\{1, 3, 5, 7, 9, ...\}|$
3. $|\mathbb{N}| < |\mathbb{R}|$
4. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$

Equivalence Relations

Prove or disprove the following statements about equivalence relations:

1. Let R be a relation on the set of integers $\mathbb{Z}$ defined by $a\,R\,b$ if and only if $a-b$ is divisible by 3. Prove that R is an equivalence relation on $\mathbb{Z}$, and describe the equivalence classes of R.
2. Let R be a relation on the set of all strings over the alphabet $\{a, b\}$ defined by $x\,R\,y$ if and only if x and y have the same length. Prove that R is an equivalence relation and describe the equivalence classes of R.
3. Let R be a relation on $\mathbb{R}$ defined by $a\,R\,b$ if and only if $a^2 = b^2$. Prove that R is an equivalence relation, and describe the equivalence classes of R.
4. Let R be a relation on the set of all people, where $a\,R\,b$ if and only if a and b have the same birth year. Prove that R is an equivalence relation on the set of all people and describe the equivalence classes of R.
5. Let R be a relation on the set of all points in the plane $\mathbb{R}^2$ defined by $(x_1, y_1)\,R\,(x_2, y_2)$ if and only if $x_1 = x_2$ or $y_1 = y_2$. Prove that R is an equivalence relation and describe the equivalence classes of R.

Units Digit

Solve the following problems related to units digits:

1. Find the units digit of 7^{100}.

2. Find the units digit of 3^{50}.

3. Find the units digit of 12^{1234}.

4. Find the units digit of 2^{987}.

5. Find the units digit of 9^{999}.